Affine Normalization of Symmetric Objects ⋆

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Abstract. A new method of normalization is used for the construction of the affine moment invariants. The affine transform is decomposed into translation, scaling, stretching, two rotations and mirror reflection. The object is successively normalized to these elementary transforms by means of low order moments. After normalization, other moments of normalized object can be used as affine invariant features of the original object. We pay special attention to the normalization of symmetric objects.

1 Introduction

Affine moment invariants as features for object recognition have been studied for many years. They were introduced independently by Reiss [1] and Flusser and Suk [2], who published its explicit forms and proved their applicability in simple recognition tasks. In their work, they decomposed the affine transform into translation, anisotropic scaling and two skews. The systems of invariants were derived by direct solving Cayley-Aronhold differential equation [2], by tensor method [3] or, equivalently, by graph method [4]). The invariants are in form of polynomials of moments.

The normalization performs an alternative approach to deriving invariants. First, the object is brought into certain "normalized" or "canonical" position, which is independent of the actual position of the original object. In this way, the influence of affine transformation is eliminated. Since the normalized position is the same for all objects differing from each other just by affine transform, the moments of normalized object are in fact affine invariants of the original one. We emphasize that no actual spatial transformation of the original object is necessary. Such a transformation would slow down the process and would introduce resampling errors. Instead, the moments of normalized objects can be calculated directly from the original one using the normalization constraints. These constraints are often formulated by means of low-order moments.

The idea of normalization was successfully used in [5], but only normalization to rotation was considered in that paper. Affine normalization was firstly

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described in [6], where two different affine decompositions were used (XSR decomposition, i.e. skew, anisotropic scaling and rotation , and XYS decomposition, i.e. two skews and anisotropic scaling). However, this approach leads to some ambiguities, which were studied in [7] in detail.

Pei and Lin [8] presented a method similar to ours. Their paper contains detailed derivation of the normalization to the first rotation and to anisotropic scaling, but they do not consider the problems with the symmetric objects and with the mirror reflection. This is a serious weakness because in many applications we have to classify man-made or specific natural objects which are very often symmetrical. Since many moments of symmetrical objects are zero, the normalization constraints may be not well defined.

Shen and Ip [9] used so called generalized complex moments computed in polar coordinates and analyzed their behavior in recognition of symmetrical objects. Heikkila [10] used Cholesky factorization of the second order moment matrix to define the normalization constraints.

We present a new, simpler way of normalization to the affine transformation, which is based both on traditional geometric as well as complex moments. The method is well defined also for objects having n-fold rotation symmetry, which is its main advantage.

2 Normalization of the image with respect to the affine transform

The affine transform

$$\begin{aligned} x' &= a_0 + a_1 x + a_2 y, \\ y' &= b_0 + b_1 x + b_2 y \end{aligned} \tag{1}$$

can be decomposed into six simple one-parameter transforms and one non-parameter

$$\begin{array}{llll}
\text{Horizontal and} & \text{vertical translation :} & \text{Scaling :} & \text{First rotation :} \\
u = x - x_0 & u = x & u = \omega x & u = x\cos\alpha - y\sin\alpha \\
v = y & v = y - y_0 & v = \omega y & v = x\sin\alpha + y\cos\alpha \\
 & & & \\
\text{Stretching :} & \text{Second rotation :} & \text{Mirror reflection :} & \\
u = \delta x & u = x\cos\rho - y\sin\rho & u = x & \\
v = \frac{1}{\delta} y & v = x\sin\rho + y\cos\rho & v = \pm y. &
\end{array} \tag{2}$$

Any function F of moments is invariant under these seven transformations if and only if it is invariant under the general affine transformation (1). The ordering of these one-parameter transforms can be changed, but the stretching must be between two rotations.

Each of these transforms imposes one constraint on the invariants. Traditional approach to the problem of affine invariants consists on expressing constraints in form of equations. Affine invariants are then obtained as their solutions.

Here we bring the object into normalized position. The parameters of the "normalization transforms" can be calculated by means of some object moments. Below we show how to normalize the object with respect to all seven one-parameter transforms.

2.1 Normalization to translation and scaling

We can easily normalize the image with respect to translation just by shifting it such that its centroid

$$x_c = \frac{m_{10}}{m_{00}}, \qquad y_c = \frac{m_{01}}{m_{00}}. \tag{3}$$

is zero. Practically, this is ensured by using central moments

$$\mu_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_c)^p (y - y_c)^q f(x,y) dx dy, \tag{4}$$

instead of geometric moments

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y) \mathrm{d}x \mathrm{d}y. \tag{5}$$

The normalization to the scaling is also simple. The scaling parameter ω can be recovered from μ_{00}

$$\omega = 1/\sqrt{\mu_{00}}. \tag{6}$$

The scale-normalized moments are then defined as

$$\nu_{pq} = \mu_{pq}/\mu_{00}^{\frac{p+q+2}{2}}. \tag{7}$$

2.2 Normalization to the first rotation and stretching

Normalization to the rotation can advantageously be done by complex moments. Complex moment is defined as

$$c_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x + iy)^p (x - iy)^q f(x,y) dx dy \ , \tag{8}$$

where i denotes imaginary unit. Each complex moment can be expressed in terms of geometric moments m_{pq} as

$$c_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q} \binom{p}{k}\binom{q}{j} (-1)^{q-j} \cdot i^{p+q-k-j} \cdot m_{k+j,p+q-k-j} \ . \tag{9}$$

We can use normalized complex moments computed from the normalized moments ν_{pq} to get translation and scaling invariance.

When rotating the image, its complex moments preserve their magnitudes while their phases are shifted. More precisely,

$$c'_{pq} = e^{i(p-q)\alpha} c_{pq}, \tag{10}$$

where α is the rotation angle measured counterclockwise.

The simplest normalization constraint is to require c'_{pq} to be real and positive. This is always possible to achieve (provided that $c_{pq} \neq 0$) by rotating the image by angle α

$$\alpha = -\frac{1}{p-q} \arctan\left(\frac{\Im(c_{pq})}{\Re(c_{pq})}\right), \tag{11}$$

where $\Re(c_{pq})$ and $\Im(c_{pq})$ denote real and imaginary parts of c_{pq}, respectively.

Generally, any non-zero c_{pq} can be used for this kind of normalization. Because of stability, we try to keep its order as low as possible. Since c_{10} was already used for translation normalization, the lowest moment we can employ is c_{20}. It leads to well known "principal axes normalization", where the angle is given as

$$\alpha = -\frac{1}{2} \arctan\left(\frac{\Im(c_{20})}{\Re(c_{20})}\right) = -\frac{1}{2} \arctan\left(\frac{2\mu_{11}}{\mu_{20} - \mu_{02}}\right). \tag{12}$$

If the c_{20} is zero, we consider the object is already normalized and set $\alpha = 0$.

Normalization to stretching can be done by imposing an additional constraint on second order moments. We require that $\mu'_{20} = \mu'_{02}$. The corresponding normalizing coefficient δ is then given as

$$\delta = \sqrt{\frac{\mu_{20} + \mu_{02} - \sqrt{(\mu_{20} - \mu_{02})^2 + 4\mu_{11}^2}}{2\sqrt{\mu_{20}\mu_{02} - \mu_{11}^2}}} \tag{13}$$

(this is well defined because $\mu_{20}\mu_{02} - \mu_{11}^2$ is always non-zero for non-degenerate 2-D objects).

After this normalization the complex moment c'_{20} becomes zero and cannot be further used for another normalization.

The moments of the normalized image to the first rotation and stretching can be computed from the moments of the original by means of (9) and (10) as

$$\mu'_{pq} = \delta^{p-q} \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^k \ \sin^{p-k+j} \alpha \ \cos^{q+k-j} \alpha \ \nu_{k+j,p+q-k-j} \ . \tag{14}$$

2.3 Normalization to the second rotation

Normalization to the second rotation is a critical step, namely for symmetric objects. We propose a normalization by one complex moment, which must be of course nonzero. However, many moments of symmetric objects equal zero. The selection of the normalizing moment must be done very carefully, especially in a discrete case where some moments which should be theoretically zero may appear in certain positions of the object as nonzero because of quantization effect.

Let us consider an object having n–fold rotation symmetry. Then all its complex moments with non-integer $(p-q)/n$ equal zero. To prove this, let us rotate the object around its origin by $2\pi/n$. Due to its symmetry, the rotated object must be the same as the original. In particular, it must hold $c_{pq}^{rot} = c_{pq}$ for any p and q. On the other hand, it follows from eq. (10)) that

$$c_{pq}^{rot} = e^{-2\pi i(p-q)/n} \cdot c_{pq}.$$

Since $(p-q)/n$ is assumed not to be an integer, this equation can be fulfilled only if $c_{pq} = 0$. Particularly, if an object has circular symmetry (i.e. $n = \infty$), the only nonzero moments can be c_{pp}'s.

The moment we use for normalization is found as follows. Let us consider a set of complex moments $\{c_{pq} | p > q, p+q \le r\}$ except those moments which were used in previous normalization steps. We sort this set according to the moment orders and, among the moments of the same order, according to $p-q$. We get a sequence of complex moments $c_{21}, c_{30}, c_{31}, c_{40}, c_{32}, c_{41}, c_{50}, c_{42}, c_{51}, c_{60}$, etc. The first nonzero moment in this sequence is selected for normalization. (In practice, "nonzero moment" means that its magnitude exceeds some threshold.) If all the moments in the sequence are zero, we consider the object circular symmetric and no normalization is necessary.

Thanks to the proper ordering of moments, c_{21} is always selected for non-symmetric objects. For symmetric objects the order of the selected moment is kept as low as possible. This is a favorable property of the method because low-order moments are more robust to noise than the higher-order ones.

Once the normalizing moment is determined, the normalizing angle ρ is calculated similarly as (11)

$$\rho = -\frac{1}{p-q} \arctan\left(\frac{\Im(c_{pq})}{\Re(c_{pq})}\right). \tag{15}$$

Finally, the moments of the object normalized to the second rotation are calculated by means of a similar formula as in the case of the first rotation

$$\tau_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^k \ \sin^{p-k+j} \rho \ \cos^{q+k-j} \rho \ \mu'_{k+j,p+q-k-j} \ , \tag{16}$$

but here the moments normalized to the first rotation and stretching μ'_{pq} must be used on the right-hand side.

The moments τ_{pq} are new affine moment invariants of the original object. Note that some of them have "prescribed" values due to normalization, regardless of the object itself:

$$\tau_{00} = 1, \ \tau_{10} = 0, \ \tau_{01} = 0, \ \tau_{02} = \tau_{20}, \ \tau_{11} = 0, \ \tau_{03} = -\tau_{21}. \tag{17}$$

All other moments (and also τ_{20} and τ_{21}) can be used as features for invariant object recognition.

2.4 Normalization to mirror reflection

Although the general affine transform (1) may contain mirror reflection, normalization to the mirror reflection should be done separately from the other transformations in (2) for practical reasons. In most affine deformations occurring in practice, any mirror reflection cannot be present in principle and we want to classify mirrored images into different classes (in character recognition we certainly want to distinguish capital S and a question mark for instance). Normalization to mirror reflection is not desirable in those cases.

If we still want to normalize objects to mirror reflection, we can do that, after all normalization mentioned above, as follows. We find the first non-zero moment (normalized to scaling, stretching and both rotations) with an odd q-index. If it is negative, then we change the signs of all moments with odd q-indices. If it is positive or if all normalized moments up to the chosen order with odd q-indices are zero, no action is required.

3 Numerical experiments

To illustrate the performance of the method, we carried out an experiment with simple patterns having different number of folds. In Fig. 1 (top row), one can see six objects whose numbers of folds are 1, 2, 3, 4, 5, and ∞, respectively. In the middle row you can see these patterns being deformed by an affine transformation with parameters $a_0 = 0$, $a_1 = -1$, $a_2 = 1$, $b_0 = 0$, $b_1 = 0$, and $b_2 = 1$. For the both sets of objects the values of the normalized moments were calculated as described in Section 2. The moment values of the original patterns are shown in Table 1. The last line of the table shows which complex moment was used for the normalization to the second rotation. The moment values of the transformed patterns were almost exactly the same – the maximum absolute error was $3 \cdot 10^{-11}$, which demonstrate an excellent performance of the proposed method even if the test objects were symmetric.

In the bottom row of Fig. 1 the normalized positions of the test patterns are shown. We recall that this is for illustration only; transforming the objects is not required for calculation of the normalized moments.

The last line of Table 1 illustrates the influence of spatial quantization in the discrete domain. Theoretically, in case of the three-point star we would need to use c_{30}, in case of the five-point star c_{50} should be used, and the circle would not require any normalization. However, in the discrete domain the symmetry is violated. That is why the algorithm selected other moments for normalization.

In the second experiment, we tested the behavior of the method in a difficult situation. The cross (see Fig. 2 left) has four folds of symmetry, so one would expect to choose c_{40} for the normalization to the second rotation. However, we deliberately set up the proportions of the cross such that $c_{40} \doteq 0$. Since in the discrete domain it is impossible to reach exactly $c_{40} = 0$, we repeated this experiment three times with slightly different dimensions of the cross. The cross deformed by an affine transform is shown in Fig. 2 right. In all three cases

moment	Letter F	Compass	3-point star	Square	5-point star	Circle
τ_{30}	-0.5843	0	0.6247	0	-0.0011	0
τ_{21}	0.2774	0	0.1394	0	0.0024	0
τ_{12}	0.5293	0	-1.2528	0	-0.0038	0
τ_{40}	1.3603	1.013	1.4748	1.2	1.265	1
τ_{31}	-0.0766	0	-0.0002	0	-0.0068	0
τ_{22}	0.9545	0.9371	1.4791	0.6	1.2664	0.9999
τ_{13}	0.1270	0	-0.0001	0	0.0106	0
τ_{04}	1.0592	0.8972	1.48	1.2	1.2641	1
	c_{21}	c_{31}	c_{21}	c_{40}	c_{21}	c_{40}

Table 1. The values of the normalized moments of the test patterns. (The values were scaled to eliminate different dynamic range of moments of different orders.) The complex moment used for the normalization to the second rotation is shown in the last line.

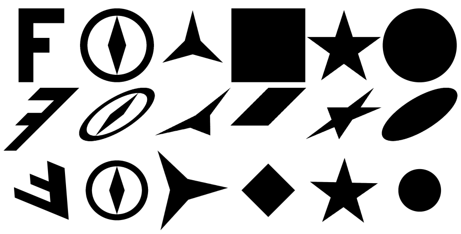

Fig. 1. The test patterns: the originals (top row), the distorted patterns (middle row), the patterns in the normalized positions (bottom row).

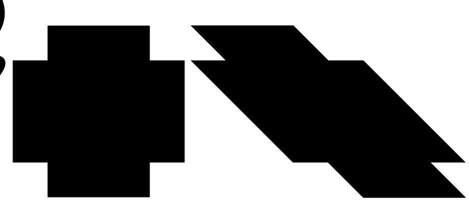

Fig. 2. The cross: the original (left) and distorted (right).

the method performed very well. A proper non-zero moment was selected for normalization (c_{51} once and c_{40} twice) and the values of the normalized moments of the original and deformed crosses were almost the same.

4 Conclusion

We presented a new way of image normalization with respect to unknown affine transform. In addition to simplicity, the main advantage of the method is their ability to handle symmetric as well as non-symmetric objects. Unlike the Shen and Ip's method [9], which was also developed for symmetric objects and has been considered as the best one, our method does not require prior knowledge of the number of folds. This is a significant improvement because its detection (either by autocorrelation or by polar Fourier analysis) is always time-consuming and sometimes very difficult.

The experiments in the paper show the performance of our method on artificial binary images to demonstrate the main features of the method. In practice, the method can be applied without any modifications also to graylevel images regardless of their symmetry/non-symmetry. The only potential drawback of our method is that in certain rare situations it might become unstable, which means that a small change of the image results in a significant change of its normalized position. This is, however, a common weakness of all geometric normalization methods.

Once the image has been normalized, its moments can be used as affine invariants for recognition. Comparing to traditional affine moment invariants [2], [3], the presented method has a big theoretical advantage. The construction of the invariants is straightforward and their structure is easy to understand. Thanks to this, we can immediately resolve the problem of finding minimum complete and independent set of invariants. For the invariants [2] and [3], this problem has not been resolved yet. Here, each moment which was not used in normalization constraints, generates just one affine invariant. Independence and completeness of such invariants follow from the independence and completeness of the moments themselves. Using minimum complete and independent set of invariants yields maximum possible recognition power at minimum computational cost.

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