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CLOSED-LOOP STRUCTURE OF DECOUPLABLE LINEAR MULTIVARIABLE SYSTEMS

Javier Ruiz-León, Jorge Luis Orozco and Ofelia Begovich

Considering a controllable, square, linear multivariable system, which is decouplable by static state feedback, we completely characterize in this paper the structure of the decoupled closed-loop system. The family of all attainable transfer function matrices for the decoupled closed-loop system is characterized, which also completely establishes all possible combinations of attainable finite pole and zero structures. The set of assignable poles as well as the set of fixed decoupling poles are determined, and decoupling is achieved avoiding unnecessary cancellations of invariant zeros. For a particular attainable decoupled closed-loop structure, it is shown how to find the corresponding state feedback, and it is proved that this feedback is unique if and only if the system is controllable.

Keywords: linear systems, multivariable systems, feedback control, pole and zero placement problems

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1. INTRODUCTION

Roughly speaking, decoupling of dynamic systems implies that each input of the system influences one and only one output. From the practical point of view it is of interest to achieve decoupling because it is often desirable to control the outputs of the system independently.

In this work we are interested in the row-by-row decoupling of linear multivariable systems with the same number of inputs and outputs (square systems) by static state feedback. The solution to this famous problem was first established in [2], based on the nonsingularity of a matrix constructed from the system matrices. The structural conditions of solvability in terms of the infinite structure of the system can be found in [1]. The decoupling problem with stability of square systems has been solved in [10] using a geometric approach, and in [12] using an algebraic approach.

Even though there exist many results concerning this problem, most of the contributions in the literature about decoupling focus mainly on the necessary and sufficient conditions to solve the problem, but they usually do not consider neither the issue of what the structure of the decoupled closed-loop system may be (aside from the diagonality of the closed-loop transfer function matrix) nor the characteristics of the decoupling state feedback. Actually, in order to simplify the

problem, a common consideration is that the diagonal entries of the closed-loop transfer function matrix are supposed to be of the form $1/s^j$, where j is a positive integer, which is also referred to as *integrator decoupling*. Of course, no pole locations to obtain adequate system dynamics are considered within this approach, not to speak of the problems (for instance, internal stability) which may be caused by possible pole-zero cancellations. Achieving first decoupling, for example in integrator decoupling form, and after that trying to assign the poles of the system can be a difficult problem, since the state feedback designed to solve the pole-assignment will usually destroy the diagonality of the closed-loop transfer function matrix. Then, the more reasonable approach seems to be to achieve both objectives using the same state feedback. Considering pole-zero cancellations, it is well known that in order to decouple a linear system, it may be necessary to cancel some invariant zeros of the system with closed-loop poles, but that not necessarily all invariant system zeros have to be cancelled. Then, a complete characterization of the decoupled closed-loop system should provide the whole set of finite pole-zero structures which can be obtained for the closed-loop system, avoiding unnecessary cancellations of invariant zeros.

A first attempt to study the structure of the decoupled closed-loop system was presented in [2], where the authors characterized the class of all feedback matrices which decouple a system, and the number of closed-loop poles which can be assigned. Their conditions, however, are cumbersome and difficult to apply, there is no connection whatsoever of these conditions to the structure of the system, and they show how to assign only a number of poles equal to the sum of the system infinite zero orders, which is in general less than the true number of assignable poles. The problem of block decoupling and pole assignment was tackled in [14] using a geometric approach, and the authors presented necessary and sufficient conditions to solve this problem in two special cases, based on the concept of controllability subspaces and their properties. Fixed decoupling poles for minimal systems were proved in [6] to be equal to the interconnection transmission zeros, as defined in this reference.

In this paper we completely describe the closed-loop structure of a decouplable system, which is considered to be controllable but not necessarily observable (i. e. we are not restricted to minimal systems), thus providing the whole set of decoupled closed-loop systems which can be obtained by static state feedback. A characterization of the set of all attainable transfer function matrices for the decoupled closed-loop system is presented, which also establishes all possible combinations of finite closed-loop pole and zero structures. The set of assignable poles as well as the set of fixed decoupling poles are determined, and decoupling is achieved avoiding unnecessary cancellations of invariant zeros. It is also shown that the corresponding state feedback for a particular attainable closed-loop structure is unique if and only if the system is controllable, and a simple procedure is provided to obtain this state feedback.

Observe that, strictly speaking, the problem solved in this paper can not be considered as a generalization of the problem of decoupling with stability (properly defined in Section 2), since we are concerned with the characterization of all possible

combinations of finite pole and zero structures for the decoupled closed-loop system, and not only with internal stable modes. However, the conditions for decoupling with stability can be easily derived from the results presented here (see Lemma 2). Besides stability, the characterization of the whole set of decoupled closed-loop systems can be further used to determine decoupling with appropriate response shaping.

The approach used in this paper is a polynomial-based structural approach. The conditions presented in our results are simple, they have a nice interpretation in terms of system structure, and allow for simple design computations.

After introducing some preliminaries in Section 2, the structure of the decoupled closed-loop system is described in Section 3, and the decoupling state feedback is presented in Section 4. An illustrative example is presented in Section 5, and we end up with some conclusions.

2. PRELIMINARIES

2.1. Problem statement and known solutions to decoupling

We consider in this work linear multivariable systems with the same number of inputs and ouputs, described by

$$(A, B, C) \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are, respectively, the state, input and output vectors of the system, and

$$T(s) = C(sI - A)^{-1}B$$

is the transfer function matrix of the system. Further, the system (A, B, C) is supposed to be controllable, but not necessarily observable. Another assumption, necessary for decoupling, is that the system is invertible, which implies that the system transfer function matrix $T(s)$ is nonsingular.

The system (A, B, C) is said to be row-by-row decouplable by static state feedback (or simply, decouplable) if there exists a state feedback

$$(F, G): \quad u(t) = Fx(t) + Gv(t),$$

where $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{m \times m}$ are constant matrices with G nonsingular, and $v(t)$ is a new input vector, such that the input $v_i(t)$ controls the output $y_i(t)$, $i = 1, \ldots, m$, without affecting the other outputs.

From the input-output point of view, the previous formulation is equivalent to the existence of a state feedback (F, G) such that the transfer function $T_{F,G}(s)$ of the closed-loop system $(A + BF, BG, C)$ is a nonsingular diagonal matrix, i.e. there exists a state feedback (F, G) such that

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \text{diag}\{w_1(s), \ldots, w_m(s)\} =: W(s) \quad (1)$$

where $w_i(s) \neq 0$, $i = 1, \ldots, m$, are strictly proper rational functions.

The solution to the decoupling problem in terms of the infinite structure of the system is given by the following result [1].

Proposition 1.   The system (A, B, C) is decouplable if and only if

$$\sum_{i=1}^{m} n_i' = \sum_{i=1}^{m} n_i \tag{2}$$

where $\{n_1', \ldots, n_m'\}$ are the infinite zero orders, and $\{n_1, \ldots, n_m\}$ are the row infinite zero orders of the system.

For the definition and properties of infinite zero orders, see for instance [13]. As it can be seen from Proposition 1, the necessary and sufficient conditions for decoupling depend only on the infinite (global and row) structure of the system. The finite structure (finite zeros and poles), on the other hand, plays an important role concerning the general structure of the decoupled closed-loop system. The system zeros which are fundamental to our study are the invariant zeros of the system. Transmission zeros will be also mentioned. Even though invariant system zeros are briefly introduced in the next section, presenting the definitions and properties of system zeros is out of the scope of this work. For a comprehensive treatment, see for instance [9]. Let us just mention that transmission zeros are related to the system transfer function matrix, and they can be considered as "input-output zeros", while invariant zeros can be considered as "internal zeros". Invariant zeros contain the transmission zeros and both sets coincide if the system is controllable and observable.

If the stability issue is considered in the problem formulation, then the system (A, B, C) is said to be decouplable with stability if it is decouplable and the closed-loop system $(A + BF, BG, C)$ is internally stable, i. e. the eigenvalues of the matrix $(A + BF)$ are located in the open left half complex plane.

The solution to the decoupling problem with stability in terms of the infinite and unstable structure of the system is given by the following result ([10, 12]).

Proposition 2.   The system (A, B, C) is decouplable with stability if and only if (1) holds, and the number of global invariant and row invariant unstable zeros[1] of the system (multiplicities included) is the same.

2.2. Decoupling and cancellation of system zeros

It is well known that in the process of decoupling a linear system, some of the transmission zeros of the system may be cancelled by assigning closed-loop poles to the position of these zeros. It is important, however, to make the distinction between transmission zeros that have to be cancelled in order to achieve decoupling, and transmission zeros which are not necessary to cancel (see Example 1). Concerning controllable and non-observable, or non-minimal systems, instead of transmission zeros it is necessary to consider the invariant zeros of the system, which may appear as transmission zeros of the decoupled closed-loop system.

In practical designs, cancellation of invariant zeros is usually avoided because of potential internal instability caused by hidden system dynamics and undesirable pole

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[1]Invariant unstable zeros are invariant zeros located in the closed right half complex plane.

locations, which is dramatically important in the case of unstable invariant zeros. Thus, if the main objective is to decouple the system, it is important at least to know the number of poles which can be freely assigned, and the number of poles which have to be cancelled with invariant zeros in order to achieve decoupling, i.e. the so-called *fixed decoupling poles.*[2] This information would provide a complete characterization of the closed-loop structure and complete pole assignment of a decouplable system.

2.3. Feedback realizability of precompensators

The approach used in the next section to characterize the set of all transfer function matrices for the decoupled closed-loop system is related to the problem of feedback realizability of dynamic precompensators, which is introduced next. First, let us define a biproper matrix: a nonsingular proper rational matrix is said to be biproper if its inverse is also proper. We have [13] that a proper rational matrix, say $V(s)$, is biproper if and only if

$$\lim_{s \to \infty} V(s)$$

is a constant and nonsingular matrix.

Let us consider a state feedback (F, G) acting on the system (A, B, C), and the corresponding closed-loop transfer function matrix

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG.$$

After some manipulations on the last equation, we obtain

$$T_{F,G}(s) = C(sI - A)^{-1}B[I - F(sI - A)^{-1}B]^{-1}G \tag{3}$$

where $T(s) = C(sI - A)^{-1}B$ is the transfer function of the system (A, B, C), and the matrix

$$[I - F(sI - A)^{-1}B]^{-1}G$$

appearing on the right side of (3) is easily seen to be a biproper matrix. Then the effect of a state feedback acting on (A, B, C) can be represented in transfer function terms as a biproper matrix postmultiplying the system transfer function $T(s)$.

The converse problem, i.e. under which conditions a proper matrix postmultiplying $T(s)$ can be realized using state feedback, is known as feedback realizability of precompensators. Then, a given proper compensator, say $Q(s)$, will be said to be feedback realizable if there exists a state feedback (F, G) such that

$$Q(s) = [I - F(sI - A)^{-1}B]^{-1}G. \tag{4}$$

The following result [3] states the conditions for a proper compensator to be realizable.

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[2]Strictly speaking, cancelled frequency values are not system poles, since they do not appear in the system transfer function matrix. Then, it should be more appropriate to speak of fixed decoupling modes instead of fixed decoupling poles, where poles are a subset of the system modes, and both sets are equal if the system is controllable and observable. For simplicity, we make no distinction in this paper between modes and poles.

Proposition 3. Let the matrices $\bar{N}(s)$ and $D(s)$ be a right coprime matrix fraction description (MFD) of the system (A, B, I_n), and let $Q(s)$ be a nonsingular compensator. Then $Q(s)$ is state feedback realizable on (A, B, I_n) if and only if

— $Q(s)$ is biproper, and

— $Q^{-1}(s)D(s)$ is a polynomial matrix.

3. STRUCTURE OF THE DECOUPLED CLOSED-LOOP SYSTEM

The conditions for decoupling a linear multivariable system (A, B, C) are intimately connected to the structure of the so-called system matrix [11]

$$P(s) = \left[\begin{array}{cc} sI - A & B \\ C & 0 \end{array} \right] \tag{5}$$

related to the structure of the matrices

$$P_i(s) = \left[\begin{array}{cc} sI - A & B \\ c_i & 0 \end{array} \right], \quad i = 1, \ldots, m, \tag{6}$$

where c_i, $i = 1, \ldots, m$, is the ith row of matrix C.

Indeed, the system (A, B, C) is decouplable if and only if the infinite structure of $P(s)$ coincides with the infinite structure of the matrices $P_i(s)$, i.e. if and only if (2) holds, where $\{n'_1, \ldots, n'_m\}$ are the infinite zero orders of $P(s)$ (infinite zero orders of the system), and $\{n_1, \ldots, n_m\}$ are the infinite zero orders of $P_1(s), \ldots, P_m(s)$ (row infinite zero orders of the system), see Proposition 1.

If the system is decouplable, then it is decouplable with stability if and only if the number of unstable zeros of $P(s)$ (unstable invariant zeros of the system), multiplicities included, is equal to the number of unstable zeros of $P_1(s), \ldots, P_m(s)$ (row unstable invariant zeros of the system), taken all together, see Proposition 2.

The invariant zeros of the system are the finite zeros of matrix $P(s)$, i.e. the roots of the invariant polynomials of $P(s)$ [9], while the row invariant zeros are the finite zeros of matrices $P_1(s), \ldots, P_m(s)$.

The general structure of the decoupled closed-loop system depends also on the structure of matrices (5) and (6), as it will be shown. First, let us introduce the following preliminary result.

Lemma 1. Let (A, B, C) be a square controllable system, and let c_i be the ith row of matrix C, $i = 1, \ldots, m$. Then, the matrix

$$P_i(s) = \left[\begin{array}{cc} sI - A & B \\ c_i & 0 \end{array} \right]$$

can have at most one non-unit invariant polynomial.

P r o o f. The invariant polynomials of $P_i(s)$ can be obtained as

$$\lambda_j(s) = \frac{\Delta_j(s)}{\Delta_{j-1}(s)}, \quad j = 1, \ldots, n+1,$$

where

$$\Delta_0(s) := 0,$$

$$\Delta_j(s) := \text{monic greatest common divisor (gcd) of all } j \times j \text{ minors of } P_i(s),$$
$$j = 1, \dots, n+1,$$

are the determinantal divisors of $P_i(s)$ (see for instance [5]). Since the system is controllable, at least the first n determinantal divisors of $P_i(s)$ are all units. This can be seen from the fact that the Smith form of $\begin{bmatrix} sI - A & B \end{bmatrix}$ is $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Then, the only possible non-unit invariant polynomial of $P_i(s)$ is the last one, which is equal to $\Delta_{n+1}(s)$. $\qquad\square$

Let us denote by $z_i(s)$ the last invariant polynomial of $P_i(s)$, $i = 1, \dots, m$. It can be seen that any finite zero of $P_i(s)$ is also a zero of the matrix $P(s)$ given by (5), but that a zero of $P(s)$ is not necessarily a zero of $P_i(s)$. In other words, any row invariant zero is an invariant zero of the system, but an invariant zero is not necessarily a row invariant zero. Then the product of the polynomials $\prod_{i=1}^{m} z_i(s)$ divides exactly $\prod_{i=1}^{n+m} \epsilon_i(s)$, where $\epsilon_i(s)$ are the invariant polynomials of $P(s)$.

The invariant zeros of the system (global or row invariant zeros) can also be obtained from a matrix fraction description (MFD) of the system as follows. Let $\bar{N}(s)$ and $D(s)$ be a right coprime MFD of (A, B, I_n). Then the matrices $N(s) := C\bar{N}(s)$ and $D(s)$ form a right MFD of (A, B, C). Observe that $N(s)$ and $D(s)$ are not necessarily right coprime, since we are not restricted to minimal systems. The invariant zeros of the system can be obtained from the invariant polynomials of the numerator matrix $N(s)$. In the case of row invariant zeros, the previously defined polynomial $z_i(s)$ corresponds then to the invariant polynomial of the ith row of $N(s)$, i.e. $z_i(s)$ is the monic gcd of all entries in the ith row of $N(s)$. This fact will be used in the proof of Theorem 1.

The family of all attainable transfer function matrices for the decoupled closed-loop system is characterized by the following result.

Theorem 1. Let (A, B, C) be a square, controllable, and decouplable system. Then, there exists a state feedback (F, G) which decouples the system, such that the transfer function of the decoupled closed-loop system is of the form

$$W(s) = C(sI - A - BF)^{-1}BG = \begin{bmatrix} k_1 \frac{z_1(s)}{a_1(s)} & & \\ & \ddots & \\ & & k_m \frac{z_m(s)}{a_m(s)} \end{bmatrix} \tag{7}$$

where $k_1, \dots, k_m$, are real numbers, $z_i(s)$ is the last invariant polynomial of the matrix $P_i(s)$, $i = 1, \dots, m$, as introduced before, $a_1(s), \dots, a_m(s)$, are monic polynomials with arbitrary roots, satisfying

$$\deg a_i(s) - \deg z_i(s) = n_i, \quad i = 1, \dots, m, \tag{8}$$

and $n_1, \dots, n_m$, are the row infinite zero orders of the system.

P r o o f . We will prove the result by showing that the set of compensators given by

$$Q(s) := T^{-1}(s)W(s), \tag{9}$$

and producing (7) as transfer function matrix, are the set of all feedback realizable compensators that decouple the system.

The degree constraint (8) is easily seen to hold, since the decoupling state feedback (F, G) does not modify the infinite zero structure of the system, and in particular it does not modify the row infinite zero orders. Since the system is decouplable, then there exists a biproper matrix $U(s)$, such that

$$T(s) = \text{diag}\left\{\frac{1}{s^{n_1}}, \cdots, \frac{1}{s^{n_m}}\right\}U(s).$$

Then, we have from (9) that

$$Q(s) = U^{-1}(s)\text{diag}\left\{s^{n_1}, \ldots, s^{n_m}\right\}W(s).$$

From the last equation and the degree constraint (8), it can be seen that $Q(s)$ is a biproper matrix.

Let $\bar{N}(s), D(s)$ be a right coprime matrix fraction description of (A, B, I_n). Then, by Proposition 3 the matrix $Q(s)$ will be proved to be state feedback realizable if $Q^{-1}(s)D(s)$ is a polynomial matrix.

We have that

$$T(s)Q(s) = C\bar{N}(s)D^{-1}(s)Q(s) = W(s),$$

then

$$Q^{-1}(s)D(s) = W^{-1}(s)N(s) = \begin{bmatrix} \frac{a_1(s)}{k_1 z_1(s)} & & \\ & \ddots & \\ & & \frac{a_m(s)}{k_m z_m(s)} \end{bmatrix} N(s)$$

where $N(s) = C\bar{N}(s)$. Since $z_i(s)$ is the monic gcd of all entries in the ith row of $N(s)$, then it can be seen that $Q^{-1}(s)D(s)$ is a polynomial matrix, implying that the matrix $Q(s)$ is state feedback realizable.

To see that any other matrix not contained in the set (7) can not be the transfer function of the decoupled closed-loop system, observe that the state feedback can not introduce finite zeros, and therefore no other polynomial different from $z_i(s)$ (not considering possible cancellations between $z_i(s)$ and $a_i(s)$), which contains the row invariant zeros of the system, can appear as numerators in (7). Alternatively, the corresponding compensator to get any matrix not contained in (7) is not feedback realizable.   □

Theorem 1 completely characterizes the set of all matrices which can be obtained as transfer function matrices for the decoupled closed-loop system of a decouplable system. This characterization provides also all the set of possible finite pole-zero structures for the decoupled closed-loop system. The set of fixed decoupling poles of the system are given by the following result.

Theorem 2. The fixed decoupling poles of the system correspond to the roots of the polynomial

$$\delta(s) := \frac{\prod_{i=1}^{n+m} \epsilon_i(s)}{\prod_{i=1}^{m} z_i(s)} \tag{10}$$

where $\epsilon_1(s), \ldots, \epsilon_{n+m}(s)$, are the invariant polynomials of $P(s)$, and $z_i(s)$ is the last invariant polynomial of $P_i(s)$, $i = 1, \ldots, m$.

P r o o f . The set of invariant zeros of (A, B, C) are the roots of the polynomials $\epsilon_i(s)$, and it is evident from (7) that the only frequency values that can be finite zeros of the decoupled closed-loop system are the roots of the polynomials $z_i(s)$. If $\delta(s)$ is a polynomial different from 1, then some of the poles of the system (the fixed decoupling poles) must be located at the positions of the roots of $\delta(s)$ producing cancellation with invariant zeros of the system. □

Remark 1. From the previous result, it can be seen that the fixed decoupling poles correspond to invariant zeros which are not row invariant zeros of the system. Observe also that there are no fixed decoupling poles (all system poles can be assigned) if the system has no invariant zeros, or if all invariant zeros (multiplicities included) are also row invariant zeros of the system.

Corollary 1. It follows from Theorem 2 that the number of poles which can be arbitrarily assigned while decoupling the system is equal to

$$n - \deg \delta(s), \tag{11}$$

where n is the order of the system and $\delta(s)$ is given by (10).

Well-known results about decoupling with stability can be readily obtained from the results presented previously, as follows.

Lemma 2. The system (A, B, C) is decouplable with stability if and only if there are no fixed unstable poles, i. e. if and only if the polynomial $\delta(s)$ given by (10) has no roots in the closed right half complex plane.

P r o o f . The result follows since this condition is equivalent to saying that the system is decouplable with stability if and only if it is decouplable, and the set of unstable invariant zeros of the system (multiplicities included) coincide with the unstable row invariant zeros of the system (see Proposition 2). □

4. DECOUPLING STATE FEEDBACK

Concerning the decoupling state feedback, it will be shown next that for a particular choice of transfer function matrix from the set (7), say $W_1(s)$, the corresponding state feedback producing $W_1(s)$ is unique if and only if the system is controllable, which is the case of the systems we are considering in this paper. First, the next preliminary result will be presented.

Lemma 3. The system (A, B, C) is controllable if and only if it does not exist a constant vector q different from zero such that

$$q(sI - A)^{-1}B = 0. \tag{12}$$

P r o o f. Evident from the controllability of the system. □

Theorem 3. Let (A, B, C) be a decouplable system, and let $W_1(s)$ be a particular matrix from the set (7). Then, the state feedback (F, G) producing $W_1(s)$ as the transfer function of the decoupled closed-loop system is unique if and only if the system is controllable.

P r o o f. Write $Q(s) = T^{-1}(s)W_1(s)$ as

$$Q(s) = Q_0 + Q_{sp}(s) \tag{13}$$

where Q_0 is a constant matrix, and $Q_{sp}(s)$ is the strictly proper part of $Q(s)$.

Then, from (4), and since $F(sI - A)^{-1}B$ is strictly proper, matrix G is uniquely given by

$$G = \lim_{s \to \infty} Q(s) = Q_0. \tag{14}$$

Since the system is decouplable for $W_1(s)$, then there exists a constant matrix F such that

$$F(sI - A)^{-1}B = I_m - GW_1^{-1}(s)T(s) \tag{15}$$

where G is given by (14).

If the system is controllable, by Lemma 3 the matrix $(sI - A)^{-1}B$ has no left constant kernel different from zero, thus matrix F is unique. □

The decoupling state feedback (F, G) can also be obtained from the constant solution to a polynomial matrix equation as follows. If the system is decouplable, then there exists a constant solution X, Y, with X nonsingular [8] to the polynomial matrix equation

$$XD(s) + Y\bar{N}(s) = Q^{-1}(s)D(s) \tag{16}$$

where $\bar{N}(s)$, $D(s)$ is a right coprime matrix fraction description of (A, B, I_n), with $D(s)$ column reduced. Then, the state feedback (F, G), given by

$$F = -X^{-1}Y, \quad G = X^{-1}, \tag{17}$$

produces $W(s)$ as the transfer function of the decoupled closed-loop system.

From the previous results, it follows that if the system is controllable, then the solution X, Y to (16) with the aforementioned properties is unique. The uniqueness of such a solution to a polynomial matrix equation like (16) has already been proved in [4]; see also [7].

5. EXAMPLE

The following example is presented in order to illustrate the results of this paper.

Example 1. Let the controllable system (A, B, C) be given by

$$A = \begin{bmatrix} -2 & 3 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

whose transfer function is

$$T(s) = \begin{bmatrix} \frac{1}{(s-2)(s+2)} & 0 \\ \frac{s-1}{(s-2)(s+2)^3} & \frac{s+1}{(s+2)^2} \end{bmatrix}.$$

Since the row infinite zero orders of the system

$$n_1 = 2, \quad n_2 = 1,$$

coincide with the infinite zero orders, then the system is decouplable. Further, we have that

$$z_1(s) = 1, \qquad z_2(s) = s - 1,$$

$$\epsilon_1(s) = \cdots = \epsilon_6(s) = 1, \qquad \epsilon_7(s) = (s+1)(s-1),$$

and

$$\delta(s) = \frac{(s+1)(s-1)}{s-1} = s + 1.$$

Then, the set of matrices which can be obtained as transfer function matrices for the decoupled closed-loop system is given by

$$W(s) = \begin{bmatrix} \frac{k_1}{(s+\alpha_1)(s+\alpha_2)} & 0 \\ 0 & \frac{k_2(s-1)}{(s+\alpha_3)(s+\alpha_4)} \end{bmatrix},$$

and there exists a fixed decoupling pole at $s = -1$, i.e. the system invariant zero at $s = -1$ has to be cancelled in order to decouple the system, while it is not necessary to cancel the invariant zero at $s = 1$. Observe that $s = 1$ is an invariant row and global zero of the system, which is not evident from the system transfer function, since the system is not observable; thus, this zero can also appear in $W(s)$ using a state feedback which decouples the system. Notice also that the system is decouplable with stability, since there are no fixed unstable poles, and that 4 out of the 5 system poles can be arbitrarily assigned. To see the importance

of characterizing the set of all attainable pole and zero finite structures for the decoupled system, observe that if the invariant zero at $s = 1$ does not appear as a zero of the closed-loop transfer function matrix (because it remains non observable or it is cancelled later on "by mistake") then the decoupled system would be internally unstable.

Let us choose a pole-zero finite structure corresponding to the following matrix

$$W_1(s) = \left[\begin{array}{cc} \frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s-1}{(s+2)^2} \end{array} \right]$$

i. e. we aim to obtain a decoupled and internally stable closed-loop system, with poles at the positions specified by $W_1(s)$.

Then, the unique state feedback producing $W_1(s)$ is computed as described in the paper as

$$F = \left[\begin{array}{ccccc} -3 & -6 & 3 & 9 & 6 \\ -2 & -7 & 0 & 1 & -1 \end{array} \right], \qquad G = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].$$

6. CONCLUSIONS

In this paper, we completely characterized the closed-loop structure of a linear square multivariable system decouplable by static state feedback. A characterization of all matrices that can be obtained as transfer function matrices for the closed-loop decoupled system was presented. From this result, all possible combinations of attainable finite closed-loop pole and zero structures of the system can be readily established. The set of assignable modes was determined, as well as the set of fixed decoupling modes. The conditions presented in our results are simple, they have a nice interpretation in terms of system structure, and allow for simple design computations.

Based on the results of the present paper, in [15] a numerical algorithm has been developed to solve the problem of decoupling and complete pole assignment.

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José Javier Ruiz León, Jorge Luis Orozco and Ofelia Begovich, Centro de Investigación y de Estudios Avanzados, Unidad Guadalajara (CINVESTAV-IPN), P. O. Box 31-438, Plaza la Luna, 44550, Guadalajara, Jalisco. México.

e-mails: jruiz@@gdl.cinvestav.mx, jorozco@@gdl.cinvestav.mx, obegovi@@gdl.cinvestav.mx