CONSTRAINED STABILIZATION
OF A DYNAMIC SYSTEM: A CASE STUDY

F. Blanchini[1], S. Cotterli, G. Koruza, S. Miani, R. Siagri
and L. Tubaro

In this work we consider the problem of determining and implementing a state feedback stabilizing control law for a laboratory two-tank dynamic system in the presence of state and control constraints. We do this by exploiting the properties of the polyhedral Lyapunov functions, i. e. Lyapunov functions whose level surfaces are polyhedra, in view of their capability of providing an arbitrarily good approximation of the maximal set of attraction, which is the largest set of initial states which can be brought to the origin with a guaranteed convergence speed.

We will first recall the basic theoretical background necessary for the scope and then we will report and analyze the results of the practical implementation on a two-tank laboratory system of a linear variable-structure and a quantized control law proposed in literature. Finally an heuristic procedure for the determination of a static linear gain will be presented.

1. INTRODUCTION

In the practical implementation of state feedback control laws there are normally several aspects which the designer has to keep in consideration and which impose restrictions on the allowable closed loop behavior. For instance a certain robustness of the closed loop system is desirable if not necessary to guarantee a stable functioning under different operating conditions which might be for example caused by effectively different set points, component obsolescence, neglected nonlinearities or high frequencies modes. Another issue which has surely to be taken into account is most often the presence of constraints on the control values and on the state variables. The former usually derives from saturation effects of the actuators whereas the latter normally comes from the necessity of keeping the states in a region in which the linearized model represents a good approximation of the real plant or might even be imposed by safety considerations.

The constrained control stabilization is by itself a challenging matter and in this context the designer can either analyze the effects of saturating a stabilizing control law or he can include the constraints in the controller requirements. If stability is the

---

[1]Corresponding author.

only matter of concern then the first approach is indeed the easiest, although this advantage is balanced by the extremely restricted set of initial states which can be asymptotically driven to the origin [9, 10], say the attraction set. Moreover if state constraints and uncertainties have to be considered then the first approach shows up its deficiencies so that the second approach appears definitely as the most preferable one. In this second class there are several techniques which can be followed to purse the desired performance specification while satisfying the imposed constraints and among these one of the approaches which can be used to overcome these limitations is that based on invariant regions [1, 2, 7, 8, 11, 12, 13, 15].

The key idea which lies behind this approach is that of determining a set of initial conditions starting from which the state evolution can be brought to the origin while assuring that no control and state constraint violation occur. This is quite a standard approach and practically amounts to determining a candidate Lyapunov function for the constrained system which can be made decreasing along the system trajectories by a proper choice of the feedback control. Of course there is a certain freedom in the choice of such Lyapunov functions. From the existing literature it turns out that the class of quadratic functions has been the most investigated one mainly due to the elegant and powerful results existing in this area. Although this class is well established and capable of furnishing simple linear control laws, it is not perfectly suited for constrained control synthesis problem due to its conservativity.

For this reason in the last years several authors [2, 5, 7, 12, 13, 15] have put their attention on the class of polyhedral functions (say functions whose level surfaces are polyhedrons in $\mathbb{R}^n$) and the associated polyhedral invariant sets. These functions have their strength in their capability of well representing linear constraints on state and control variables while being representable by a finite number of parameters.

In this work we will focus on the problem of determining a state feedback stabilizing control law for a constrained dynamic laboratory system affected by structured memoryless uncertainties. First the solution will be given in terms of polyhedral Lyapunov functions and then we will discuss the issues connected with the implementation of two nonlinear control laws already proposed in literature.

In Section 2 we will report some preliminary definitions and in the following section we will briefly report some known results concerning the constrained stabilization of dynamic systems by means of polyhedral Lyapunov functions. Then, based on these results, in Section 4 we will focus our attention on the determination of stabilizing control laws whose application to the system under consideration will be reported in Section 5. Finally in Section 6 we will report some final considerations and the directions for further research in this area.

Schematically, the outline of the present paper will be the following:
– Definitions
– Brief summary of theoretical results on constrained control via polyhedral invariant sets.
– Description of the linear variable structure and discontinuous control.
– Determination of a polyhedral Lyapunov function and implementation of the cited control laws on a two-tank laboratory system.
– Analysis of the results and final considerations.

2. DEFINITIONS

We will denote with $conv(S)$ the convex hull of a set $S \subseteq \mathbb{R}^n$. We will call C-set a closed and convex set containing the origin as an interior point. Given a C-set $P \subseteq \mathbb{R}^n$ we will denote with $\lambda P = \{y = \lambda x, x \in P\}$ and with ∂P its border. A C-set P will be said symmetric if $x \in P$ implies $-x \in P$. In this work we will be dealing with symmetric polyhedral C-sets which can be represented in terms of their delimiting planes as

$$P = \{x \in \mathbb{R}^n : |F_i x| \leq 1, i = 1, \ldots, s\},$$

(each F_i represents an n-dimensional row vector) as well as by their dual representation

$$P = conv(v_1, \ldots, v_k) = conv(V),$$

in terms of vertex set $V = \{v_1\, v_2 \ldots v_r\}$, which will be denoted by $vert\{P\}$. For these sets it is possible to introduce a compact notation using component-wise vector inequalities with which the set expression becomes

$$P = \{x : |Fx| \leq \bar{1}\}$$

or, using the dual notation,

$$P = \{x = V\alpha,\, \alpha \in \mathbb{R}^r,\, \alpha \geq 0,\, \|\alpha\|_1 \leq 1\},$$

where F is an $s \times n$ full column rank matrix having rows F_i and V is the full row rank matrix having the vertices v_i as column defined above, and $\bar{1} = [1 \ldots 1]^T$ represents an s-dimensional unitary column vector.

It is known that every symmetric C-set P induces a norm $\Psi_P(\cdot)$ on $\mathbb{R}^n$ defined as

$$\Psi_P(x) = \min\{\lambda : x \in \lambda P\}.$$

For a polyhedral set $P = \{x : |Fx| \leq \bar{1}\}$ the above expression can be simplified as

$$\Psi_P(x) = \max_i |F_i x|$$

and in this case we will denote by $I(x)$ the set of indexes for which $|F_i x|$ is maximum

$$I(x) = \{i : |F_i x| = \Psi_P(x)\}. \tag{1}$$

3. SET–INDUCED LYAPUNOV FUNCTIONS

In this section we will recall some of the results concerning polyhedral Lyapunov functions and their use in the stabilization of uncertain linear dynamic time-invariant systems in the presence of control and, possibly, state constraints. Consider a continuous-time uncertain dynamic system of the form

$$\dot{x}(t) = A(w)\, x(t) + B(w)\, u(t) \tag{2}$$

where the state and control values are constrained to belong to the C-sets $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^p$ for every $t \geq 0$ and the matrices $A(w)$ and $B(w)$ belong to the polytopes of matrices

$$A(w) = \sum_{i=1}^{p} w_i A_i, \quad B(w) = \sum_{i=1}^{p} w_i B_i.$$

The uncertainty $w(t)$ is assumed to be piecewise continuous and such that

$$w \in \mathcal{W} = \left\{ w : \sum_{i=1}^{p} w_i = 1, \ w_i \geq 0 \right\}. \tag{3}$$

For this system we want to provide a stabilizing control law and a region of initial states starting from which the constraints are never violated for the closed loop system's motion for every possible uncertainty sequence, in other words we want to provide a domain of attraction whose definition is now reported.

Definition 3.1. [4] The C-set $S \subset X$ is a domain of attraction (with speed of convergence β) for system (2) if there exists $\beta > 0$ such that for all $x_0 \in S$ there exists a piecewise continuous control function $u(\cdot) : \mathbb{R} \to U$ such that the trajectory $x(t)$ with initial condition $x(0) = x_0$ corresponding to $u(t)$ is such that

$$\Psi_S \left(x(t) \right) \leq e^{-\beta t} \Psi_S \left(x(0) \right) \tag{4}$$

for every possible $w(t)$ as in (3). If we take $\beta = 0$, the set S is simply said to be U-invariant [7].

If we temporarily assume that there is no uncertainty then it is immediate that a first solution can be obtained by selecting a stabilizing linear static state feedback control law $u = Kx$ and then picking as set of initial states the ellipsoidal region $X_0 = \{x : x^T P x \leq d\}$, where P is the solution of the Lyapunov equation of the closed loop system and $d \geq 0$ is the maximal value such that $X_0 \subset (X_K \cap X)$, being $X_K = \{x : Kx \in U\}$. Unfortunately an inappropriate choice of the gain K might result in a very small or either empty if uncertainties are considered) set of attraction whereas we are normally interested in determining, given the constraint sets X and U, the maximal set of initial states which can be asymptotically taken to the origin.

One of the possible ways to proceed is that of trying to maximize the set of attraction as done in [4]. The results contained in the above reference allow to provide an arbitrarily close approximation of the maximal domain of attraction by exploiting the relation existing between the original system (2) and its discrete-time approximating system, the Euler Approximating System (EAS), which is defined as

$$x(k+1) = [I + \tau A(w)] x(k) + \tau B(w) u(k), \quad \tau > 0. \tag{5}$$

For this system, in the presence of control and state constraints, it is possible to give a definition of domain attraction which is almost identical to the one given

in the continuous-time case apart from the replacement of the constant $\beta > 0$ with $0 < \lambda \leq 1$ and the decreasing condition (4) which becomes

$$\Psi_S\left(x(k)\right) \leq \lambda^k \Psi_S\left(x(0)\right). \tag{6}$$

After these necessary definitions we are now able to summarize the main results concerning the constrained control of continuous-time dynamic systems.

1. If there exists a C-set S which is a domain of attraction for (2) with a speed of convergence $\beta > 0$ then for all $\beta' < \beta$ there exists $\tau > 0$ such that the set S is contractive for the EAS with $\lambda' = 1 - \tau\beta'$.

2. If, for some $0 \leq \lambda < 1$, there exists a λ-contractive C-set P for the EAS (5) then P is a domain of attraction for (2) with $\beta = \frac{1-\lambda}{\tau}$. Moreover, there exists a Lipschitz feedback control function $\Phi : P \to U$ assuring condition (4).

3. For every ϵ_1, $\epsilon_2 > 0$ the set $S_{\bar{\beta}}$ (the largest domain of attraction in X for (2) with speed of convergence $\bar{\beta} > 0$) can always be approximated with a polyhedral C-set P such that $(1 - \epsilon_1)S_{\bar{\beta}} \subset P \subset X$ and such that P is a domain of attraction for (2) with speed of convergence β, with $\bar{\beta} - \epsilon_2 < \beta < \bar{\beta}$. Moreover the control can be expressed in a feedback form $u = \Phi(x)$ where Φ is a Lipschitz function on P.

4. It is possible to determine such polyhedral functions by means of a numerical procedure which can be applied to the EAS of (2).

In simpler words the meaning of the above results is the following: for a given $\beta > 0$ we can get an arbitrarily close approximation of the largest domain of attraction (with speed of convergence β) for (2) by applying the numerical procedure suggested in [4] to the EAS (5) for an appropriate choice of the parameter τ which finally depends mainly on how close we want this approximation to be.

4. LINEAR VARIABLE STRUCTURE AND DISCONTINUOUS CONTROL LAW

Once a polyhedral approximation of the domain of attraction for (2) with a certain speed of convergence has been found, a feedback control law has to be provided. In this section we focus our attention on the determination of two stabilizing control laws whose practical implementation will be presented in the next section. Let then

$$P = \{x : |F_i x| \leq \bar{1}, \ i = 1, \ldots, s\}$$

be the contractive set which resulted from applying the mentioned procedure to the EAS of (2). To each vertex v_i of P remains associated a control value u_i (actually provided by the procedure itself). The set P can be naturally partitioned in simplicial sectors S_h each delimited by the origin and n vertices $v_{h_1} \ldots v_{h_n}$ which lay on the same face $cx = 1$ delimiting the set P (hence the row vector c is either F_i or $-F_i$, for some i) [7]. The partition can be made in such a way that two of these sectors have intersection with empty interior and the union of all the sectors is P. Thus every $x \in P$ belongs necessarily to at least one of these sectors so that it is

possible to define a mapping $h = h(x)$ from the state space to the set of the sectors indexes, where $h(x)$ is the index of a sector containing x. Now, if to each of these sectors we associate the gain K^h defined as

$$K^h = [u_{h_1} \ldots u_{h_n}] \, [v_{h_1} \ldots v_{h_n}]^{-1}$$

where the matrix $[v_{h_1} \ldots v_{h_n}]$ is invertible by construction, we have that the linear variable structure control law defined as

$$u_{vs}(x) = K^{h(x)} x$$

stabilizes the system and guarantees condition (4) for every initial state $x(0) \in P$. Moreover it can be shown [3] that this control law is continuous.

It is quite obvious that such partition, to reduce the computational load of the proposed control law, should be computed off line so that on-line the algorithm has just to check to which sector the state belongs. If the contractive set under consideration is simplicial, say every face contains exactly n vertices, the determination of the partition is immediate and the required map is indeed $h(x) = I(x)$ as defined in (1). Unfortunately the construction of the maximal λ-contractive set most often generates a non-simplicial set $P \subset \mathbb{R}^n$ (say a set whose delimiting planes contain more than n vertices, see [6]) hence such a partition results in being an essential point for the practical implementation of the controller (apart obviously from the two dimensional case where every polyhedron is simplicial). One possible way to derive the proceeding partition is by means of a 'stretching' procedure which, starting from the original polyhedron P generates a supporting polyhedron $\tilde{P}$ [6] which is nothing but a polyhedral representation of the mentioned partition. We skip the description of such procedure for the sake of brevity (the interested readers can contact the corresponding author for the Fortran code almost in its final form).

The major drawback of computing the auxiliary polytope and using its expression in the on-line implementation of the control law is unfortunately given by the high number of simplicial sectors of the supporting polyhedron. To avoid the burdens deriving from this the authors have recently proposed a discontinuous control law applicable to single input continuous-time systems (the extension to the multi-input case is under development) and which relies solely on the contractive region P and which is now reported.

Suppose a β-contractive symmetric region $P = \{x : |Fx| \leq \bar{1}\}$ for a single input continuous-time system has been found and that the control constraint C-set can be written as $U = [u_{\min}, u_{\max}]$. To derive this new control law we rewrite the symmetric polyhedron $P = \{x : |Fx| \leq \bar{1}\}$ in its unsymmetric form, say $P = \{x : [F^T \ -F^T]^T x \leq \bar{1}\} = \{x : \tilde{F}x \leq \bar{1}\}$ and, in analogy with the symmetric case, we define $\tilde{I}(x) = \{i : \tilde{F}_i x = \max_j \tilde{F}_j x\}$. We now define the mapping

$$\bar{I}(x) = \min_{i \in \tilde{I}(x)} i \tag{7}$$

which associates (arbitrarily) to every $x \in P$ a single index corresponding to a sector of P and for every x we consider the following min-max problem:

$$\nu = \min_{u \in U} \max_{w \in W} F_{\bar{I}(x)}(A(w)\, x + B(w)\, u)$$

and let $u_{\bar{I}(x)}$ be the control value for which the minimum is reached. The right hand side of the last equation is the derivative of the given polyhedral function along the system trajectories (apart from proper subspaces of $I\!R^n$) and, being linear in all its terms, it is clear that $u_{\bar{I}(x)}$ is either $u_{\min}$ or $u_{\max}$ (or the value 0 if there is more than one minimizer). In this way the control law $u(x) = u_{\bar{I}(x)}$ remains defined on the whole state space (this is actually the main reason for the introduction of the mapping $\bar{I}(x)$) and can be proved [4] to be such that

$$\Psi_P(x) \leq e^{-\beta t} \Psi_P(x(0))$$

for every initial state $x(0) \in P$.

As a final comment we point out that the cited control law is suitable to handle the case of quantized control devices (see [4]).

5. APPLICATION OF THE CONTROL TO THE TWO TANK SYSTEM

The system we considered is a laboratory two-tank plant whose structure is that reported in the scheme in Figure 1.
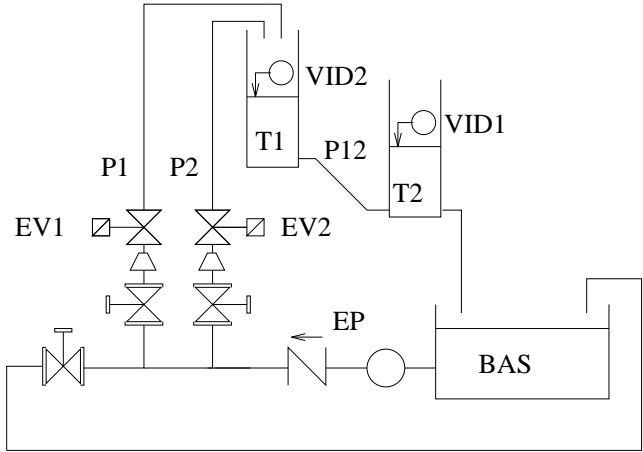

Fig. 1. Plant schematic representation.

It is formed by the electric pump EP whose job is that of supplying water to the two parallel pipes P1 and P2 whose flow can be either 0 or $U_{\max}$ and is regulated by two on-off electro-valves EV1 and EV2 which are commanded by the signals coming from the digital board BRD1 (not reported in Figure 1). The two parallel pipes bring water to the first tank T1 which is connected, through P12, to an identical tank T2 positioned at a lower level. From T2 the water flows out to the recirculation basin BA. The two identical variable inductance devices VID1 and VID2, together with a demodulating circuit in BRD1, allow the computer to acquire the water levels of the two tanks. These levels are the state variables of the system.

If we denote by h_1 and h_2 the water levels of the two tanks and we choose as linearization point the steady state value $[h_{10} \ h_{20}]^T$ corresponding to the constant

input $u_0 = .02985$ and we set $x_1(t) = h_1(t) - h_{10}(t)$ and $x_2(t) = h_2(t) - h_{20}(t)$, we get a linearized time-invariant system whose state and input matrix A and B are

$$A = \begin{bmatrix} -\frac{\alpha}{2\sqrt{h_{10}-h_{20}}} & \frac{\alpha}{2\sqrt{h_{10}-h_{20}}} \\ \frac{\alpha}{2\sqrt{h_{10}-h_{20}}} & -\frac{\alpha}{2\sqrt{h_{10}-h_{20}}} - \frac{\beta}{2\sqrt{h_{20}}} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the parameters entering the above matrix are $\alpha = .08409$, $\beta = .04711$, $h_{10} = .5274$, $h_{20} = .4014$. To keep into account the effects due to the non linear part of the system we considered the uncertain system described by

$$A(\xi, \eta) = \begin{bmatrix} -\xi & \xi \\ \xi & -(\xi + \eta) \end{bmatrix} \quad B(\xi, \eta) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $\xi = .118 \pm .05$ and $\eta = .038 \pm .01$. The state and control constraint sets we considered are respectively given by $X = \{[x_1 \ x_2]^T : |x_1| \le .1, |x_2| \le .1\}$ and $U = \{-U_{\max}, U_{\max}\}$, where $U_{\max} = .02985$. Starting from X we computed the maximal .2-contractive region, by using the corresponding EAS with $\tau = 1$ and $\lambda = .8$. The region representation in terms of planes is given by $P = \{x : |Fx| \le 1\}$ where
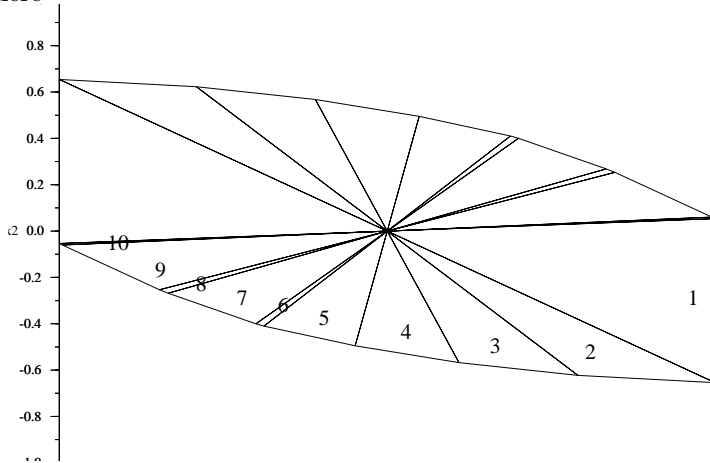

Fig. 2. The maximal β-contractive region, with β=0.2.

$$F = \begin{bmatrix} 1.000 & 0.000 \\ -0.1299 & -1.727 \\ -0.2842 & -1.871 \\ -0.4429 & -1.932 \\ -0.5833 & -1.905 \\ -0.6903 & -1.806 \\ -0.8258 & -1.671 \\ -0.8716 & -1.557 \\ -0.9236 & -1.414 \\ -0.9295 & -1.317 \end{bmatrix}$$

and it is ordered in a way such that each row i of F delimits the sector i according to Figure 2.

This region is formed by 20 symmetric sectors and (as it is always the case in two dimensions) is simplicial. Hence the computation of the sector gains is immediate and results in 10 different gains which are reported in the matrix K

$$K = \begin{bmatrix} -0.2839 & -0.3003 \\ -1.035 & -1.449 \\ -0.0855 & -0.5613 \\ -0.1329 & -0.5796 \\ -0.1750 & -0.5713 \\ -0.2071 & -0.5419 \\ -0.2477 & -0.5012 \\ -0.2614 & -0.4672 \\ -0.2771 & -0.4243 \\ -0.2788 & -0.3964 \end{bmatrix}$$

which again is ordered in a way such that the ith row of K corresponds to the ith sector of P. The result of the implementation of the variable structure control law $u(x) = K^{I(x)}x$ is reported in Figure 3.

We let the reader note that in this simple experiment we didn't force the initial state to belong to the set P. This can be immediately seen from the fact that the control saturates for the first 20 seconds. After this period the system is maintained inside the region and converges asymptotically to the steady state value (the origin of the linearized system) with the assigned contractivity speed.

For this same plant we also implemented the discontinuous control law but, as one can see from the experimental results in Figure 4, due to the extremely low sampling frequency (1 Hz) the system exhibits a limit cycle thus not converging to the origin (this is anyway in accordance with the theory of sliding modes, see [14]). Although for two dimensional systems the computational load associated to the discontinuous control law is essentially the same of the linear variable structure control and is given by the on-line determination of the sector to which the state belongs through (7), we stress once again that for systems of higher dimensions this is not the case since the ratio between the number of facets delimiting the auxiliary simplicial polyhedron and that of the "natural" contractive polyhedron grows exponentially.
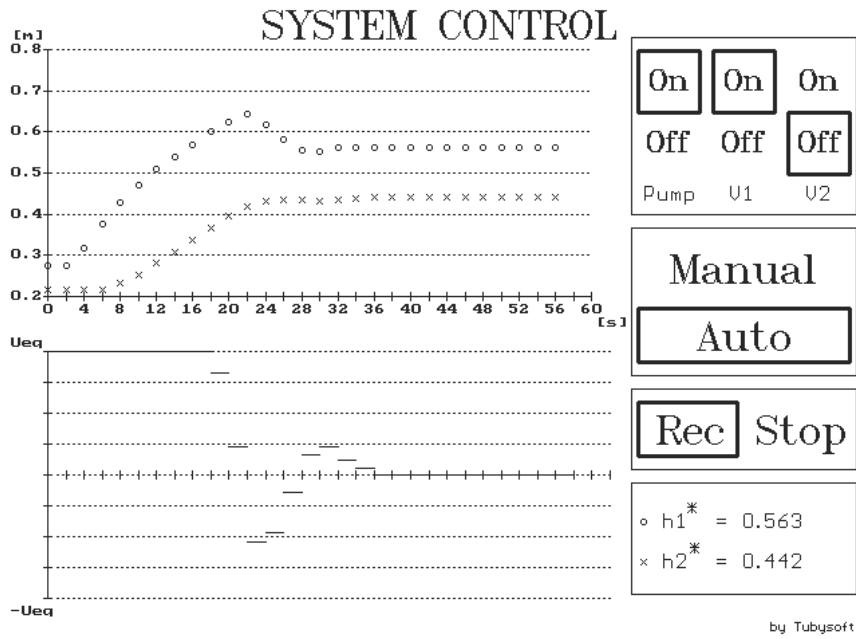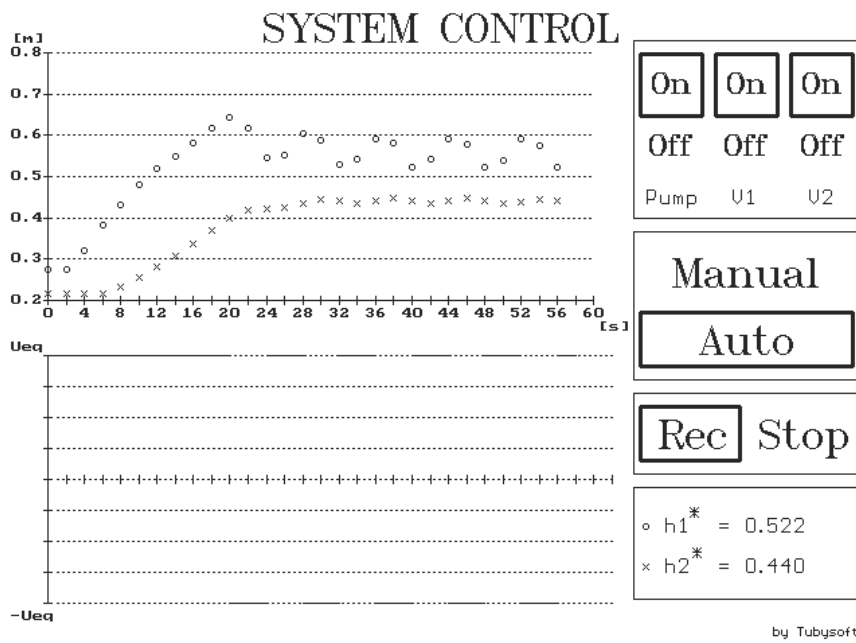
Fig. 3. Variable structure control.

Fig. 4. Bang-Bang control system evolution.

Thus on one side we have an efficient control law whose implementation might be unrealistic for high dimensional systems whereas the discontinuous one provides reasonably good results if the sampling time is short enough. To try to reduce the control complexity while maintaining a sufficiently large domain of attraction the authors have considered an heuristic procedure which consists in considering a simple linear control law whose gain is obtained by averaging the gains of the previously reported variable structure control law. Although not yet supported by any theoretical interpretation (currently under research), in most of the cases this way of proceeding seems to furnish a good compromise between the controller complexity and the dimension of the domain of attraction.

In our case the average gain is given by $k = [\ -.2984 \ \ -.5792 \ ]$ and the maximal β-contractive region, with $\beta = .2$, of the closed loop system included in the non-saturation set $X \cap X_U$, where $X_U = \{x : |kx| \le .3\}$, resulted in the internal region in Figure 5. This set is obviously smaller than the largest 0.2-contractive domain of attraction reported in Figure 2 and its existence assures a speed of convergence $\beta = 0.2$ for the closed-loop system with the obtained linear control. We also computed the largest invariant set of the closed loop system when the above linear control is applied (to this aim it is sufficient to apply the cited Fortran code to the system without input and $A_{cl}(w) = A(w) + B * K$), which resulted in the external region in Figure 5. This set represents the set of all the initial states starting from which the closed-loop system trajectories will never get out of the state and control constraints sets (i.e. outside the set $X \cap X_U$).
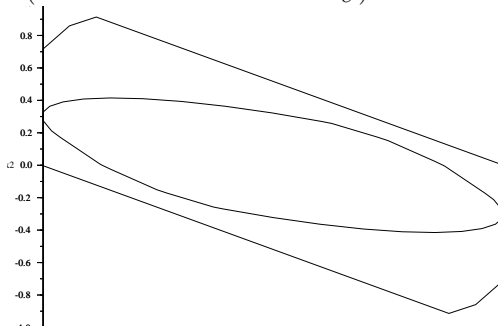

Fig. 5. The largest β-contractive and invariant sets with $u = kx$.

6. CONCLUDING REMARKS

In this work we have considered an experimental laboratory plant on which two set induced nonlinear control laws have been implemented. First we have recalled some known results concerning the constrained stabilization of the class of systems under consideration. Then the application of the proposed techniques to the laboratory control system has been presented together with some heuristic considerations regarding the possibility of deriving a simple linear control law. This gain can be obtained by averaging those of the various sectors concurring in the determination of the nonlinear control law and in most of the case the authors have seen that this results in a quite good closed loop behavior.

Some directions for further research in this area concern the possibility of reducing the complexity of the proposed controllers as well as that of solving output feedback stabilization.

R E F E R E N C E S

[1] A. Benzaouia and C. Burgat: The regulator problem for a class of linear systems with constrained control. Systems Control Lett. *10* (1988), 357–363.

[2] F. Blanchini: Constrained control for uncertain linear systems. J. Optim. Theory Appl. *71* (1991), 3, 465–484.

[3] F. Blanchini: Nonquadratic Lyapunov functions for robust control. Automatica *31* (1995), 3, 451–461.

[4] F. Blanchini and S. Miani: Constrained stabilization of continuous–time linear systems. Systems Control Lett. *28* (1996), 95–102.

[5] R. K. Brayton and C. H. Tong: Constructive stability and asymptotic stability of dynamical systems. IEEE Trans. Circuits and Systems *CAS-27* (1980), 11, 1121–1130.

[6] B. Grumbaum: Convex Polytopes. Wiley, New York 1967.

[7] P. O. Gutman and M. Cwikel: Admissible sets and feedback control for discrete–time linear systems with bounded control and states. IEEE Trans. Automat. Control *AC-31* (1986), 4, 373–376.

[8] P. O. Gutman and P. Hagander: A new design of constrained controllers for linear systems. IEEE Trans. Automat. Control *AC-30* (1985), 22–33.

[9] A. T. Fuller: In–the–large stability of relay and saturating control systems with linear controllers. Internat. J. Control *10* (1969), 457–480.

[10] H. J. Sussman, E. D. Sontag and Y. Yang: A general result on the stabilization of linear systems using bounded controls. IEEE Trans. Automat. Control *AC-39* (1994), 12, 2411–2425.

[11] M. Sznaier: A set–induced norm approach to the robust control of constrained linear systems. SIAM J. Control Optim. *31* (1993), 3, 733–746.

[12] K. Tan and E. Gilbert: Linear systems with state and control constraints: the theory and the applications of the maximal output admissible sets. IEEE Trans. Automat. Control *AC-36* (1991), 9, 1008–1020.

[13] S. Tarbouriech and C. Burgat: Positively invariant sets for constrained continuous–time systems with cone properties. IEEE Trans. Automat. Control *39* (1994), 401–405.

[14] I. Utkin: Sliding Modes and Their Applications in Variable Structure Systems. Mir Publisher, Moscow 1974.

[15] M. Vassilaki and G. Bitsoris: Constrained regulation of linear continuous–time dynamical systems. Systems Control Lett. *13* (1989), 247–252.

F. Blanchini, G. Koruza and L. Tubaro, Dipartimento di Matematica e Informatica, Università degli Studi di Udine, Via delle Scienze 208, 33100 Udine. Italy.
e-mails: blanchini, koruza, tubaro@uniud.it

S. Cotterli and R. Siagri, Eurotech, Maiano del Friuli (Ud). Italy.

S. Miani, Dipartimento di Elettronica e Informatica, Università degli Studi di Padova, Via Gradenigo 6/a, 35131 Padova. Italy.
e-mail: miani@dimi.uniud.it