



Contributions to the theory of weakly distributive complete Boolean algebras

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Abstract. We investigate complete Boolean algebras that carry a continuous submeasure.

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1. Introduction

A complete Boolean algebra B is a *measure algebra* if it carries a strictly positive σ -additive measure. B is a *Maharam algebra* if it carries a strictly positive continuous submeasure. Every measure algebra is a Maharam algebra and every Maharam algebra is weakly distributive and satisfies the countable chain condition (ccc).

The 1937 problem of von Neumann from the Scottish book [1] asks if every weakly distributive ccc Boolean algebra is a measure algebra. In [2], Balcar, Jech and Pazák proved that it is consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra, and in [3], Talagrand proved that there exists a Maharam algebra that is not a measure algebra.

For more details on the history see [4].

The present paper investigates *diagonal properties* of complete Boolean algebras. These properties are related to the weak distributive law and to the existence of a continuous submeasure. We apply these properties to get a closer look at Maharam algebras, in particular those that are not measure algebras.

2. Diagonal properties

A *Boolean algebra* is a set B with Boolean operations $a \vee b$ and $-a$ and constants $\mathbf{0}$ and $\mathbf{1}$. If every subset A of B has a least upper bound $\bigvee A$ and the greatest lower bound $\bigwedge A$ then B is *complete* Boolean algebra. An *antichain* in B is a nonempty set $A \subset B$ such that distinct $a, b \in A$ are *disjoint*, i.e. $a \wedge b = \mathbf{0}$. B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

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Let B be a complete Boolean algebra. B is *weakly distributive* (more precisely, (ω, ω) -weakly distributive), if whenever $A_0, A_1, \dots, A_n, \dots$ ($n \in \omega$) are countable maximal antichains, then there exists a dense set D such that each $d \in D$ meets only finitely many elements of each A_n .

Definition. Let B be a complete Boolean algebra.

- (1) B has the diagonal property if whenever $A_0, A_1, \dots, A_n, \dots$ ($n \in \omega$) are countable maximal antichains, then each A_n has a finite subset E_n such that

$$\lim_n \bigvee \{a : a \in E_n\} = \mathbf{1}.$$

- (2) B has the strategic diagonal property if Player II has a winning strategy in the infinite game where Player I plays maximal antichains A_0, A_1, A_2, \dots , Player II plays finite sets $E_0 \subset A_0, E_1 \subset A_1, E_2 \subset A_2, \dots$ and Player II wins if and only if

$$\lim_n \bigvee \{a : a \in E_n\} = \mathbf{1}.$$

- (3) B has the uniform diagonal property if there exist functions $F_0, F_1, \dots, F_n, \dots$ ($n \in \omega$) acting on maximal antichains such that whenever A_0, A_1, A_2, \dots are maximal antichains and for each $n, E_n = F_n(A_n)$, then

$$\lim_n \bigvee \{a : a \in E_n\} = \mathbf{1}.$$

Clearly, the uniform diagonal property implies the strategic diagonal property, which in turn implies the diagonal property.

It has been long well known that if B satisfies ccc then the diagonal property is equivalent to weak distributivity.

If B satisfies ccc then either the strategic diagonal property or the uniform diagonal property is equivalent to the existence of a continuous submeasure (see Fremlin [5] or Balcar - Jech [4]).

In [6] (Lemma 3.5) it is shown that the diagonal property implies the \mathfrak{b} -chain condition, where \mathfrak{b} is the *bounding number*, the least cardinal \mathfrak{b} of a family \mathcal{F} of functions from ω to ω such that for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ such that $g(n) \leq f(n)$ for infinitely many n :

Theorem 1. [6] *A complete Boolean algebra B satisfies the diagonal property if and only if it is weakly distributive and satisfies the \mathfrak{b} -chain condition.*

Proof. First let B be weakly distributive and \mathfrak{b} -cc. Let $A_n = \{a_{nk} : k \in \omega\}, n \in \omega$, be countable maximal antichains. By weak distributivity,

$$\bigvee_{f \in \omega^\omega} \liminf_n \bigvee \{a_{nk} : k \leq f(n)\} = \mathbf{1},$$

and by \mathfrak{b} -cc there is a family \mathcal{F} of size less than \mathfrak{b} such that

$$\bigvee_{f \in \mathcal{F}} \liminf_n \bigvee \{a_{nk} : k \leq f(n)\} = \mathbf{1}.$$

Let $g \in \omega^\omega$ be an upper bound of \mathcal{F} under eventual domination. It follows that

$$\liminf_n \bigvee \{a_{nk} : k \leq g(n)\} = \mathbf{1}$$

proving that B has the diagonal property.

Conversely, if B does not have the \mathfrak{b} -chain condition then B contains $\mathcal{P}(\mathfrak{b})$ as a complete subalgebra, and it suffices to prove that $\mathcal{P}(\mathfrak{b})$ does not have the diagonal property.

Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of functions from ω to ω , and let, for each $n, k \in \omega$,

$$a_{n,k} = \{ \alpha < \mathfrak{b} : f_\alpha(n) = k \},$$

$$A_n = \{ a_{nk} : k \in \omega \}.$$

Each A_n is a maximal antichain in $\mathcal{P}(\mathfrak{b})$, and if $g \in \omega^\omega$, then there exist an $\alpha < \mathfrak{b}$ such that $g(n) < f_\alpha(n)$ for infinitely many n . Hence for infinitely many n ,

$$\alpha \notin \bigcup \{ a_{nk} : k \leq g(n) \},$$

and so $\alpha \notin \liminf_n \bigcup \{ a_{nk} : k \leq g(n) \}$. So $\mathcal{P}(\mathfrak{b})$ does not have the diagonal property. \square

We now show that the strategic diagonal property implies ccc, and so both the strategic diagonal property and the uniform diagonal property are equivalent to B being Maharam algebra.

If B does not have ccc then it contains $\mathcal{P}(\omega_1)$ as a complete subalgebra, so it is enough to show that $\mathcal{P}(\omega_1)$ does not have the strategic diagonal property.

Theorem 2. *The algebra $\mathcal{P}(\omega_1)$ does not have the strategic diagonal property. Consequently, if B is a complete Boolean algebra with the strategic diagonal property then B satisfies the countable chain condition.*

Proof. For each $\beta < \omega_1$, let f_β be a one-to-one function from β into ω . For $\alpha < \omega_1$ and $n \in \omega$, let

$$a_{\alpha,n} = \{ \beta < \omega_1 : f_\beta(\alpha) = n \},$$

$$A_\alpha = \{ a_{\alpha,n} : n \in \omega \}.$$

Now assume that σ is a strategy for Player II in the game (2). We shall find α_n , $n \in \omega$, such that if Player I plays A_{α_n} and Player II uses σ , then II loses.

Let $\omega_1^{<\omega}$ denote the set of all increasing finite sequences of countable ordinals. For $s \in \omega_1^{<\omega}$, $s = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$, let $F(s) = k$ be such that if Player II applies σ to $\langle A_{\alpha_0}, \dots, A_{\alpha_{n-1}} \rangle$ resulting in a finite set $E \subset A_{\alpha_{n-1}}$, then $E \subset \{a_{\alpha_{n-1},0}, \dots, a_{\alpha_{n-1},k}\}$. For each $s \in \omega_1^{<\omega}$ there exists some k such that the set

$$W_s = \{ \alpha < \omega_1 : F(s \hat{\ } \alpha) = k \}$$

is uncountable. Let C_s be the set of all $\beta < \omega_1$ such that $W_s \cap \beta$ is unbounded in β . C_s is a closed unbounded set.

Let C be the diagonal intersection of the C_s , i.e.

$$C = \{ \beta < \omega_1 : (\forall s \in \beta^{<\omega}) \beta \in C_s \}.$$

C is closed unbounded, so let $\beta > 0$ be some $\beta \in C$.

We shall construct $\{\alpha_n\}_n$ such that for every n , if $E_n = \sigma(A_{\alpha_0}, \dots, A_{\alpha_n})$ then $\beta \notin \bigcup \{a : a \in E_n\}$. This witnesses that II loses the game and so σ is not a winning strategy.

We construct α_n by induction. For $n \in \omega$, assume that $\alpha_0, \dots, \alpha_{n-1}$ have been found, and let $s = \langle \alpha_0, \dots, \alpha_{n-1} \rangle$. Let k be such that $F(s \hat{\ } \alpha) = k$ for all $\alpha \in W_s$. Since $W_s \cap \beta$ is infinite and f_β is one-to-one, there exists some $\alpha \in W_s$ such that $f_\beta(\alpha) > k$. We let $\alpha_n = \alpha$. Since $\beta \notin a_{\alpha,0} \cup \dots \cup a_{\alpha,k}$ we have $\beta \notin \bigcup \{a : a \in E_n\}$ where $E_n = \sigma(A_{\alpha_0}, \dots, A_{\alpha_n})$

Thus if Player I plays A_{α_n} , $n \in \omega$, Player II loses: we have

$$\beta \notin \bigcup_{n=0}^{\infty} \bigcup \{a : a \in E_n\}$$

proving that it is not the case that $\lim_n \bigcup \{a : a \in E_n\} = \omega_1$. \square

3. Forcing iteration

Let us consider the operation on complete Boolean algebras corresponding to iterated forcing: if B is a complete Boolean algebra and \dot{C} is a complete Boolean algebra in V^B , then $B * \dot{C}$ is a complete Boolean algebra that produces the iterated forcing model $(V^B)^{\dot{C}}$. For the details, see [7].

It is well known that if B is a measure algebra and if \dot{C} is a measure algebra in V^B then $B * \dot{C}$ is a measure algebra. Similarly, if B is a ccc and weakly distributive and \dot{C} is ccc and weakly distributive in V^B , then $B * \dot{C}$ is ccc and weakly distributive. We prove the same for Maharam algebras:

Theorem 3. *If B is a Maharam algebra and if \dot{C} is a Maharam algebra in V^B , then $B * \dot{C}$ is a Maharam algebra.*

According to Fremlin's notes [5], Theorem 3 was also proved by Farah by a different argument.

If f and g are functions from ω to ω , we say that g dominates f , $f <^* g$, if for some $N \in \omega$, $f(n) < g(n)$ for all $n \geq N$.

A Boolean-valued name \dot{a} for a natural number corresponds to a (countable indexed) partition of $\mathbf{1}$, namely

$$\{ \|\dot{a} = k\| : k \in \omega \}.$$

A Boolean-valued name \dot{f} for a function from ω to ω corresponds to a matrix of partitions $\{A_n : n \in \omega\}$, where

$$A_n = \{ \|\dot{f}(n) = k\| : k \in \omega \}.$$

If \dot{f} is a name for a function from ω to ω and if $g : \omega \rightarrow \omega$, then

$$\|\dot{f} <^* g\| = \mathbf{1} \quad \text{iff} \quad \lim_n \|\dot{f}(n) <^* g(n)\| = \mathbf{1}.$$

Using these observations, we can reformulate the diagonal properties as follows, obtaining another characterization of ccc weakly distributive and Maharam algebras (see [4, pp. 258, 259 and 261]).

Let B be a complete Boolean algebra. Then

- (1) B has the diagonal property if and only if for every name \dot{f} for a function from ω to ω there exists a function $g : \omega \rightarrow \omega$ such that $\|\dot{f} <^* g\| = \mathbf{1}$.
- (2) B has the strategic diagonal property if Player II has a winning strategy in the game where I plays names $\dot{f}(0), \dot{f}(1), \dot{f}(2), \dots$ for integers and II plays integers $g(0), g(1), g(2), \dots$ and II wins if and only if $\|\dot{f} <^* g\| = \mathbf{1}$.
- (3) B has the uniform diagonal property if and only if there exist functions $F_n, n \in \omega$, acting on names for natural numbers such that for every name \dot{f} for a function from ω to ω , we have

$$\|\dot{f} < g\| = \mathbf{1},$$

where $g : \omega \rightarrow \omega$ is obtained as follows: for every n , $g(n) = F_n(\dot{f}(n))$.

Theorem 4. *The diagonal properties are preserved under two-step forcing iteration:*

*If B has the diagonal property (resp. the uniform diagonal property) and if \dot{C} has, in V^B , the diagonal property (resp. the uniform diagonal property) then $B * \dot{C}$ has the diagonal (resp. uniform diagonal) property.*

Theorem (3) now follows.

Proof. First we give a proof for the diagonal property which is somewhat simpler.

Let $D = B * \dot{C}$ and assume that both B and \dot{C} (in V^B) have the diagonal property. Then if \dot{f} is a D -name, then (since \dot{f} corresponds to a B -name for a \dot{C} -name) V^B satisfies that there is some \dot{g} such that $\|\dot{f} <^* \dot{g}\|_{\dot{C}} = \mathbf{1}$. Now \dot{g}

is a B -name, so there exists some h such that $\|\dot{g} <^* h\|_B = \mathbf{1}$. It follows that $\|\dot{f} <^* \dot{g}\|_D = \mathbf{1}$ and $\|\dot{g} <^* h\|_D = \mathbf{1}$, hence $\|\dot{f} <^* h\|_D = \mathbf{1}$.

Now let $D = B * \dot{C}$ and assume that both B and \dot{C} (in V^B) have the uniform diagonal property.

There exist functions F_n acting on B -names for natural numbers that witness the domination of each B -name \dot{f} , and similarly, in V^B there are functions \dot{G}_n acting on \dot{C} -names for natural numbers. We obtain functions H_n for the algebra $D = B * \dot{C}$ as follows: If \dot{a} is a D -name for a natural number, apply \dot{G}_n inside V^B to get a B -valued natural number \dot{b} , and then let $H_n(\dot{a}) = F_n(\dot{b})$. (In other words, $H_n(\dot{a}) = F_n(\dot{G}_n(\dot{a}))$ where \dot{a} on the right hand side is considered a B -name for a \dot{C} -name.) Then we verify that the functions H_n witness the uniform diagonal property of D . \square

4. Pathological Maharam algebras

Let us call a Maharam algebra *pathological* if it is not a measure algebra. Talagrand's construction produces an example of pathological Maharam algebra B of size 2^{\aleph_0} . The following theorem implies that there are arbitrary large pathological Maharam algebras.

Theorem 5. *If B is a measure algebra and \dot{C} is a pathological Maharam algebra in V^B then $B * \dot{C}$ is pathological.*

*Moreover, $B * \dot{C}$ has a continuous submeasure that extends the given measure on B .*

The algebra $B * \dot{C}$ contains a measure algebra (B) as a complete subalgebra. It is not known whether there exists a Maharam algebra that does not contain a measure algebra as a complete subalgebra.

Proof. Let $D = B * \dot{C}$. The algebra D is a Maharam algebra by Theorem (3). If D were a measure algebra, then $\dot{C} = D : B$ would also be a measure algebra: this is proved e.g. in Kunen's paper [8]. Since Theorem (5) claims a little more, we give a proof that D is pathological by explicitly describing the pathological submeasure.

Let B be a measure algebra. There is a measure space M and σ -additive measure μ on M such that B is isomorphic to the σ -algebra of Borel sets in M mod the ideal of μ -null sets. An argument due to Dana Scott gives a representation of real numbers in V^B by real-valued measurable function on M : a name $\dot{r} \in V^B$ for a real number corresponds to a measurable function $f : M \rightarrow \mathbb{R}$ such that for every real q

$$\|\dot{r} \geq q\|_B = \{x \in M : f(x) \geq q\}.$$

Now let \dot{C} be a B -name for a pathological Maharam algebra, and let \dot{m} be a name for a continuous submeasure on \dot{C} that is not uniformly exhaustive. That

is, for every $q \in B^+$ there exist a $p \in B^+$, $p < q$, and an $\varepsilon > 0$ such that for every $k \in \omega$,

$$p \Vdash \exists \text{ disjoint } c_1, \dots, c_k \text{ with } \dot{m}(c_i) \geq \varepsilon, \quad i = 1, \dots, k. \quad (1)$$

Let $D = B * \dot{C}$. We define a function $\nu : D \rightarrow [0, 1]$ as follows: An element of D is a B -name \dot{c} for an element of \dot{C} . The value $\dot{m}(\dot{c})$ is a B -name for a real number in $[0, 1]$, hence represented by a measurable function $f : M \rightarrow [0, 1]$. We let

$$\nu(\dot{c}) = \int_M f d\mu.$$

It is a routine argument to verify that ν is a subadditive function on D , and $\nu(\dot{c}) > 0$ when $\dot{c} \neq \mathbf{0}$. Moreover, if $\|\dot{c} \in B\| = \mathbf{1}$, then let b be the unique $b \in B$ such that $\|\dot{c} = \mathbf{1}\| = b$ and $\|\dot{c} = \mathbf{0}\| = -b$. We have $\nu(\dot{c}) = \int_b d\mu = \mu(b)$.

We'll show that ν is continuous and is not uniformly exhaustive.

First let $\dot{c}_0 \geq \dot{c}_1 \geq \dots \geq \dot{c}_n \geq \dots$ be a sequence in D such that $\bigwedge_{n \in \omega} \dot{c}_n = \mathbf{0}$. Let $\varepsilon > 0$. We shall find an $N \in \omega$ such that $\nu(\dot{c}_n) < 2\varepsilon$, for every $n \geq N$.

In V^B , the sequence $\{\dot{c}_n\}_{n \in \omega}$ is a decreasing sequence and $\|\bigwedge_{n \in \omega} \dot{c}_n = \mathbf{0}\| = \mathbf{1}$, and since \dot{m} is continuous, we have $\|\lim_n \dot{m}(\dot{c}_n) = \mathbf{0}\| = \mathbf{1}$. Thus there exists a maximal antichain $\{p_k : k \in \omega\}$ and for each k a number n_k such that

$$p_k \Vdash \dot{m}(\dot{c}_n) < \varepsilon, \text{ for all } n \geq n_k.$$

Now let K be such that

$$\mu \left(\bigvee_{k=K+1}^{\infty} p_k \right) < \varepsilon,$$

and let $N = \max\{n_k : k \leq K\}$. Let $a = p_0 \vee \dots \vee p_K$ and $b = \bigvee_{k=K+1}^{\infty} p_k$. For each n , let f_n be a measurable function representing $\dot{m}(\dot{c}_n)$. If $n \geq N$, we have

$$\nu(\dot{c}_n) = \int_M f_n d\mu = \int_a f_n d\mu + \int_b f_n d\mu < \int_a \varepsilon d\mu + \int_b 1 d\mu \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Thus ν is continuous.

In order to show that ν is not uniformly exhaustive, we first find some $p \neq \mathbf{0}$ and $\varepsilon > 0$ such that Eq. (1) holds for every $k \in \omega$. Thus for every k there exist $\dot{c}_1, \dots, \dot{c}_k \in D$ such that $p \Vdash \dot{c}_1, \dots, \dot{c}_k$ are pairwise disjoint and $p \Vdash \dot{m}(\dot{c}_i) \geq \varepsilon$ for $i = 1, \dots, k$. We may assume that $-p \Vdash \dot{c}_i = \mathbf{0}$ for all i , and so $\dot{c}_1, \dots, \dot{c}_k$ are pairwise disjoint. Moreover, for each $i = 1, \dots, k$,

$$\nu(\dot{c}_i) \geq \int_p \varepsilon d\mu = \varepsilon \cdot \mu(p).$$

Thus ν is not uniformly exhaustive. \square

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