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SOLUBLE APPROXIMATION OF LINEAR SYSTEMS IN MAX–PLUS ALGEBRA

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We propose an efficient method for finding a Chebyshev-best soluble approximation to an insoluble system of linear equations over max-plus algebra.

Keywords: discrete-event dynamic systems, max-plus algebra, systems of linear equations, approximation

AMS Subject Classification: 93B25, 15A06, 06F05, 37M99

1. INTRODUCTION

It is well-known [1, 4] that the structure of many discrete-event dynamic systems may be represented by square matrices A over the *max-plus semiring*

$$\Re = (\{-\infty\} \cup R, \oplus, \otimes) = (\{-\infty\} \cup R, max, +).$$

For example, if the initial event-times of such a system are represented by a vector $\mathbf{s}$, then the event-times after r stages are given by the rth term of the *orbit*

$$\{A^{(r)} \otimes \mathbf{s}(r = 1, 2, \ldots)\} \quad \text{where} \quad A^{(r)} = A \otimes A \otimes \ldots \otimes A(r\text{-fold}).$$

The *reachability problem* asks whether $\mathbf{s}$ can be chosen so that the orbit contains a given vector $\mathbf{b}$. Clearly, the answer is affirmative if and only if event-times $\mathbf{b}$ can be achieved after one stage from suitable previous event-times, so algebraically the reachability problem produces the linear-equations problem: to solve $A \otimes \mathbf{x} = \mathbf{b}$.

In a practical situation, the data may be such that an exact solution is not possible. In [4] it was shown how to find the maximum solution to the inequality $A \otimes \mathbf{x} \leq \mathbf{b}$ – the so-called *principal solution* – from which may be inferred the Chebyshev-least perturbation of $\mathbf{b}$ necessary to make the system $A \otimes \mathbf{x} = \mathbf{b}$ soluble. Some necessary facts relevant to this are reviewed in the next section.

In [5], the same problem was solved for the related algebraic system *fuzzy algebra*. The question of achieving solubility by modifying the matrix A was examined for fuzzy algebra in [2], while for both fuzzy algebra and $\Re$ the search for solubility by omitting equations was shown in [3] to lead to an NP-complete problem.

In the present paper, we consider how solubility may be achieved for a system $A \otimes \mathbf{x} = \mathbf{b}$ over $\Re$ if both A and $\mathbf{b}$ may be perturbed. Specifically, we seek a Chebyshev-least perturbation, consistent with solubility, of the matrix $[A, \mathbf{b}]$.

2. PRELIMINARIES

In the system $\Re$, we write $a^{(r)}$ to denote the r-fold product $a \otimes \ldots \otimes a$. Since the operation $\otimes$ represents arithmetical addition, $a^{(r)}$ has the value ra. $a^{(-1)}$ is the multiplicative inverse in $\Re$, hence $a^{(-1)} = -a$.

The system $\Re$ is embeddable in the self-dual system

$$\Im = (\{-\infty\} \cup R \cup \{+\infty\}, \oplus, \otimes, \oplus', \otimes') = (\{-\infty\} \cup R \cup \{+\infty\}, \max, +, \min, +)$$

where the operations $\otimes, \otimes'$, representing arithmetical addition, differ only in that

$$-\infty \otimes +\infty = -\infty, \qquad -\infty \otimes' +\infty = +\infty.$$

The set of all m by n matrices over $\Im$ will be denoted by $\Im(m,n)$, the set of all $m-$vectors by $\Im(m)$ and the operations $\oplus, \otimes$ and $\oplus', \otimes'$ are extended to matrix algebra in the usual way. Matrices will be denoted by upper-case italics and vectors by lower-case bold letters.

For any matrix $A = [a_{ij}] \in \Im(m,n)$, the *conjugate matrix* is $A^* = [-a_{ji}] \in \Im(n,m)$ obtained by negation and transposition. We shall use the following properties of conjugation (compare [4, p. 5])

$$(A^*)^* = A \text{ and } (A \otimes B)^* = B^* \otimes' A^*. \tag{1}$$

A set of linear inequalities $A \otimes \mathbf{x} \leq \mathbf{b}$ over $\Re$ always possesses a solution. The greatest is

$$\mathbf{x}^p(A, \mathbf{b}) = A^* \otimes' \mathbf{b}. \tag{2}$$

This *principal solution* is calculated in $\Im$ but lies in $\Re$. It is also the greatest solution of $A \otimes \mathbf{x} = \mathbf{b}$ if and only if any solution exists (see [4, p. 5] and [1, p. 112]).

For brevity, in what follows, the symbol $[A, \mathbf{b}]$ for $A \in \Im(m,n), \mathbf{b} \in \Im(m)$ represents the $m \times (n+1)$ matrix obtained by appending column $\mathbf{b}$ as column $n+1$ to matrix A.

Definition 1. Given two matrices $P, Q \in \Im(m,n)$, their Chebyshev distance will be denoted by $\Delta(P,Q) = \max_{i,j} |p_{ij} - q_{ij}|$.

Definition 2. For two given integers m, n denote the family of all soluble max-plus linear systems with n unknowns and m equations by

$$\mathcal{S}(m,n) = \{(A, \mathbf{b}); A \in \Im(m,n), \mathbf{b} \in \Im(m);\ \text{system } A \otimes \mathbf{x} = \mathbf{b} \text{ is soluble}\}.$$

A Chebyshev-best soluble approximation of an insoluble system

$$A \otimes \mathbf{x} = \mathbf{b}, A \in \Im(m,n), \mathbf{b} \in \Im(m)$$

is a pair $A' \in \Im(m,n), \mathbf{b}' \in \Im(m)$ such that $(A', \mathbf{b}') \in \mathcal{S}(m,n)$ and

$$\Delta([A', \mathbf{b}'], [A, \mathbf{b}]) \leq \Delta([A'', \mathbf{b}''], [A, \mathbf{b}])$$

for each pair $(A'', \mathbf{b}'') \in \mathcal{S}(m,n)$.

Let us denote by

$$\delta^{+}(B \otimes \mathbf{x}; \mathbf{b}) = \max_{i}\{(B \otimes \mathbf{x})_i - b_i\}$$

and by

$$\delta^{-}(B \otimes \mathbf{x}; \mathbf{b}) = \min_{i}\{(B \otimes \mathbf{x})_i - b_i\}$$

the extreme positive and the extreme negative deviation of $B \otimes \mathbf{x}$ from $\mathbf{b}$, respectively. In notation of max-plus algebra

$$\delta^{+}(B \otimes \mathbf{x}; \mathbf{b}) = \mathbf{b}^{*} \otimes (B \otimes \mathbf{x})$$

and

$$\delta^{-}(B \otimes \mathbf{x}; \mathbf{b}) = \mathbf{b}^{*} \otimes' (B \otimes \mathbf{x}).$$

Note that if $\hat{\mathbf{x}} = \mathbf{x}^{p}(B, \mathbf{b})$ then $\delta^{+}(B \otimes \hat{\mathbf{x}}; \mathbf{b}) = 0$ and $\delta^{-}(B \otimes \hat{\mathbf{x}}; \mathbf{b}) \leq 0$, moreover $\delta^{-}(B \otimes \hat{\mathbf{x}}, \mathbf{b}) = 0$ if and only if the system $B \otimes \mathbf{x} = \mathbf{b}$ is soluble.

Theorem 1. Let $A \in \Im(m,n)$ and $\mathbf{b} \in \Im(m)$ be such that $(A, \mathbf{b}) \notin \mathcal{S}(m,n)$; let us define

$$\delta = (\delta^{-}(A \otimes \mathbf{x}^{p}(A, \mathbf{b}); \mathbf{b}))^{(1/4)}. \tag{3}$$

If $B \in \Im(m,n)$ is such that $\Delta(A, B) \leq \delta$, i. e.

$$\delta^{(-1)} \otimes A \leq B \leq \delta \otimes A,$$

then $\Delta(B \otimes \mathbf{x}, \mathbf{b}) \geq \delta$ for each $\mathbf{x} \in \Im(n)$, with equality only if $(\mathbf{x}^{p}(A, \mathbf{b}))^{*} \otimes \mathbf{x} = \delta^{(2)}$.

P r o o f. Let $(\mathbf{x}^{p}(A, \mathbf{b}))^{*} \otimes \mathbf{x} = \varepsilon^{(2)}$. This means that $\max_j\{x_j - (\mathbf{x}^{p}(A, \mathbf{b}))_j\} = \varepsilon^{(2)}$, hence for each j $x_j \leq \varepsilon^{(2)} + (\mathbf{x}^{p}(A, \mathbf{b}))_j$; or in max-plus algebra notation $\mathbf{x} \leq \varepsilon^{(2)} \otimes \mathbf{x}^{p}(A, \mathbf{b})$. Two cases arise:

1. $\varepsilon \geq \delta$. Since $B \geq \delta^{(-1)} \otimes A$, we have

$$\begin{aligned}
\delta^{+}(B \otimes \mathbf{x}, \mathbf{b}) &= \mathbf{b}^{*} \otimes (B \otimes \mathbf{x}) \geq \\
&\geq \delta^{(-1)} \otimes \mathbf{b}^{*} \otimes (A \otimes \mathbf{x}) = \\
&= \delta^{(-1)} \otimes (A^{*} \otimes' \mathbf{b})^{*} \otimes \mathbf{x} = \ \text{(by (1) and associativity of } \otimes\text{)} \\
&= \delta^{(-1)} \otimes (\mathbf{x}^{p}(A, \mathbf{b}))^{*} \otimes \mathbf{x} = \ \text{(by (2))} \\
&= \delta^{(-1)} \otimes \varepsilon^{(2)} \geq \delta.
\end{aligned}$$

2. $\varepsilon < \delta$. Since $B \le \delta \otimes A$ and $\mathbf{x} \le \varepsilon^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})$, we have

$$\begin{aligned}
\delta^-(B \otimes \mathbf{x}, \mathbf{b}) &= \mathbf{b}^* \otimes' (B \otimes \mathbf{x}) \le \\
&\le \mathbf{b}^* \otimes' (\delta \otimes A \otimes \varepsilon^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})) = \\
&= \delta \otimes \varepsilon^{(2)} \otimes \mathbf{b}^* \otimes' (A \otimes \mathbf{x}^p(A, \mathbf{b})) = \quad \text{(by commutativity of scalar multiplication)} \\
&= \delta \otimes \varepsilon^{(2)} \otimes \delta^{(-4)} < \quad \text{(by (3))} \\
&< \delta^{(-1)}.
\end{aligned}$$

Hence either $\delta^+(B \otimes \mathbf{x}, \mathbf{b}) \ge \delta$ or $\delta^-(B \otimes \mathbf{x}, \mathbf{b}) < \delta^{(-1)}$ and so $\Delta(B \otimes \mathbf{x}; \mathbf{b}) \ge \delta$. □

3. ALGORITHM APPROXIMATION

Input: Matrix $A \in \Im(m, n)$, vector $\mathbf{b} \in \Im(m)$.

Output: A pair $(A', \mathbf{b}') \in \mathcal{S}(m, n)$ with $\Delta([A, \mathbf{b}], [A', \mathbf{b}'])$ smallest possible.

Step 1. Find the principal solution $\mathbf{x}^p(A, \mathbf{b})$ and $\delta := (\Delta(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}))^{(1/4)}$.

Step 2. $\hat{\mathbf{x}} := \delta^{(2)} \otimes \mathbf{x}^p(A, \mathbf{b})$.

Step 3. **For each** row i **with** $b_i^* \otimes' (A \otimes \hat{\mathbf{x}})_i = \varepsilon_i^{(2)}$ **do** (comment $|\varepsilon_i| \le \delta$)
begin $b_i' := \varepsilon_i \otimes b_i$; **for all** j **do** $a_{ij}' = \varepsilon_i^{(-1)} \otimes a_{ij}$ **end**.

Example. Suppose the following matrix A and vector $\mathbf{b}$ are given.

$$A = \begin{pmatrix} 10 & -1 & 11 \\ 9 & 11 & 5 \\ 5 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}; \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$$

We compute successively

$$\mathbf{x}^p(A, \mathbf{b}) = \begin{pmatrix} -10 & -9 & -5 & -1 \\ 1 & -11 & 0 & 2 \\ -11 & -5 & -2 & 0 \end{pmatrix} \otimes' \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -8 \\ -9 \end{pmatrix}; A \otimes \mathbf{x}^p(A, \mathbf{b}) = \begin{pmatrix} 2 \\ 3 \\ -3 \\ -7 \end{pmatrix}$$

so the Chebyshev error is $\Delta(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}) = \delta^{(4)} = 8$ and it is achieved in row 4. Now,

$$\hat{\mathbf{x}} = \begin{pmatrix} -4 \\ -4 \\ -5 \end{pmatrix}; A \otimes \hat{\mathbf{x}} = \begin{pmatrix} 6 \\ 7 \\ 1 \\ -3 \end{pmatrix}; \varepsilon^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 0 \\ -4 \end{pmatrix}; A' = \begin{pmatrix} 8 & -3 & 9 \\ 7 & 9 & 3 \\ 5 & 0 & 2 \\ 3 & 0 & 2 \end{pmatrix}; \mathbf{b}' = \begin{pmatrix} 4 \\ 5 \\ 1 \\ -1 \end{pmatrix}.$$

Theorem 2. Algorithm APPROXIMATION correctly finds in $O(mn)$ steps a Chebyshev-best soluble approximation of system $A \otimes \mathbf{x} = \mathbf{b}, A \in \Im(m, n), \mathbf{b} \in \Im(m)$ over max-plus algebra.

P r o o f. Notice, that for $\hat{x}$ defined in the second step of the algorithm, $\delta^+(\delta^{(2)} \otimes A \otimes x^p(A, b); b) = \delta^{(2)}$, $\delta^-(\delta^{(2)} \otimes A \otimes x^p(A, b); b) = \delta^{(-2)}$, and hence $\Delta(A\hat{x}, b) = \delta^{(2)}$.

Then, system $A' \otimes \mathbf{x} = \mathbf{b}'$ is soluble, $\hat{\mathbf{x}}$ being a solution. Further, $\Delta([A, \mathbf{b}], [A', \mathbf{b}']) \le \delta$. Moreover, Theorem 1 shows that it is impossible to find a soluble system $A'' \otimes \mathbf{x} = \mathbf{b}''$ with Chebyshev error $\triangle([A, b], [A'', b''])$ smaller than δ.

The complexity bound is trivial. □

In conclusion, we recall [4, p. 5] the important property of $\mathbf{x}^p(A, \mathbf{b})$ that no $\mathbf{x}$ can have both

$$\delta^+(A \otimes \mathbf{x}, \mathbf{b}) \le 0 \text{ (i.e. } A \otimes \mathbf{x} \le \mathbf{b})$$

and

$$\delta^-(A \otimes \mathbf{x}, \mathbf{b}) > \delta^-(A \otimes \mathbf{x}^p(A, \mathbf{b}), \mathbf{b}) = \delta^{(-4)}.$$

Setting $\mathbf{x} = \delta^{(-2)} \otimes \mathbf{y}$, it follows that no $\mathbf{y}$ can have $\triangle(A \otimes \mathbf{y}, \mathbf{b}) < \delta^{(-2)}$ (see also [6]). In other words, to produce a soluble approximation if only $\mathbf{b}$ may be perturbed incurs at best a Chebyshev error double that incurred at best if both A and $\mathbf{b}$ may be perturbed.

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