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ON THE INITIAL VALUE PROBLEM FOR TWO-DIMENSIONAL SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS

Jiří Šremr

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ON THE INITIAL VALUE PROBLEM FOR TWO–DIMENSIONAL SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS

JIŘÍ ŠREMR

We establish new efficient conditions sufficient for the unique solvability of the Cauchy problem for two-dimensional systems of linear functional differential equations with monotone operators.

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1. Introduction and Notation

On the interval $[a,b]$, we consider two–dimensional differential system

$$u_i'(t) = \sigma_{i1}\,\ell_{i1}(u_1)(t) + \sigma_{i2}\,\ell_{i2}(u_2)(t) + q_i(t) \qquad (i=1,2) \tag{1.1}$$

with the initial conditions

$$u_1(a) = c_1, \qquad u_2(a) = c_2\,, \tag{1.2}$$

where $\ell_{ik} : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are linear nondecreasing operators, $\sigma_{ik} \in \{-1,1\}$, $q_i \in L([a,b];\mathbb{R})$, and $c_i \in \mathbb{R}$ $(i,k=1,2)$. Under a solution of the problem (1.1), (1.2) is understood an absolutely continuous vector function $u=(u_1,u_2)^T : [a,b] \to \mathbb{R}^2$ satisfying (1.1) almost everywhere on $[a,b]$ and verifying also the initial conditions (1.2).

The problem on the solvability of the Cauchy problem for linear functional differential equations and their systems has been studied by many authors (see, e.g., [1, 7, 9, 10, 12, 17] and references therein). There are a lot of interested results but only a few efficient conditions is known at present. Furthermore, most them is available for the one-dimmensional case only or for the systems with the so–called Volterra operators (see, e.g., [3–5, 7, 9, 12]). Let us mention that the efficient conditions guaranteeing the unique solvability of the initial value problem for n-dimensional systems of linear functional difefrential equations are given, e.g., in [2, 10, 11, 13, 14].

In this paper, we establish new efficient condition sufficient for the unique solvability of the problem (1.1), (1.2) for any disposition of the numbers $\sigma_{ij} \in \{-1,1\}$ $(i,j=1,2)$. The integral conditions given in Theorems 2.1–2.11 are optimal in a certain sense which is shown by counter–examples constructed in the last part of the paper.

The following notation is used throughout the paper:

1. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

2. $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ equipped with the norm
$$\|u\|_C = \max\big\{|u(t)| : t \in [a,b]\big\}.$$

3. $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h : [a,b] \to \mathbb{R}$ equipped with the norm
$$\|h\|_L = \int_a^b |h(s)|ds.$$

4. $L\big([a,b];\mathbb{R}_+\big) = \big\{h \in L([a,b];\mathbb{R}) : h(t) \geq 0 \text{ for a.a. } t \in [a,b]\big\}$.

5. An operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is said to be nondecreasing if the inequality
$$\ell(u_1)(t) \leq \ell(u_2)(t) \quad \text{for a.a.} \quad t \in [a,b]$$
holds for every functions $u_1, u_2 \in C([a,b];\mathbb{R})$ such that
$$u_1(t) \leq u_2(t) \quad \text{for} \quad t \in [a,b].$$

6. $\mathcal{P}_{ab}$ is the set of linear nondecreasing operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

2. Main Results

In this section, we present the main results of the paper. The proofs are given later, in Section 3. Theorems formulated in Subsections 2.1–2.6 contain the efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) for any disposition of the numbers $\sigma_{ij} \in \{-1, 1\}$ $(i, j = 1, 2)$. Recall that the operators ℓ_{ij} are supposed to be linear and nondecreasing, i.e., such that $\ell_{ij} \in \mathcal{P}_{ab}$ for $i, j = 1, 2$.

Put
$$A_{ij} = \int_a^b \ell_{ij}(1)(s)ds \quad \text{for} \quad i, j = 1, 2 \tag{2.1}$$
and
$$\varphi(s) = \begin{cases} 1 & \text{for} \quad s \in [0, 1[\\ 1 - \frac{1}{4}(s-1)^2 & \text{for} \quad s \in [1, 3[\end{cases}. \tag{2.2}$$

2.1. The case $\sigma_{11} = 1$, $\sigma_{22} = 1$, $\sigma_{12}\sigma_{21} > 0$

Theorem 2.1. *Let* $\sigma_{11} = 1$, $\sigma_{22} = 1$, *and* $\sigma_{12}\sigma_{21} > 0$. *Let, moreover,*

$$A_{ii} < 1 \quad \text{for} \quad i = 1, 2 \tag{2.3}$$

and

$$A_{12}\, A_{21} < (1 - A_{11})(1 - A_{22}), \tag{2.4}$$

where the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.1. Neither one of the strict inequalities (2.3) and (2.4) can be replaced by the nonstrict one (see Examples 4.1 and 4.3).

Remark 2.2. Let H_1 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < (1 - x)(1 - y)$$

(see Fig. 2.1). According to Theorem 2.1, the problem (1.1), (1.2) is uniquely solvable

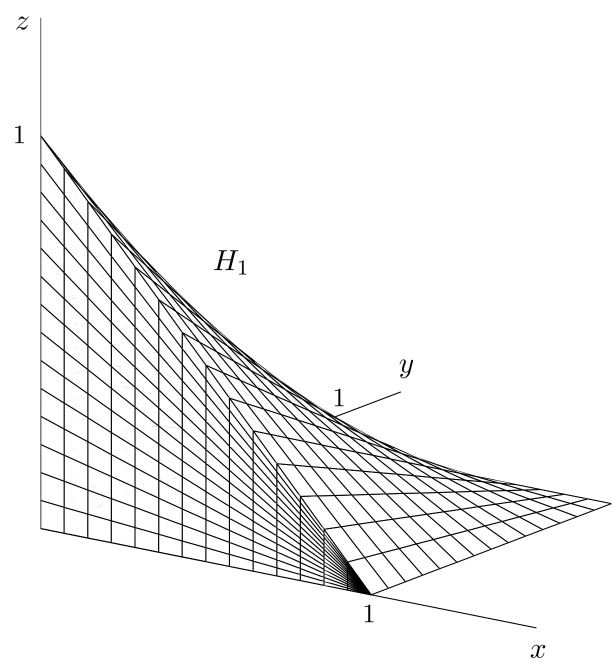

Fig. 2.1.

if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in H_1\,.$$

Remark 2.3. It should be noted that Theorem 2.1 can be derived as a consequence of Corollary 1.3.1 given in [10]. However, we shall prove this theorem using the technique common for all the statements of this paper.

Remark 2.4. According to Corollary 3.2 of [16], if $\sigma_{11} = 1, \sigma_{22} = 1$, $\sigma_{12}\sigma_{21} > 0$, and

$$A_{11} + A_{12} < 1, \qquad A_{21} + A_{22} < 1, \tag{2.5}$$

where the numbers A_{ij} $(i, j = 1, 2)$ are defined by (2.1), then the problem (1.1), (1.2) has a unique solution $(u_1, u_2)^T$. Moreover, this solution satisfies

$$u_1(t) \geq 0, \quad \sigma_{12}u_2(t) \geq 0 \quad \text{for} \quad t \in [a, b]$$

provided that $c_1 \geq 0$, $\sigma_{12}c_2 \geq 0$, and

$$q_1(t) \geq 0, \quad \sigma_{12}q_2(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

If the assumption (2.5) is weakened to the assumptions (2.3), (2.4) then the problem (1.1), (1.2) has still a unique solution but no information about sign of this solution is guaranteed in general.

2.2. The case $\sigma_{11} = 1$, $\sigma_{22} = 1$, $\sigma_{12}\sigma_{21} < 0$

Theorem 2.2. *Let $\sigma_{11} = 1$, $\sigma_{22} = 1$, and $\sigma_{12}\sigma_{21} < 0$. Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < 4\sqrt{(1 - A_{11})(1 - A_{22})} + \left(\sqrt{1 - A_{11}} + \sqrt{1 - A_{22}}\right)^2, \tag{2.6}$$

where the numbers A_{ij} $(i, j = 1, 2)$ are defined by (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.5. The strict inequalities (2.3) in Theorem 2.2 cannot be replaced by the nonstrict ones (see Example 4.1). Furthermore, the strict inequality (2.6) cannot be replaced by the nonstrict one provided $A_{11} = A_{22}$ (see Example 4.4).

Remark 2.6. Let H_2 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < 4\sqrt{(1 - x)(1 - y)} + \left(\sqrt{1 - x} + \sqrt{1 - y}\right)^2$$

(see Fig. 2.2). According to Theorem 2.2, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_2\,.$$

2.3. The case $\sigma_{11}\sigma_{22} < 0$, $\sigma_{12}\sigma_{21} > 0$

At first, we consider the case, where $\sigma_{11} = 1$ and $\sigma_{22} = -1$.

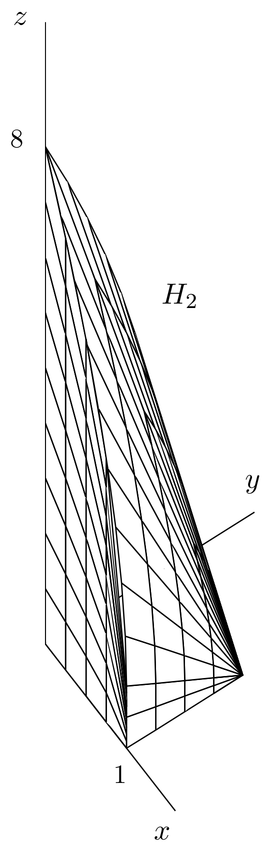

Fig. 2.2.

Theorem 2.3. *Let* $\sigma_{11} = 1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} > 0$. *Let, moreover,*

$$A_{11} < 1, \qquad A_{22} < 3, \tag{2.7}$$

and

$$A_{12}A_{21} < (1 - A_{11})\varphi(A_{22}), \tag{2.8}$$

where the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1) *and the function* φ *is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.7. Neither one of the strict inequalities (2.7) and (2.8) can be replaced by the nonstrict one (see Examples 4.1, 4.2, 4.5, and 4.6).

Remark 2.8. Let H_3 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 3, \quad z < (1 - x)\varphi(y)$$

(see Fig. 2.3). According to Theorem 2.3, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_3\,.$$

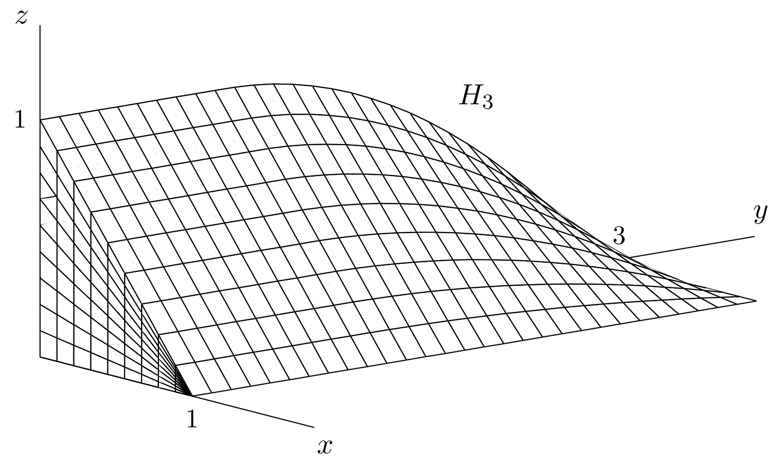

Fig. 2.3.

The next statement concerning the case, where $\sigma_{11} = -1$ and $\sigma_{22} = 1$, follows immediately from Theorem 2.3.

Theorem 2.4. *Let* $\sigma_{11} = -1$, $\sigma_{22} = 1$, *and* $\sigma_{12}\sigma_{21} > 0$. *Let, moreover,*

$$A_{11} < 3, \qquad A_{22} < 1, \tag{2.9}$$

and

$$A_{12}A_{21} < (1 - A_{22})\varphi(A_{11}),$$

where the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1) *and the function* φ *is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

2.4. The case $\boldsymbol{\sigma_{11}\sigma_{22} < 0}$, $\boldsymbol{\sigma_{12}\sigma_{21} < 0}$

At first, we consider the case, where $\sigma_{11} = 1$ and $\sigma_{22} = -1$.

Theorem 2.5. *Let* $\sigma_{11} = 1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.7) *be satisfied and*

$$A_{12}A_{21} < (1 - A_{11})(3 - A_{22}), \tag{2.10}$$

where the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.9. The strict inequalities (2.7) cannot be replaced by the nonstrict ones (see Examples 4.1 and 4.2). Furthermore, the strict inequality (2.10) cannot be replaced by the nonstrict one provided $1 < A_{22} < 3$ (see Example 4.7).

Remark 2.10. Let H_4 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 3, \quad z < (1 - x)(3 - y)$$

(see Fig. 2.4). According to Theorem 2.5, the problem (1.1), (1.2) is uniquely solvable

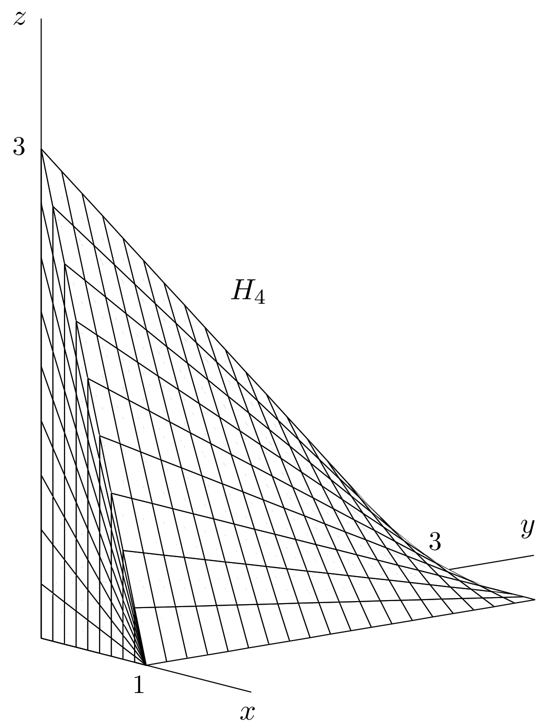

Fig. 2.4.

if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in H_4\,.$$

Example 4.7 shows that Theorem 2.5 is optimal whenever $1 < A_{22} < 3$. If $A_{22} \leq 1$ then the theorem mentioned can be improved. For example, the next theorem improves Theorem 2.5 if A_{22} is close to zero.

Theorem 2.6. *Let* $\sigma_{11} = 1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < \frac{\omega(1-A_{11})\big[1 + A_{22}(1-A_{22})\big]}{1 - A_{11} + \omega A_{22}}\,, \tag{2.11}$$

where

$$\omega = 4\sqrt{1-A_{11}} + \left(1 + \sqrt{(1-A_{11})(1-A_{22})}\right)^2 \tag{2.12}$$

and the numbers A_{ij} $(i,j=1,2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.11. If $A_{22} = 0$ then the inequality (2.11) can be rewritten as

$$A_{12}A_{21} < 4\sqrt{1-A_{11}} + \left(1 + \sqrt{1-A_{11}}\right)^2,$$

which coincides with the assumption (2.6) of Theorem 2.2.

Remark 2.12. Let $\widetilde{H}_4$ be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < \frac{\widetilde{\omega}(1-x)\big[1 + y(1-y)\big]}{1 - x + \widetilde{\omega} y}\,,$$

where

$$\widetilde{\omega} = 4\sqrt{1-x} + \Big(1 + \sqrt{(1-x)(1-y)}\Big)^2$$

(see Fig. 2.5). According to Theorem 2.6, the problem (1.1), (1.2) is uniquely solvable

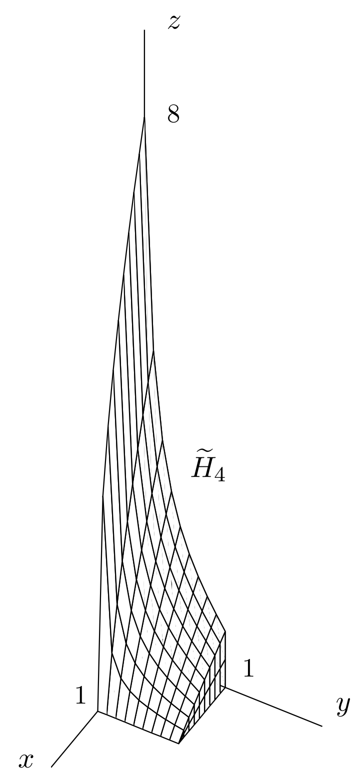

Fig. 2.5.

if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds \right) \in \widetilde{H}_4\,.$$

The next statements concerning the case, where $\sigma_{11} = -1$ and $\sigma_{22} = 1$, follow immediately from Theorems 2.5 and 2.6.

Theorem 2.7. *Let* $\sigma_{11} = -1$, $\sigma_{22} = 1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.9) *be satisfied and*

$$A_{12}A_{21} < (1 - A_{22})(3 - A_{11}),$$

where the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Theorem 2.8. *Let* $\sigma_{11} = -1$, $\sigma_{22} = 1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < \frac{\omega_0(1 - A_{22})\big[1 + A_{11}(1 - A_{11})\big]}{1 - A_{22} + \omega_0 A_{11}}\,,$$

where

$$\omega_0 = 4\sqrt{1 - A_{22}} + \Big(1 + \sqrt{(1 - A_{11})(1 - A_{22})}\Big)^2$$

and the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

2.5. The case $\boldsymbol{\sigma_{11} = -1}$, $\boldsymbol{\sigma_{22} = -1}$, $\boldsymbol{\sigma_{12}\sigma_{21} > 0}$

Theorem 2.9. *Let* $\sigma_{11} = -1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} > 0$. *Let, moreover,*

$$A_{ii} < 3 \quad \textit{for} \quad i = 1, 2 \tag{2.13}$$

and

$$A_{12}A_{21} < \frac{1}{\omega}\,\varphi(A_{11})\varphi(A_{22}), \tag{2.14}$$

where

$$\omega = \max\big\{1, A_{11}(A_{22} - 1), A_{22}(A_{11} - 1)\big\}, \tag{2.15}$$

the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1) *and the function* φ *is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.13. The strict inequalities (2.13) cannot be replaced by the nonstrict ones (see Example 4.2). Furthermore, the strict inequality (2.14) cannot be replaced by the nonstrict one provided $\omega = 1$ (see Examples 4.8–4.10).

Remark 2.14. Let H_5 be the set of triplets $(x, y, z) \in \mathbb{R}_+^3$ satisfying

$$x < 3, \quad y < 3, \quad z < \frac{\varphi(x)\varphi(y)}{\max\{1, x(y - 1), y(x - 1)\}}$$

(see Fig. 2.6). According to Theorem 2.9, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_5\,.$$

2.6. The case $\boldsymbol{\sigma_{11} = -1}$, $\boldsymbol{\sigma_{22} = -1}$, $\boldsymbol{\sigma_{12}\sigma_{21} < 0}$

Theorem 2.10. *Let* $\sigma_{11} = -1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.13) *be satisfied and*

$$A_{12}A_{21} < \frac{1}{\omega}\Big(3 - \max\{A_{11}, A_{22}\}\Big)\varphi\Big(\min\{A_{11}, A_{22}\}\Big), \tag{2.16}$$

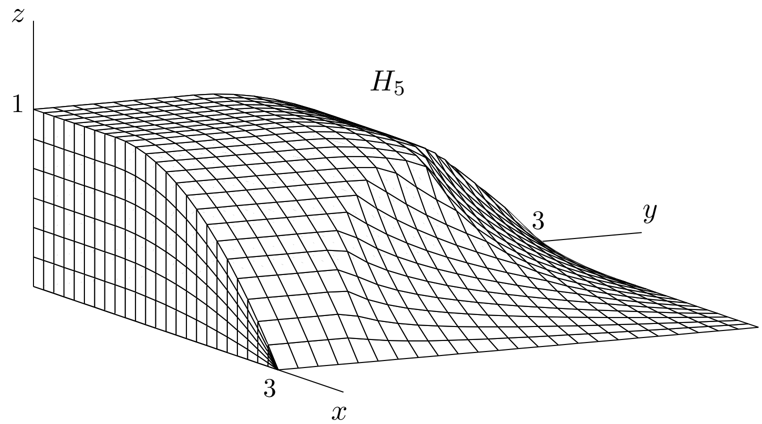

Fig. 2.6.

where

$$\omega = \max\big\{1, 3(A_{11}-1), 3(A_{22}-1)\big\}, \tag{2.17}$$

the numbers A_{ij} $(i,j=1,2)$ *are defined by* (2.1) *and the function* φ *is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.15. The strict inequalities (2.13) cannot be replaced by the nonstrict ones (see Example 4.2). Futhermore, the strict inequality (2.16) cannot be replaced by the nonstrict one provided that $\omega = 1$ and $\max\{A_{11}, A_{22}\} > 1$ (see Examples 4.11 and 4.12).

Remark 2.16. Let H_6 be the set of triplets $(x,y,z) \in \mathbb{R}_+^3$ satisfying

$$x < 3, \quad y < 3, \quad z < \frac{\big(3-\max\{x,y\}\big)\varphi\big(\min\{x,y\}\big)}{\max\{1, 3(x-1), 3(y-1)\}}$$

(see Fig. 2.7). According to Theorem 2.10, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in H_6\,.$$

If $\max\{A_{11}, A_{22}\} \le 1$ then the assumption (2.16) of Theorem 2.10 can be improved. For example, the next theorem improves Theorem 2.10 if $\max\{A_{11}, A_{22}\}$ is close to zero.

Theorem 2.11. *Let* $\sigma_{11} = -1$, $\sigma_{22} = -1$, *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < \frac{\omega_0}{\omega_0\big(A_{11}+A_{22}-A_{11}A_{22}\big)+A_{11}A_{22}+1}\,, \tag{2.18}$$

where

$$\omega_0 = 4 + \Big(\sqrt{1-A_{11}} + \sqrt{1-A_{22}}\Big)^2 \tag{2.19}$$

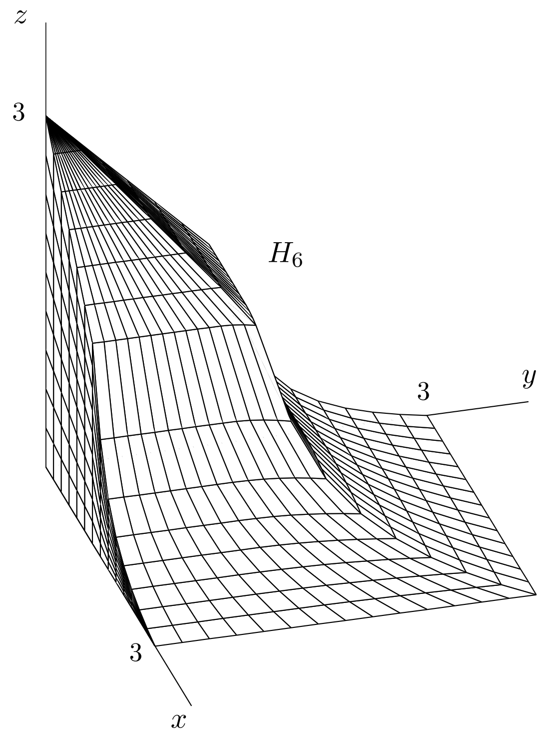

Fig. 2.7.

and the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1)*. Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.17. If $A_{11} = A_{22} = 0$ then the inequality (2.18) can be rewritten as

$$A_{12}A_{21} < 8,$$

which coincides with the assumption (2.6) of Theorem 2.2.

Remark 2.18. Let $\widetilde{H}_6$ be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < \frac{\widetilde{\omega}_0}{\widetilde{\omega}_0(x + y - xy) + xy + 1}\,,$$

where

$$\widetilde{\omega}_0 = 4 + \left(\sqrt{1-x} + \sqrt{1-y}\right)^2$$

(see Fig. 2.8). According to Theorem 2.11, the problem (1.1), (1.2) is uniquely solvable if $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ are such that

$$\left(\int_a^b \ell_{11}(1)(s)ds\,, \int_a^b \ell_{22}(1)(s)ds\,, \int_a^b \ell_{12}(1)(s)ds \int_a^b \ell_{21}(1)(s)ds\right) \in \widetilde{H}_6\,.$$

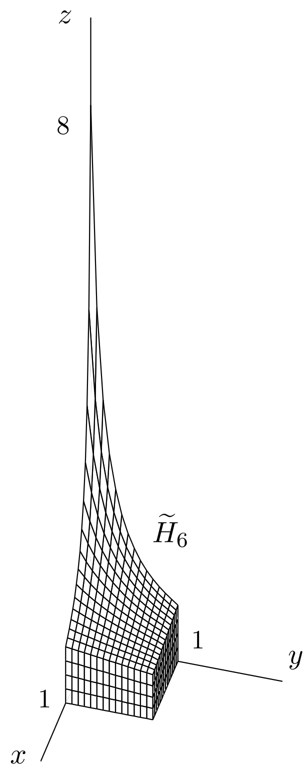

Fig. 2.8.

3. Proofs of the Main Results

In this section, we shall prove all the statements formulated above. Recall that the numbers A_{ij} $(i, j = 1, 2)$ are defined by (2.1) and the function φ is given by (2.2).

It is well–known from the general theory of boundary value problems for functional differential equations (see, e.g., [8, 10, 11, 15]) that the following lemma is true.

Lemma 3.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u_i'(t) = \sigma_{i1}\,\ell_{i1}(u_1)(t) + \sigma_{i2}\,\ell_{i2}(u_2)(t) \qquad (i = 1, 2), \tag{3.1}$$

$$u_1(a) = 0, \qquad u_2(a) = 0 \tag{3.2}$$

has only the trivial solution.

In order to simplify the discussion in the proofs below, we formulate the following obvious lemma.

Lemma 3.2. $(u_1, u_2)^T$ *is a solution of the problem* (3.1), (3.2) *if and only if* $(u_1, -u_2)^T$ *is a solution of the problem*

$$v_i'(t) = (-1)^{i-1}\sigma_{i1}\,\ell_{i1}(v_1)(t) + (-1)^{i}\sigma_{i2}\,\ell_{i2}(v_2)(t) \qquad (i = 1, 2), \tag{3.3}$$

$$v_1(a) = 0, \qquad v_2(a) = 0\,. \tag{3.4}$$

Lemma 3.3 (Remark 1.1 in [6]). *Let $\ell \in \mathcal{P}_{ab}$ be such that*

$$\int_a^b \ell(1)(s)ds < 1.$$

Then every absolutely continuous function $u : [a,b] \to \mathbb{R}$ such that

$$u'(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a,b], \qquad u(a) \geq 0$$

satisfies $u(t) \geq 0$ for $t \in [a,b]$.

Now we are in position to prove Theorems 2.1–2.11.

Proof of Theorem 2.1. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_i'(t) = \ell_{i1}(u_1)(t) + \ell_{i2}(u_2)(t) \qquad (i = 1,2) \tag{3.5}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.5), (3.2). If the inequality

$$u_i(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{3.6}$$

holds for some $i \in \{1,2\}$ then, by virtue of (2.3), the assumption $\ell_{3-i\,i} \in \mathcal{P}_{ab}$, and Lemma 3.3, we get

$$u_{3-i}(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{3.7}$$

Consequently, the functions u_1 and u_2 satisfy one of the following cases.

(a) Both functions u_1 and u_2 do not change their signs. Then, without loss of generality, we can assume that (3.6) holds for $i = 1,2$.

(b) Both functions u_1 and u_2 change their signs.

Put

$$M_i = \max\{u_i(t) : t \in [a,b]\} \qquad (i = 1,2) \tag{3.8}$$

and choose $\alpha_i \in [a,b]$ $(i = 1,2)$ such that

$$u_i(\alpha_i) = M_i \quad \text{for} \quad i = 1,2. \tag{3.9}$$

Obviously, in both cases (a) and (b), we have

$$M_1 \geq 0, \quad M_2 \geq 0, \quad M_1 + M_2 > 0. \tag{3.10}$$

The integration of (3.5) from a to α_i, in view of (3.8)–(3.10), and the assumptions $\ell_{i1}, \ell_{i2} \in \mathcal{P}_{ab}$, yields

$$M_i = \int_a^{\alpha_i} \ell_{i1}(u_1)(s)ds + \int_a^{\alpha_i} \ell_{i2}(u_2)(s)ds \leq$$

$$\leq M_1 \int\limits_a^{\alpha_i} \ell_{i1}(1)(s)ds + M_2 \int\limits_a^{\alpha_i} \ell_{i2}(1)(s)ds \leq M_1 A_{i1} + M_2 A_{i2} \quad (i = 1, 2). \tag{3.11}$$

By virtue of (2.3) and (3.10), we get from (3.11) that

$$0 \leq M_i\big(1 - A_{ii}\big) \leq M_{3-i} A_{i\,3-i} \qquad (i = 1, 2). \tag{3.12}$$

Using (2.3) and (3.10) once again, (3.12) implies $M_1 > 0$, $M_2 > 0$, and

$$(1 - A_{11})(1 - A_{22}) \leq A_{12} A_{21}\,,$$

which contradicts (2.4).

The contradiction obtained proves that the problem (3.5), (3.2) has only the trivial solution. □

Proof of Theorem 2.2. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.13$_1$}$$
$$u_2'(t) = -\ell_{21}(u_1)(t) + \ell_{22}(u_2)(t) \tag{3.13$_2$}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.13$_1$), (3.13$_2$), (3.2). It is clear that u_1 and u_2 satisfy one of the following items.

(a) One of the functions u_1 and u_2 is of a constant sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \geq 0$ for $t \in [a, b]$.

(b) Both functions u_1 and u_2 change their signs.

Case (a): $u_1(t) \geq 0$ for $t \in [a, b]$. In view of (2.3) and the assumption $\ell_{21} \in \mathcal{P}_{ab}$, Lemma 3.3 yields $u_2(t) \leq 0$ for $t \in [a, b]$. Now, by virtue of (2.3) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 again implies $u_1(t) \leq 0$ for $t \in [a, b]$. Consequently, $u_1 \equiv 0$ and Lemma 3.3 once again results in $u_2 \equiv 0$, a contradiction.

Case (b): u_1 and u_2 change their signs. Put

$$M_i = \max\big\{u_i(t) : t \in [a, b]\big\}, \quad m_i = -\min\big\{u_i(t) : t \in [a, b]\big\} \quad (i = 1, 2) \tag{3.14}$$

and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities

$$u_i(\alpha_i) = M_i\,, \qquad u_i(\beta_i) = -m_i \tag{3.15$_i$}$$

are satisfied for $i = 1, 2$. Obviously,

$$M_i > 0, \quad m_i > 0 \quad \text{for} \quad i = 1, 2. \tag{3.16}$$

Furthermore, we denote

$$B_{ij} = \int\limits_a^{\min\{\alpha_i,\beta_i\}} \ell_{ij}(1)(s)ds, \qquad D_{ij} = \int\limits_{\min\{\alpha_i,\beta_i\}}^{\max\{\alpha_i,\beta_i\}} \ell_{ij}(1)(s)ds \qquad (i, j = 1, 2). \tag{3.17}$$

It is clear that

$$B_{ij} + D_{ij} \le A_{ij} \quad \text{for} \quad i,j = 1,2. \tag{3.18}$$

According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$. The integrations of (3.13_1) from a to α_1 and from α_1 to β_1, in view of (3.14), (3.15_1), (3.17), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, result in

$$M_1 = \int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \le$$

$$\le M_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds = M_1 B_{11} + M_2 B_{12}$$

and

$$M_1 + m_1 = -\int_{\alpha_1}^{\beta_1} \ell_{11}(u_1)(s)ds - \int_{\alpha_1}^{\beta_1} \ell_{12}(u_2)(s)ds \le$$

$$\le m_1 \int_{\alpha_1}^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_{\alpha_1}^{\beta_1} \ell_{12}(1)(s)ds = m_1 D_{11} + m_2 D_{12}\,.$$

The last relations, by virtue of (2.3) and (3.16), imply

$$0 < \frac{M_1}{M_2}(1 - B_{11}) + \frac{m_1}{m_2}(1 - D_{11}) + \frac{M_1}{m_2} \le B_{12} + D_{12} \le A_{12}\,. \tag{3.19}$$

On the other hand, the integrations of (3.13_2) from a to α_2 and from α_2 to β_2, on account of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, arrive at

$$M_2 = -\int_a^{\alpha_2} \ell_{21}(u_1)(s)ds + \int_a^{\alpha_2} \ell_{22}(u_2)(s)ds \le$$

$$\le m_1 \int_a^{\alpha_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\alpha_2} \ell_{22}(1)(s)ds = m_1 B_{21} + M_2 B_{22}$$

and

$$M_2 + m_2 = \int_{\alpha_2}^{\beta_2} \ell_{21}(u_1)(s)ds - \int_{\alpha_2}^{\beta_2} \ell_{22}(u_2)(s)ds \le$$

$$\le M_1 \int_{\alpha_2}^{\beta_2} \ell_{21}(1)(s)ds + m_2 \int_{\alpha_2}^{\beta_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,.$$

The last relations, by virtue of (2.3) and (3.16), yield

$$0 < \frac{M_2}{m_1}(1 - B_{22}) + \frac{m_2}{M_1}(1 - D_{22}) + \frac{M_2}{M_1} \le B_{21} + D_{21} \le A_{21}\,. \tag{3.20}$$

Now, it follows from (3.19) and (3.20) that

$$\begin{aligned} A_{12}A_{21} \ge{}& \frac{M_1}{m_1}(1 - B_{11})(1 - B_{22}) + \frac{m_2}{M_2}(1 - B_{11})(1 - D_{22}) + 1 - B_{11} + \\ &+ \frac{M_2}{m_2}(1 - D_{11})(1 - B_{22}) + \frac{m_1}{M_1}(1 - D_{11})(1 - D_{22}) + \frac{m_1 M_2}{m_2 M_1}(1 - D_{11}) + \\ &+ \frac{M_2 M_1}{m_1 m_2}(1 - B_{22}) + 1 - D_{22} + \frac{M_2}{m_2}\,. \end{aligned} \tag{3.21}$$

Using the relation

$$x + y \ge 2\sqrt{xy} \quad \text{for} \quad x \ge 0,\ y \ge 0, \tag{3.22}$$

it is easy to verify that

$$\begin{aligned} &\frac{M_1}{m_1}(1 - B_{11})(1 - B_{22}) + \frac{m_1}{M_1}(1 - D_{11})(1 - D_{22}) \ge \\ &\ge 2\sqrt{(1 - B_{11})(1 - B_{22})(1 - D_{11})(1 - D_{22})} \ge \\ &\ge 2\sqrt{(1 - B_{11} - D_{11})(1 - B_{22} - D_{22})} \ge 2\sqrt{(1 - A_{11})(1 - A_{22})}\,, \end{aligned} \tag{3.23}$$

$$\frac{m_1 M_2}{m_2 M_1}(1 - D_{11}) + \frac{M_2 M_1}{m_1 m_2}(1 - B_{22}) \ge 2\,\frac{M_2}{m_2}\sqrt{(1 - D_{11})(1 - B_{22})}\,, \tag{3.24}$$

$$\begin{aligned} &\frac{M_2}{m_2}(1 - D_{11})(1 - B_{22}) + 2\,\frac{M_2}{m_2}\sqrt{(1 - D_{11})(1 - B_{22})} + \frac{M_2}{m_2} = \\ &= \frac{M_2}{m_2}\left(\sqrt{(1 - D_{11})(1 - B_{22})} + 1\right)^2, \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} &\frac{m_2}{M_2}(1 - B_{11})(1 - D_{22}) + \frac{M_2}{m_2}\left(\sqrt{(1 - D_{11})(1 - B_{22})} + 1\right)^2 \ge \\ &\ge 2\sqrt{(1 - B_{11})(1 - D_{22})}\left(\sqrt{(1 - D_{11})(1 - B_{22})} + 1\right) \ge \\ &\ge 2\sqrt{(1 - B_{11} - D_{11})(1 - B_{22} - D_{22})} + 2\sqrt{(1 - B_{11})(1 - D_{22})} \ge \\ &\ge 2\sqrt{(1 - A_{11})(1 - A_{22})} + 2\sqrt{(1 - B_{11})(1 - D_{22})}\,. \end{aligned} \tag{3.26}$$

Therefore, by virtue of (3.23)–(3.26), (3.21) implies

$$\begin{aligned} &A_{12}A_{21} \ge \\ &\ge 4\sqrt{(1 - A_{11})(1 - A_{22})} + 1 - B_{11} + 2\sqrt{(1 - B_{11})(1 - D_{22})} + 1 - D_{22} \ge \end{aligned}$$

$$\geq 4\sqrt{(1-A_{11})(1-A_{22})} + \left(\sqrt{1-A_{11}} + \sqrt{1-A_{22}}\right)^2,$$

which contradicts (2.6).

The contradictions obtained in (a) and (b) prove that the problem (3.13_1), (3.13_2), (3.2) has only the trivial solution. ☐

Proof of Theorem 2.3. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.27_1}$$

$$u_2'(t) = \ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.27_2}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.27_1), (3.27_2), (3.2). Define the numbers M_i, m_i $(i = 1, 2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.15_i) are satisfied for $i = 1, 2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.17). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

The integrations of (3.27_1) from a to α_1 and from a to β_1, in view of (3.14), (3.15_1), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield

$$\begin{aligned} M_1 = \int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \leq \\ \leq M_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds \leq M_1 A_{11} + M_2 A_{12} \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} m_1 = -\int_a^{\beta_1} \ell_{11}(u_1)(s)ds - \int_a^{\beta_1} \ell_{12}(u_2)(s)ds \leq \\ \leq m_1 \int_a^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_a^{\beta_1} \ell_{12}(1)(s)ds \leq m_1 A_{11} + m_2 A_{12}\,. \end{aligned} \tag{3.29}$$

Now we shall divide the discussion into the following two cases.

(a) The function u_2 is of a constant sign. Then, without loss of generality we can assume that $u_2(t) \geq 0$ for $t \in [a, b]$.

(b) The function u_2 changes its sign.

Case (a): $u_2(t) \geq 0$ for $t \in [a,b]$. In view of (2.7) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \geq 0$ for $t \in [a,b]$. Consequently, (3.10) is true. The integration of (3.27_2) from a to α_2, on account of (3.14), (3.15_2), and the assumption $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yields

$$M_2 = \int_a^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_a^{\alpha_2} \ell_{22}(u_2)(s)ds \leq M_1 \int_a^{\alpha_2} \ell_{21}(1)(s)ds \leq M_1 A_{21}\,. \tag{3.30}$$

According to (2.7) and (3.10), it follows from (3.28) and (3.30) that

$$0 \leq M_1(1 - A_{11}) \leq M_2 A_{12}\,, \qquad 0 \leq M_2 \leq M_1 A_{21}. \tag{3.31}$$

Using (2.7) and (3.10) once again, the last relations imply $M_1 > 0$, $M_2 > 0$, and

$$A_{12}A_{21} \geq 1 - A_{11} \geq (1 - A_{11})\varphi(A_{22}),$$

which contradicts (2.8).

Case (b): u_2 changes its sign. It is clear that

$$M_2 > 0, \qquad m_2 > 0. \tag{3.32}$$

We can assume without loss of generality that $\beta_2 < \alpha_2$. The integrations of (3.27_2) from a to β_2 and from β_2 to α_2, in view of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} m_2 = -\int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \leq \\ \leq m_1 \int_a^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = m_1 B_{21} + M_2 B_{22} \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} M_2 + m_2 = \int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq \\ \leq M_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,. \end{aligned} \tag{3.34}$$

On the other hand, using (2.7) and (3.32), from (3.28) and (3.29) we get

$$\frac{M_1}{M_2} \leq \frac{A_{12}}{1 - A_{11}}\,, \qquad \frac{m_1}{m_2} \leq \frac{A_{12}}{1 - A_{11}}\,. \tag{3.35}$$

If we take the assumption (2.8) into account, (3.35) yields

$$\frac{m_1}{m_2} B_{21} \leq \frac{A_{12}A_{21}}{1 - A_{11}} < 1, \qquad \frac{M_1}{M_2} D_{21} \leq \frac{A_{12}A_{21}}{1 - A_{11}} < 1.$$

Consequently, it follows from (3.33) and (3.34) that

$$0 < 1 - \frac{m_1}{m_2}\, B_{21} \le \frac{M_2}{m_2}\, B_{22}\,, \qquad 0 < 1 - \frac{M_1}{M_2}\, D_{21} \le \frac{m_2}{M_2}\,(D_{22}-1),$$

whence we get $D_{22} > 1$ and

$$\left(1 - \frac{m_1}{m_2}\, B_{21}\right)\left(1 - \frac{M_1}{M_2}\, D_{21}\right) \le B_{22}\big(D_{22}-1\big).$$

Therefore,

$$1 - \frac{m_1}{m_2}\, B_{21} - \frac{M_1}{M_2}\, D_{21} \le \frac{1}{4}\big(B_{22}+D_{22}-1\big)^2 \le \frac{1}{4}\big(A_{22}-1\big)^2,$$

which, together with (3.35), results in

$$\begin{aligned} \varphi(A_{22}) = 1 - \frac{1}{4}\big(A_{22}-1\big)^2 \le \frac{m_1}{m_2}\, B_{21} + \frac{M_1}{M_2}\, D_{21} \le \\ \le \frac{A_{12}}{1-A_{11}}\,(B_{21}+D_{21}) \le \frac{A_{12}A_{21}}{1-A_{11}}\,. \end{aligned}$$

But this contradicts (2.8).

The contradictions obtained in (a) and (b) prove that the problem (3.27_1), (3.27_2), (3.2) has only the trivial solution. □

Proof of Theorem 2.4. The validity of the theorem follows immediately from Theorem 2.3. □

Proof of Theorem 2.5. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = \ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.36_1}$$

$$u_2'(t) = -\ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.36_2}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1,u_2)^T$ is a nontrivial solution of the problem (3.36_1), (3.36_2), (3.2). It is clear that one of the following items is satisfied.

(a) The function u_2 is of a constant sign. Then, without loss of generality, we can assume that $u_2(t) \ge 0$ for $t \in [a,b]$.

(b) The function u_2 changes its sign.

Case (a): $u_2(t) \ge 0$ for $t \in [a,b]$. In view of (2.7) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \ge 0$ for $t \in [a,b]$. Therefore, by virtue of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.36_2) yields $u_2'(t) \le 0$ for $t \in [a,b]$. Consequently, $u_2 \equiv 0$ and Lemma 3.3 once again results in $u_1 \equiv 0$, which is a contradiction.

Case (b): u_2 changes its sign. Define the numbers M_i, m_i $(i = 1,2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a,b]$ $(i=1,2)$ such that the equalities (3.15_i) are satisfied for $i=1,2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i,j=1,2)$ be given by (3.17). It is clear that

$$M_1 \geq 0, \quad m_1 \geq 0, \quad M_2 > 0, \quad m_2 > 0.$$

We can assume without loss of generality that $\beta_2 < \alpha_2$. The integrations of (3.36_2) from a to β_2 and from β_2 to α_2, in view of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yield

$$\begin{aligned} m_2 = \int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \leq \\ \leq M_1 \int_a^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = M_1 B_{21} + M_2 B_{22} \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} M_2 + m_2 = -\int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq \\ \leq m_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = m_1 D_{21} + m_2 D_{22}\,. \end{aligned} \tag{3.38}$$

By virtue of (3.18) and (3.32), it follows from (3.37) and (3.38) that

$$3 - A_{22} \leq 1 + \frac{m_2}{M_2} + \frac{M_2}{m_2} - B_{22} - D_{22} \leq \frac{M_1}{M_2} B_{21} + \frac{m_1}{m_2} D_{21}\,. \tag{3.39}$$

On the other hand, the integrations of (3.36_1) from a to α_1 and from a to β_1, on account of (3.14), (3.15_1), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield (3.28) and (3.29), respectively. Using (2.7) and (3.32), from (3.28) and (3.29) we get (3.35). Consequently, (3.39) implies

$$3 - A_{22} \leq \frac{A_{12}}{1 - A_{11}} (B_{21} + D_{21}) \leq \frac{A_{12} A_{21}}{1 - A_{11}}\,,$$

which contradicts (2.10).

The contradictions obtained in (a) and (b) prove that the problem (3.36_1), (3.36_2), (3.2) has only the trivial solution. □

Proof of Theorem 2.6. If $A_{12}A_{21} < (1-A_{11})(1-A_{22})$ then the validity of the theorem follows immediately from Theorem 2.5. Therefore, suppose that

$$A_{12}A_{21} \geq (1-A_{11})(1-A_{22}). \tag{3.40}$$

According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the problem (3.36_1), (3.36_2), (3.2) has only the trivial solution.

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.36_1), (3.36_2), (3.2). Define the numbers M_i, m_i $(i = 1,2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a,b]$ $(i = 1,2)$ such that the equalities (3.15_i) are satisfied for $i = 1,2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i,j = 1,2)$ be given by (3.17). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1,2.$$

For the sake of clarity we shall devide the discussion into the following cases.

(a) The function u_2 is of a constant sign. Then, without loss of generality, we can assume that $u_2(t) \geq 0$ for $t \in [a,b]$.

(b) The function u_2 changes its sign. Then, without loss of generality, we can assume that $\beta_2 < \alpha_2$. It is clear that one of the following items is satisfied.

  (b1) $u_1(t) \geq 0$ for $t \in [a,b]$.

  (b2) $u_1(t) \leq 0$ for $t \in [a,b]$.

  (b3) The function u_1 changes its sign.

Case (a): $u_2(t) \geq 0$ for $t \in [a,b]$. In view of (2.3) and the assumption $\ell_{12} \in \mathcal{P}_{ab}$, Lemma 3.3 implies $u_1(t) \geq 0$ for $t \in [a,b]$. Therefore, by virtue of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.36_2) yields $u_2'(t) \leq 0$ for $t \in [a,b]$. Consequently, $u_2 \equiv 0$ and Lemma 3.3 once again results in $u_1 \equiv 0$, which is a contradiction.

Case (b): u_2 changes its sign and $\beta_2 < \alpha_2$. Obviously, (3.32) is true. The integrations of (3.36_2) from a to β_2 and from β_2 to α_2, in view of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yield (3.37) and (3.38), respectively. At first we note that, by virtue of (2.3), the assumption (2.11) implies

$$A_{22}\Big[A_{12}A_{21} - (1 - A_{11})(1 - A_{22})\Big] < 1 - A_{11}\,. \tag{3.41}$$

Now we are in position to discuss the cases (b1)–(b3).

Case (b1): $u_1(t) \geq 0$ for $t \in [a,b]$. This means that $m_1 = 0$. Consequently, (3.38) implies

$$M_2 \leq m_2(D_{22} - 1) \leq m_2(A_{22} - 1),$$

which, together with (2.3), contradicts (3.32).

Case (b2): $u_1(t) \leq 0$ for $t \in [a,b]$. This means that $M_1 = 0$. Consequently, (3.37) and (3.38) yield

$$M_2 \leq m_1 A_{21} - m_2(1 - A_{22}), \qquad m_2 \leq M_2 A_{22}\,. \tag{3.42}$$

On the other hand, the integration of (3.36_1) from a to β_1, in view of (3.14), (3.15_1), and the assumption $\ell_{11}, \ell_{21} \in \mathcal{P}_{ab}$, results in (3.29). If we take now (2.3) into account, it follows from (3.29) and (3.42) that

$$m_2(1 - A_{11}) \leq M_2 A_{22}(1 - A_{11}) \leq$$

$$\leq m_1 A_{21} A_{22}(1 - A_{11}) - m_2 A_{22}(1 - A_{11})(1 - A_{22}) \leq$$
$$\leq m_2 A_{12} A_{21} A_{22} - m_2 A_{22}(1 - A_{11})(1 - A_{22}).$$

Since $m_2 > 0$, we get from the last relations that

$$1 - A_{11} \leq A_{22}\Big[A_{12}A_{21} - (1 - A_{11})(1 - A_{22})\Big],$$

which contradicts (3.41).

Case (b3): u_1 changes its sign. Suppose that $\alpha_1 < \beta_1$ (the case, where $\alpha_1 > \beta_1$, can be proved analogously). Obviously, (3.16) is true. The integrations of (3.36_1) from a to α_1 and from α_1 to β_1, on account of (3.14), (3.15_1), (3.17), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield

$$M_1 = \int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \leq$$
$$\leq M_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds = M_1 B_{11} + M_2 B_{12} \quad (3.43)$$

and

$$M_1 + m_1 = -\int_{\alpha_1}^{\beta_1} \ell_{11}(u_1)(s)ds - \int_{\alpha_1}^{\beta_1} \ell_{12}(u_2)(s)ds \leq$$
$$\leq m_1 \int_{\alpha_1}^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_{\alpha_1}^{\beta_1} \ell_{12}(1)(s)ds = m_1 D_{11} + m_2 D_{12}\,, \quad (3.44)$$

respectively. By virtue of (2.3), (3.16), and (3.18), combining the inequalities (3.37), (3.38) and (3.43), (3.44), we get

$$0 < \frac{m_2}{M_1} + \frac{M_2}{m_1} + \frac{m_2}{m_1}(1 - D_{22}) \leq A_{21} + \frac{M_2}{M_1} B_{22} \quad (3.45)$$

and

$$0 < \frac{M_1}{M_2}(1 - B_{11}) + \frac{m_1}{m_2}(1 - D_{11}) + \frac{M_1}{m_2} \leq A_{12}\,, \quad (3.46)$$

respectively.

On the other hand, in view of (2.3), the relations (3.38) and (3.44) imply

$$M_2(1 - A_{11}) \leq m_2\Big[A_{12}A_{21} - (1 - A_{11})(1 - A_{22})\Big].$$

Using (3.37) and (3.40) in the last inequality, we get

$$M_2\Big(1{-}A_{11}{-}A_{22}\big[A_{12}A_{21}{-}(1{-}A_{11})(1{-}A_{22})\big]\Big) \leq M_1 A_{21}\Big[A_{12}A_{21}{-}(1{-}A_{11})(1{-}A_{22})\Big].$$

Consequently,

$$A_{21} + \frac{M_2}{M_1}\, B_{22} \leq \frac{(1-A_{11})A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]}\,, \tag{3.47}$$

because the inequality (3.41) is true.

Now, it follows from (3.45)–(3.47) that

$$\begin{aligned} \frac{(1-A_{11})A_{12}A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]} &\geq \frac{m_2}{M_2}\,(1-B_{11}) + \\ + \frac{m_1}{M_1}\,(1-D_{11}) + 1 + \frac{M_1}{m_1}\,(1-B_{11}) &+ \frac{M_2}{m_2}\,(1-D_{11}) + \frac{M_1M_2}{m_1m_2} + \\ + \frac{M_1m_2}{M_2m_1}\,(1-B_{11})(1-D_{22}) &+ (1-D_{11})(1-D_{22}) + \frac{M_1}{m_1}\,(1-D_{22}). \end{aligned} \tag{3.48}$$

Using the realition (3.22), we get

$$\frac{M_1M_2}{m_1m_2} + \frac{M_1m_2}{M_2m_1}\,(1-B_{11})(1-D_{22}) \geq 2\,\frac{M_1}{m_1}\sqrt{(1-B_{11})(1-D_{22})}\,, \tag{3.49}$$

$$\begin{aligned} \frac{M_1}{m_1}\,(1-B_{11}) + 2\,\frac{M_1}{m_1}\sqrt{(1-B_{11})(1-D_{22})} + \frac{M_1}{m_1}\,(1-D_{22}) &= \\ = \frac{M_1}{m_1}\left(\sqrt{1-B_{11}} + \sqrt{1-D_{22}}\right)^2&, \end{aligned} \tag{3.50}$$

$$\begin{aligned} \frac{M_1}{m_1}\left(\sqrt{1-B_{11}} + \sqrt{1-D_{22}}\right)^2 + \frac{m_1}{M_1}\,(1-D_{11}) &\geq \\ \geq 2\sqrt{1-D_{11}}\Big(\sqrt{1-B_{11}} + \sqrt{1-D_{22}}\Big) &\geq \\ \geq 2\sqrt{1-B_{11}-D_{11}} + 2\sqrt{(1-D_{11})(1-D_{22})} &\geq \\ \geq 2\sqrt{1-A_{11}} + 2\sqrt{(1-D_{11})(1-D_{22})}&, \end{aligned} \tag{3.51}$$

and

$$\frac{m_2}{M_2}\,(1-B_{11}) + \frac{M_2}{m_2}\,(1-D_{11}) \geq 2\sqrt{(1-B_{11})(1-D_{11})} \geq 2\sqrt{1-A_{11}}\,. \tag{3.52}$$

Finaly, in view (3.49)–(3.52), (3.48) implies

$$\begin{aligned} \frac{(1-A_{11})A_{12}A_{21}}{1-A_{11}-A_{22}\Big[A_{12}A_{21}-(1-A_{11})(1-A_{22})\Big]} &\geq \\ \geq 4\sqrt{1-A_{11}} + 1 + 2\sqrt{(1-D_{11})(1-D_{22})} + (1-D_{11})(1-D_{22}) &\geq \\ \geq 4\sqrt{1-A_{11}} + \left(1 + \sqrt{(1-A_{11})(1-A_{22})}\right)^2 &= \omega, \end{aligned}$$

which contradicts (2.11).

The contradictions obtained in (a) and (b) prove that the problem (3.36_1), (3.36_2), (3.2) has only the trivial solution. □

Proof of Theorem 2.7. The validity of the theorem follows immediately from Theorem 2.5. □

Proof of Theorem 2.8. The validity of the theorem follows immediately from Theorem 2.6. □

Proof of Theorem 2.9. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = -\ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.53_1}$$
$$u_2'(t) = \ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.53_2}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.53_1), (3.53_2), (3.2). Define the numbers M_i, m_i $(i = 1, 2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.15_i) are satisfied for $i = 1, 2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.17). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

For the sake of clarity we shall devide the discussion into the following cases.

(a) Both functions u_1 and u_2 do not change their signs and $u_1(t)u_2(t) \geq 0$ for $t \in [a, b]$. Then, without loss of generality, we can assume that

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

(b) Both functions u_1 and u_2 do not change their signs and $u_1(t)u_2(t) \leq 0$ for $t \in [a, b]$. Then, without loss of generality, we can assume that

$$u_1(t) \geq 0, \quad u_2(t) \leq 0 \quad \text{for} \quad t \in [a, b].$$

(c) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. Then, without loss of generality, we can assume that $u_1(t) \geq 0$ for $t \in [a, b]$.

(d) Both functions u_1 and u_2 change their signs. Then, without loss of generality, we can assume that $\alpha_1 < \beta_1$. Obviously, one of the following items is satisfied.

(d1) $\beta_2 < \alpha_2$ and $D_{ii} \geq 1$ for some $i \in \{1, 2\}$.

(d2) $\beta_2 < \alpha_2$ and $D_{ii} < 1$ for $i = 1, 2$.

(d3) $\beta_2 > \alpha_2$ and $D_{ii} \geq 1$ for some $i \in \{1, 2\}$.

(d4) $\beta_2 > \alpha_2$ and $D_{ii} < 1$ for $i = 1, 2$.

At first we note that the function φ satisfies

$$\varphi(A_{ii}) \leq 1 - B_{ii}(D_{ii} - 1) \quad \text{for} \quad i = 1, 2. \tag{3.54}$$

Case (a): $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for $t \in [a, b]$. Obviuously, (3.10) is true. The integration of (3.53_i) from a to α_i, in view of (3.14), (3.15_i), and the assumptions $\ell_{i1}, \ell_{i2} \in \mathcal{P}_{ab}$, yields

$$M_i = (-1)^i \int_a^{\alpha_i} \ell_{i1}(u_1)(s)ds + (-1)^{i-1} \int_a^{\alpha_i} \ell_{i2}(u_2)(s)ds \leq$$
$$\leq M_{3-i} \int_a^{\alpha_i} \ell_{i\,3-i}(1)(s)ds \leq M_{3-i}A_{i\,3-i} \quad (i = 1, 2). \tag{3.55}$$

By virtue of (3.10), (3.55) implies $M_1 > 0$, $M_2 > 0$, and $A_{12}A_{21} \geq 1$, which contradicts (2.14), because $\omega \geq 1$ and $0 < \varphi(A_{ii}) \leq 1$ for $i = 1, 2$.

Case (b): $u_1(t) \geq 0$ and $u_2(t) \leq 0$ for $t \in [a, b]$. In view of the assumptions $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$, (3.53_1) and (3.53_2) arrive at $u_1'(t) \leq 0$ for $t \in [a, b]$ and $u_2'(t) \geq 0$ for $t \in [a, b]$, respectively. Consequently, $u_1 \equiv 0$ and $u_2 \equiv 0$, a contradiction.

Case (c): $u_1(t) \geq 0$ for $t \in [a, b]$ and u_2 changes its sign. Obviously, $m_1 = 0$ and (3.32) is true. Suppose that $\beta_2 < \alpha_2$ (the case, where $\beta_2 > \alpha_2$, can be proved analogously). The integration of (3.53_1) from a to α_1, on account of (3.14), (3.15_1), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yields

$$M_1 = -\int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \leq M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds \leq M_2 A_{12}\,. \tag{3.56}$$

On the other hand, the integrations of (3.53_2) from a to β_2 and from β_2 to α_2, in view of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in

$$m_2 = -\int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \leq M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = M_2 B_{22} \tag{3.57}$$

and

$$M_2 + m_2 = \int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq$$
$$\leq M_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,, \tag{3.58}$$

respectively.

It follows from (3.56) and (3.58) that

$$M_2 \leq M_2 A_{12} A_{21} + m_2(D_{22} - 1). \tag{3.59}$$

Hence, by virtue of (2.14) and (3.32), (3.59) implies

$$0 < M_2(1 - A_{12}A_{21}) \leq m_2(D_{22} - 1). \tag{3.60}$$

Using (3.54), the relations (3.57) and (3.60) result in

$$\varphi(A_{22}) \leq 1 - B_{22}(D_{22} - 1) \leq A_{12}A_{21}\,,$$

which contradicts (2.14), because $\omega \geq 1$ and $0 < \varphi(A_{11}) \leq 1$.

Case (d): u_1 and u_2 change their signs and $\alpha_1 < \beta_1$. Obviously, (3.16) is true. The integrations of (3.53_1) from a to α_1 and from α_1 to β_1, in view of (3.14), (3.15_1), (3.17), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yield

$$\begin{aligned} M_1 = -\int_a^{\alpha_1} \ell_{11}(u_1)(s)ds + \int_a^{\alpha_1} \ell_{12}(u_2)(s)ds \leq \\ \leq m_1 \int_a^{\alpha_1} \ell_{11}(1)(s)ds + M_2 \int_a^{\alpha_1} \ell_{12}(1)(s)ds = m_1 B_{11} + M_2 B_{12} \end{aligned} \tag{3.61}$$

and

$$\begin{aligned} M_1 + m_1 = \int_{\alpha_1}^{\beta_1} \ell_{11}(u_1)(s)ds - \int_{\alpha_1}^{\beta_1} \ell_{12}(u_2)(s)ds \leq \\ \leq M_1 \int_{\alpha_1}^{\beta_1} \ell_{11}(1)(s)ds + m_2 \int_{\alpha_1}^{\beta_1} \ell_{12}(1)(s)ds = M_1 D_{11} + m_2 D_{12}\,. \end{aligned} \tag{3.62}$$

Furthermore, under the assumption $\beta_2 < \alpha_2$, the integrations of (3.53_2) from a to β_2 and from β_2 to α_2, in view of (3.14), (3.15_2), (3.17), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} m_2 = -\int_a^{\beta_2} \ell_{21}(u_1)(s)ds + \int_a^{\beta_2} \ell_{22}(u_2)(s)ds \leq \\ \leq m_1 \int_a^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_a^{\beta_2} \ell_{22}(1)(s)ds = m_1 B_{21} + M_2 B_{22} \end{aligned} \tag{3.63_1}$$

and

$$\begin{aligned} M_2 + m_2 = \int_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq \\ \leq M_1 \int_{\beta_2}^{\alpha_2} \ell_{21}(1)(s)ds + m_2 \int_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds = M_1 D_{21} + m_2 D_{22}\,. \end{aligned} \tag{3.64_1}$$

If $\beta_2 > \alpha_2$, we obtain in a similar manner the inequalities

$$M_2 \leq M_1 B_{21} + m_2 B_{22}\,, \tag{3.63$_2$}$$

$$M_2 + m_2 \leq m_1 D_{21} + M_2 D_{22}\,. \tag{3.64$_2$}$$

Now we are in position to discuss the cases (d1)–(d4).

Case (d1): $\beta_2 < \alpha_2$ and $D_{ii} \geq 1$ for some $i \in \{1,2\}$. Suppose that $D_{22} \geq 1$ (the case, where $D_{11} \geq 1$, can be proved analogously). Using this assumption, from (3.63_1) and (3.64_1), we get

$$m_2 \leq m_1 B_{21} + M_1 B_{22} D_{21} + m_2 B_{22}(D_{22} - 1)$$

and

$$M_2 \leq M_1 D_{21} + m_1 B_{21}(D_{22} - 1) + M_2 B_{22}(D_{22} - 1).$$

Hence, in view of (3.54), the last two inequalities yield

$$m_2 \varphi(A_{22}) \leq m_1 B_{21} + M_1 B_{22} D_{21}\,, \tag{3.65}$$

$$M_2 \varphi(A_{22}) \leq M_1 D_{21} + m_1 B_{21}(D_{22} - 1). \tag{3.66}$$

By virtue of (2.14) and (3.16), it follows from (3.61), (3.66) and (3.62), (3.65) that

$$0 < M_1 \Big[\varphi(A_{22}) - B_{12} D_{21}\Big] \leq m_1 \Big[\varphi(A_{22}) B_{11} + B_{12} B_{21}(D_{22} - 1)\Big] \tag{3.67}$$

and

$$0 < m_1 \Big[\varphi(A_{22}) - D_{12} B_{21}\Big] \leq M_1 \Big[\varphi(A_{22})(D_{11} - 1) + D_{12} D_{21} B_{22}\Big], \tag{3.68}$$

respectively. Combining (3.67) and (3.68), we get

$$\begin{aligned}\varphi^2(A_{22}) \leq \varphi(A_{22})\Big[B_{12} D_{21} + D_{12} B_{21}\Big] - B_{12} D_{12} B_{21} D_{21}\Big(1 - B_{22}(D_{22} - 1)\Big) + \\ + \varphi(A_{22})\Big[B_{12} B_{21}(D_{11} - 1)(D_{22} - 1) + D_{12} D_{21} B_{11} B_{22}\Big] + \\ + \varphi^2(A_{22}) B_{11}(D_{11} - 1).\end{aligned} \tag{3.69}$$

Since $1 - B_{ii}(D_{ii} - 1) \geq \varphi(A_{ii}) > 0$ for $i = 1, 2$ and

$$B_{12} D_{21} + D_{12} B_{21} \leq A_{12} A_{21} - B_{12} B_{21} - D_{12} D_{21}\,, \tag{3.70}$$

we obtain from (3.69) that

$$\begin{aligned}&\varphi(A_{11})\varphi(A_{22}) \leq \\ &\quad \leq A_{12} A_{21} + B_{12} B_{21}\Big[(D_{11} - 1)(D_{22} - 1) - 1\Big] + D_{12} D_{21}\Big[B_{11} B_{22} - 1\Big].\end{aligned} \tag{3.71}$$

If $(D_{11} - 1)(D_{22} - 1) \leq 1$ and $B_{11} B_{22} \leq 1$ then (3.71) implies

$$\varphi(A_{11})\varphi(A_{22}) \leq A_{12} A_{21}\,,$$

which contradicts (2.14).

If $(D_{11}-1)(D_{22}-1) \le 1$ and $B_{11}B_{22} > 1$ then, in view of (3.18) and the assumption $D_{22} \ge 1$, we obtain from (3.71) that

$$\varphi(A_{11})\varphi(A_{22}) \le A_{12}A_{21}B_{11}B_{22} \le A_{12}A_{21}B_{11}(A_{22}-D_{22}) \le A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.14).

If $(D_{11}-1)(D_{22}-1) > 1$ and $B_{11}B_{22} \le 1$ then (3.71) arrives at

$$\varphi(A_{11})\varphi(A_{22}) \le A_{12}A_{21}(D_{11}-1)(D_{22}-1) \le A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.14).

If $(D_{11}-1)(D_{22}-1) > 1$ and $B_{11}B_{22} > 1$ then (3.71) yields

$$\varphi(A_{11})\varphi(A_{22}) \le A_{12}A_{21}\Big[(D_{11}-1)(D_{22}-1) + B_{11}B_{22} - 1\Big] \le$$
$$\le A_{12}A_{21}\Big[A_{11}(D_{22}-1) + A_{11}B_{22}\Big] \le A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.14).

Case (d2): $\beta_2 < \alpha_2$ and $D_{ii} < 1$ for $i = 1, 2$. We first note that

$$B_{11}B_{22} \le (A_{ii}-D_{ii})B_{3-i\,3-i} = (A_{ii}-1)B_{3-i\,3-i} + (1-D_{ii})B_{3-i\,3-i} \qquad (3.72_i)$$

for $i = 1, 2$. By virtue of (3.16), we get from the inequalities (3.62) and (3.64_1)

$$m_1 \le m_2 D_{12} \qquad (3.73)$$

and

$$M_2 \le M_1 D_{21}\,. \qquad (3.74)$$

Therefore, in view of (2.14) and (3.16), the relations (3.62), (3.63_1), (3.74) and (3.61), (3.74) result in

$$0 < m_1\Big(1 - D_{12}B_{21}\Big) \le M_1\Big[D_{12}D_{21}B_{22} - (1-D_{11})\Big] \qquad (3.75)$$

and

$$0 < M_1\Big(1 - B_{12}D_{21}\Big) \le m_1 B_{11}\,, \qquad (3.76)$$

respectively. Combining (3.72_1), (3.75), (3.76) and taking the inequality $D_{12}D_{21} \le 1$ into account, we get

$$\Big(1 - B_{12}D_{21}\Big)\Big(1 - D_{12}B_{21}\Big) \le D_{12}D_{21}(A_{11}-1)B_{22} + (B_{22}-B_{11})(1-D_{11}). \quad (3.77)$$

On the other hand, by virtue of (2.14) and (3.16), the relations (3.61), (3.64_1), (3.73) and (3.63_1), (3.73) imply

$$0 < M_2\Big(1 - B_{12}D_{21}\Big) \le m_2\Big[D_{12}D_{21}B_{11} - (1-D_{22})\Big] \qquad (3.78)$$

and

$$0 < m_2\Big(1 - D_{12}B_{21}\Big) \leq M_2B_{22}\,, \tag{3.79}$$

respectively. Combining (3.72$_2$), (3.78), (3.79) and taking the inequality $D_{12}D_{21} \leq 1$ into account, we obtain

$$\Big(1 - B_{12}D_{21}\Big)\Big(1 - D_{12}B_{21}\Big) \leq D_{12}D_{21}(A_{22} - 1)B_{11} + (B_{11} - B_{22})(1 - D_{22}). \tag{3.80}$$

First suppose that $B_{22} \leq B_{11}$. Then, by virtue of (3.70), the inequality (3.77) arrives at

$$\begin{aligned} 1 \leq B_{12}D_{21} + D_{12}B_{21} + D_{12}D_{21}(A_{11} - 1)B_{22} \leq \\ \leq A_{12}A_{21} + D_{12}D_{21}\Big[(A_{11} - 1)B_{22} - 1\Big]. \end{aligned} \tag{3.81}$$

If $(A_{11} - 1)B_{22} \leq 1$ then (3.81) implies $1 \leq A_{12}A_{21}$, which contradicts (2.14), because $0 < \varphi(A_{ii}) \leq 1$ for $i = 1, 2$.

If $(A_{11} - 1)B_{22} > 1$ then (3.81) yields

$$1 \leq A_{12}A_{21}(A_{11} - 1)B_{22} \leq A_{12}A_{21}(A_{11} - 1)A_{22}\,,$$

which contradicts (2.14), because $0 < \varphi(A_{ii}) \leq 1$ for $i = 1, 2$.

Now suppose that $B_{22} > B_{11}$. Then, by virtue of (3.70), the inequality (3.80) results in

$$\begin{aligned} 1 \leq B_{12}D_{21} + D_{12}B_{21} + D_{12}D_{21}(A_{22} - 1)B_{11} \leq \\ \leq A_{12}A_{21} + D_{12}D_{21}\Big[(A_{22} - 1)B_{11} - 1\Big]. \end{aligned} \tag{3.82}$$

If $(A_{22} - 1)B_{11} \leq 1$ then (3.82) implies $1 \leq A_{12}A_{21}$, which contradicts (2.14), because $0 < \varphi(A_{ii}) \leq 1$ for $i = 1, 2$.

If $(A_{22} - 1)B_{11} > 1$ then (3.82) yields

$$1 \leq A_{12}A_{21}(A_{22} - 1)B_{11} \leq A_{12}A_{21}(A_{22} - 1)A_{11}\,,$$

which contradicts (2.14), because $0 < \varphi(A_{ii}) \leq 1$ for $i = 1, 2$.

Case (d3): $\beta_2 > \alpha_2$ and $D_{ii} \geq 1$ for some $i \in \{1, 2\}$. Suppose that $D_{22} \geq 1$ (the case, where $D_{11} \geq 1$, can be proved analogously). In a similar manner as in the case (d1), combining (3.61), (3.62) and (3.63$_2$), (3.64$_2$), we get

$$\begin{aligned} &\varphi(A_{11})\varphi(A_{22}) \leq \\ &\quad \leq A_{12}A_{21} + D_{12}B_{21}\Big[B_{11}(D_{22} - 1) - 1\Big] + B_{12}D_{21}\Big[B_{22}(D_{11} - 1) - 1\Big]. \end{aligned} \tag{3.83}$$

If $B_{11}(D_{22} - 1) \leq 1$ and $B_{22}(D_{11} - 1) \leq 1$ then (3.83) implies

$$\varphi(A_{11})\varphi(A_{22}) \leq A_{12}A_{21}\,,$$

which contradicts (2.14).

If $B_{11}(D_{22}-1)\le 1$ and $B_{22}(D_{11}-1)>1$ then we obtain from (3.83) that

$$\varphi(A_{11})\varphi(A_{22})\le A_{12}A_{21}B_{22}(D_{11}-1)\le A_{12}A_{21}A_{22}(A_{11}-1),$$

which contradicts (2.14).

If $B_{11}(D_{22}-1)>1$ and $B_{22}(D_{11}-1)\le 1$ then (3.83) arrives at

$$\varphi(A_{11})\varphi(A_{22})\le A_{12}A_{21}B_{11}(D_{22}-1)\le A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.14).

If $B_{11}(D_{22}-1)>1$ and $B_{22}(D_{11}-1)>1$ then (3.83) yields

$$\varphi(A_{11})\varphi(A_{22})\le A_{12}A_{21}\Big[B_{11}(D_{22}-1)+(D_{11}-1)B_{22}-1\Big]\le$$
$$\le A_{12}A_{21}\Big[A_{11}(D_{22}-1)+A_{11}B_{22}\Big]\le A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.14).

Case (d4): $\beta_2>\alpha_2$ and $D_{ii}<1$ for $i=1,2$. The inequalities (3.62) and (3.64_2) result in

$$m_1\le m_2D_{12}\,,\qquad m_2\le m_1D_{21}\,.$$

Hence, we get

$$1\le D_{12}D_{21}\le A_{12}A_{21}\,,$$

which contradicts (2.14), because $0<\varphi(A_{ii})\le 1$ for $i=1,2$.

The contradictions obtained in (a)–(d) prove that the problem (3.53_1), (3.53_2), (3.2) has only the trivial solution. □

Before we prove Theorem 2.10, we give the following lemma.

Lemma 3.4. *Let the function φ be defined by (2.2). Then, for any $0\le x\le y<3$, the inequality*

$$\big(3-y\big)\varphi(x)\le\big(3-x\big)\varphi(y)\tag{3.84}$$

is satisfied.

Proof. Let $0\le x\le y<3$ be arbitrary but fixed. It is clear that one of the following cases is satisfied:

(a) $0\le x\le y\le 1$ holds. Then

$$\big(3-y\big)\varphi(x)=3-y\le 3-x=\big(3-x\big)\varphi(y).$$

(b) $0\le x\le 1$ and $1<y<3$ are satisfied. Then we get

$$3-y\le 2\left[1-\frac{1}{4}(y-1)^2\right].$$

Consequently,

$$\big(3-y\big)\varphi(x)=3-y\le 2\left[1-\frac{1}{4}(y-1)^2\right]\le\big(3-x\big)\varphi(y).$$

(c) $1 < x \leq y < 3$ is true. Then we obtain

$$(3-y)\Big[4-(x-1)^2\Big] = (3-y)\Big[2+(x-1)\Big]\Big[2-(x-1)\Big] =$$
$$= (3-y)(1+x)(3-x) \leq (3-x)(1+y)(3-y) =$$
$$= (3-x)\Big[2+(y-1)\Big]\Big[2-(y-1)\Big] = (3-x)\Big[4-(y-1)^2\Big],$$

i.e., the inequality (3.84) holds.

□

Proof of Theorem 2.10. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = -\ell_{11}(u_1)(t) + \ell_{12}(u_2)(t), \tag{3.85$_1$}$$
$$u_2'(t) = -\ell_{21}(u_1)(t) - \ell_{22}(u_2)(t) \tag{3.85$_2$}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.85$_1$), (3.85$_2$), (3.2). Define the numbers M_i, m_i $(i = 1, 2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.15$_i$) are satisfied for $i = 1, 2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.17). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

For the sake of clarity we shall devide the discussion into the following cases.

(a) Both functions u_1 and u_2 do not change their signs. According to Lemma 3.2, we can assume without loss of generality that

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \text{for} \quad t \in [a, b].$$

(b) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \geq 0$ for $t \in [a, b]$.

(c) Both functions u_1 and u_2 change their signs. According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$. Obviously, one of the following items is satisfied:

(c1) $D_{ii} \geq 1$ for some $i \in \{1, 2\}$

(c2) $D_{ii} < 1$ for $i = 1, 2$ and

(c2.1) $m_1 D_{21} \leq m_2 B_{22}$

(c2.2) $M_1 \leq M_2 D_{12}$

(c2.3) $m_1 D_{21} > m_2 B_{22}$ and $M_1 > M_2 D_{12}$

At first we note that (3.54) is true and, by virtue of Lemma 3.4, the assumption (2.16) can be rewritten as

$$\omega A_{12}A_{21} < \Big(3 - A_{ii}\Big)\varphi\big(A_{3-i\,3-i}\big) \quad \text{for} \quad i = 1, 2. \tag{3.86}$$

Case (a): $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for $t \in [a, b]$. In view of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.85$_2$) implies $u_2'(t) \leq 0$ for $t \in [a, b]$. Therefore, $u_2 \equiv 0$ and, by virtue of the assumption $\ell_{11} \in \mathcal{P}_{ab}$, (3.85$_1$) arrives at $u_1'(t) \leq 0$ for $t \in [a, b]$. Consequently, $u_1 \equiv 0$ as well, which is a contradiction.

Case (b): $u_1(t) \geq 0$ for $t \in [a, b]$ and u_2 changes its sign. Obviously, (3.32) is true, $M_1 \geq 0$, and $m_1 = 0$. Suppose that $\alpha_2 < \beta_2$ (the case, where $\alpha_2 > \beta_2$, can be proved analogously). The integration of (3.85$_1$) from a to α_1, in view of (3.14), (3.15$_1$), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yields (3.56).

On the other hand, the integrations of (3.85$_2$) from a to α_2 and from α_2 to β_2, in view of (3.14), (3.15$_2$), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in

$$M_2 = -\int_a^{\alpha_2} \ell_{21}(u_1)(s)ds - \int_a^{\alpha_2} \ell_{22}(u_2)(s)ds \leq m_2 \int_a^{\alpha_2} \ell_{22}(1)(s)ds = m_2 B_{22} \tag{3.87}$$

and

$$\begin{aligned} M_2 + m_2 = \int_{\alpha_2}^{\beta_2} \ell_{21}(u_1)(s)ds + \int_{\alpha_2}^{\beta_2} \ell_{22}(u_2)(s)ds \leq \\ \leq M_1 \int_{\alpha_2}^{\beta_2} \ell_{21}(1)(s)ds + M_2 \int_{\alpha_2}^{\beta_2} \ell_{22}(1)(s)ds = M_1 D_{21} + M_2 D_{22}\,, \end{aligned} \tag{3.88}$$

respectively. By virtue of (3.32), combining (3.56), (3.87), and (3.88), we get

$$3 - A_{22} \leq 1 + \frac{M_2}{m_2} + \frac{m_2}{M_2} - B_{22} - D_{22} \leq \frac{M_1}{M_2} D_{21} \leq A_{12}A_{21}\,,$$

which contradicts (3.86), because $\omega \geq 1$ and $0 < \varphi(A_{11}) \leq 1$.

Case (c): u_1 and u_2 change their signs, $\alpha_1 < \beta_1$, and $\beta_2 < \alpha_2$. Obviously, (3.16) is true. The integrations of (3.85$_1$) from a to α_1 and from α_1 to β_1, in view of (3.14), (3.15$_1$), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, imply (3.61) and (3.62). On the other hand, the integrations of (3.85$_2$) from a to β_2 and from β_2 to α_2, on account of (3.14), (3.15$_2$), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in (3.37) and (3.38).

By virtue of (3.16), the relations (3.61), (3.62) and (3.37), (3.38) arrive at

$$3 - B_{11} - D_{11} \leq 1 + \frac{M_1}{m_1} + \frac{m_1}{M_1} - B_{11} - D_{11} \leq \frac{M_2}{m_1} B_{12} + \frac{m_2}{M_1} D_{12} \tag{3.89}$$

and

$$3 - B_{22} - D_{22} \leq 1 + \frac{M_2}{m_2} + \frac{m_2}{M_2} - B_{22} - D_{22} \leq \frac{M_1}{M_2} B_{21} + \frac{m_1}{m_2} D_{21}\,, \tag{3.90}$$

respectively.

Case (c1): $D_{ii} \geq 1$ *for some* $i \in \{1,2\}$. Suppose that $D_{11} \geq 1$ (the case, where $D_{22} \geq 1$, can be proved analogously). Using this assumption and combining (3.61) and (3.62), we get

$$M_1 \leq M_1 B_{11}(D_{11}-1) + m_2 B_{11} D_{12} + M_2 B_{12}$$

and

$$m_1 \leq m_1 B_{11}(D_{11}-1) + M_2(D_{11}-1)B_{12} + m_2 D_{12}\,.$$

Hence, in view of (3.54), the last two inequalities yield

$$M_1\varphi(A_{11}) \leq m_2 B_{11} D_{12} + M_2 B_{12}\,, \tag{3.91}$$

$$m_1\varphi(A_{11}) \leq M_2(D_{11}-1)B_{12} + m_2 D_{12}\,. \tag{3.92}$$

By virtue of the assumption $D_{11} \geq 1$, it follows from (3.37), (3.91) and (3.38), (3.92) that

$$M_1\Big[\varphi(A_{11}) - B_{11}D_{12}B_{21}\Big] \leq M_2\Big[B_{11}B_{22}D_{12} + B_{12}\Big] \tag{3.93}$$

and

$$m_1\Big[\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}\Big] \leq m_2\Big[(D_{11}-1)(D_{22}-1)B_{12} + D_{12}\Big], \tag{3.94}$$

respectively. Note that, in view of (3.18) and the condition $D_{11} \geq 1$, the assumption (3.86) guarantees

$$\begin{gathered} B_{11}D_{12}B_{21} \leq (A_{11}-1)A_{12}A_{21} < \frac{3-A_{22}}{3}\,\varphi(A_{11}) \leq \varphi(A_{11}), \\ (D_{11}-1)B_{12}D_{21} \leq (A_{11}-1)A_{12}A_{21} < \frac{3-A_{22}}{3}\,\varphi(A_{11}) \leq \varphi(A_{11}). \end{gathered} \tag{3.95}$$

Consequently, we get from (3.90), (3.93), and (3.94) that

$$\begin{aligned} (3-B_{22}-D_{22})&\Big[\varphi(A_{11}) - B_{11}D_{12}B_{21}\Big]\Big[\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}\Big] \leq \\ &\leq \Big[B_{11}B_{22}D_{12}B_{21} + B_{12}B_{21}\Big]\Big[\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}\Big] + \\ &+ \Big[(D_{11}-1)(D_{22}-1)B_{12}D_{21} + D_{12}D_{21}\Big]\Big[\varphi(A_{11}) - B_{11}D_{12}B_{21}\Big] \leq \\ &\leq \varphi(A_{11})\Big[B_{12}B_{21} + D_{12}D_{21} + B_{11}B_{22}D_{12}B_{21} + (D_{11}-1)(D_{22}-1)B_{12}D_{21}\Big]. \end{aligned} \tag{3.96}$$

On the other hand,

$$\begin{aligned} (3-B_{22}-D_{22})&\Big[\varphi(A_{11}) - B_{11}D_{12}B_{21}\Big]\Big[\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}\Big] \geq \\ &\geq (3-A_{22})\varphi(A_{11})^2 - \varphi(A_{11})(3-B_{22}-D_{22})B_{11}D_{12}B_{21} - \\ &- \varphi(A_{11})(3-B_{22}-D_{22})(D_{11}-1)B_{12}D_{21}\,. \end{aligned} \tag{3.97}$$

By virtue of (3.18), the inequality

$$B_{12}B_{21} + D_{12}D_{21} \le A_{12}A_{21} - D_{12}B_{21} - B_{12}D_{21} \tag{3.98}$$

is true. Consequently, (3.96) and (3.97) imply

$$\begin{aligned}(3 - A_{22})\varphi(A_{11}) \le A_{12}A_{21}+ \\ + D_{12}B_{21}\Big[(3 - D_{22})B_{11} - 1\Big] + B_{12}D_{21}\Big[(2 - B_{22})(D_{11} - 1) - 1\Big].\end{aligned} \tag{3.99}$$

If $(3 - D_{22})B_{11} \le 1$ and $(2 - B_{22})(D_{11} - 1) \le 1$ then (3.99) yields

$$(3 - A_{22})\varphi(A_{11}) \le A_{12}A_{21}\,,$$

which contradicts (3.86).

If $(3 - D_{22})B_{11} \le 1$ and $(2 - B_{22})(D_{11} - 1) > 1$ then (3.99) results in

$$(3 - A_{22})\varphi(A_{11}) \le A_{12}A_{21}(2 - B_{22})(D_{11} - 1) \le 3(A_{11} - 1)A_{12}A_{21}\,,$$

which contradicts (3.86).

If $(3 - D_{22})B_{11} > 1$ and $(2 - B_{22})(D_{11} - 1) \le 1$ then, in view of (3.18) and the assumption $D_{11} \ge 1$, we obtain from (3.99) that

$$\begin{aligned}(3 - A_{22})\varphi(A_{11}) \le A_{12}A_{21}(3 - D_{22})B_{11} \le \\ \le 3A_{12}A_{21}(A_{11} - D_{11}) \le 3(A_{11} - 1)A_{12}A_{21}\,,\end{aligned}$$

which contradicts (3.86).

If $(3 - D_{22})B_{11} > 1$ and $(2 - B_{22})(D_{11} - 1) > 1$ then (3.99) arrives at

$$\begin{aligned}(3 - A_{22})\varphi(A_{11}) \le A_{12}A_{21}\Big[(3 - D_{22})B_{11} + (2 - B_{22})(D_{11} - 1) - 1\Big] \le \\ \le A_{12}A_{21}\Big[3B_{11} + 3(D_{11} - 1)\Big] \le 3(A_{11} - 1)A_{12}A_{21}\,,\end{aligned}$$

which contradicts (3.86).

Case (c2): $D_{ii} < 1$ for $i = 1, 2$. By virtue of (3.16), the inequalities (3.62) and (3.38) result in

$$m_1 \le m_2 D_{12} \tag{3.100}$$

and

$$M_2 \le m_1 D_{21}\,, \tag{3.101}$$

respectively.

Case (c2.1): $m_1 D_{21} \le m_2 B_{22}$. Combining (3.37), (3.38) and taking (3.18) into account, we get

$$\begin{aligned}m_2 \le M_1 B_{21} + m_1 B_{22} D_{21} + m_2 B_{22}(D_{22} - 1) \le \\ \le M_1 B_{21} + m_1 (A_{22} - D_{22}) D_{21} + m_2 B_{22}(D_{22} - 1) =\end{aligned}$$

$$= M_1 B_{21} + m_1(A_{22}-1)D_{21} + (1-D_{22})\Big[m_1 D_{21} - m_2 B_{22}\Big].$$

Consequently,

$$m_2 \le M_1 B_{21} + m_1(A_{22}-1)D_{21}\,. \tag{3.102}$$

If $A_{22} \le 1$ then (3.89), (3.101), and (3.102) arrive at

$$3 - A_{11} \le 3 - B_{11} - D_{11} \le B_{12}D_{21} + D_{12}B_{21} \le A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{22}) \le 1$.

Therefore, suppose that

$$A_{22} > 1. \tag{3.103}$$

Then, using (3.62) in (3.102), we obtain

$$m_2 \le M_1 B_{21} + M_1(A_{22}-1)(D_{11}-1)D_{21} + m_2(A_{22}-1)D_{12}D_{21}\,,$$

i.e.,

$$m_2\Big[1-(A_{22}-1)D_{12}D_{21}\Big] \le M_1\Big[B_{21} - (A_{22}-1)(1-D_{11})D_{21}\Big]. \tag{3.104}$$

Note that the assumption (3.86) guarantees

$$(A_{22}-1)D_{12}D_{21} \le (A_{22}-1)A_{12}A_{21} < \frac{3-A_{22}}{3}\,\varphi(A_{11}) < 1.$$

Consequently, we get from (3.89), (3.101), and (3.104) that

$$\begin{aligned}\Big(3 - B_{11} - D_{11}\Big)\Big[1-(A_{22}-1)D_{12}D_{21}\Big] \le \\ \le \Big[1-(A_{22}-1)D_{12}D_{21}\Big]B_{12}D_{21} + D_{12}B_{21} - (A_{22}-1)(1-D_{11})D_{12}D_{21} \le \\ \le B_{12}D_{21} + D_{12}B_{21} - (A_{22}-1)(1-D_{11})D_{12}D_{21}\,. \end{aligned} \tag{3.105}$$

By virtue of the inequality

$$B_{12}D_{21} + D_{12}B_{21} \le A_{12}A_{21} - B_{12}B_{21} - D_{12}D_{21}\,, \tag{3.106}$$

(3.105) implies

$$3 - A_{11} \le A_{12}A_{21} + D_{12}D_{21}\Big[(A_{22}-1)(2-B_{11}) - 1\Big]. \tag{3.107}$$

If $(A_{22}-1)(2-B_{11}) \le 1$ then (3.107) results in

$$3 - A_{11} \le A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{22}) \le 1$.

If $(A_{22}-1)(2-B_{11}) > 1$ then (3.107) yields

$$3 - A_{11} \le A_{12}A_{21}(A_{22}-1)(2-B_{11}) \le 3(A_{22}-1)A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{22}) \le 1$.

Case (c2.2): $M_1 \leq M_2 D_{12}$. Using (3.100), we get from (3.90) that

$$3 - A_{22} \leq 3 - B_{22} - D_{22} \leq D_{12}B_{21} + D_{12}D_{21} = D_{12}(B_{21} + D_{21}) \leq A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{11}) \leq 1$.

Case (c2.3): $m_1 D_{21} > m_2 B_{22}$ *and* $M_1 > M_2 D_{12}$. We first note that, under the assumption $D_{12} = 0$, (3.89) and (3.101) yield

$$3 - A_{11} \leq 3 - B_{11} - D_{11} \leq B_{12}D_{21} \leq A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{22}) \leq 1$. Therefore, suppose that $D_{12} > 0$. Then we have

$$\frac{M_2}{M_1} < \frac{1}{D_{12}}\,. \tag{3.108}$$

Note also that (3.100) and the assumption $m_1 D_{21} > m_2 B_{22}$ guarantee

$$D_{12}D_{21} > B_{22}\,. \tag{3.109}$$

It follows from (3.37) and (3.108) that

$$\frac{m_2}{M_1} \leq B_{21} + \frac{M_2}{M_1}\,B_{22} \leq B_{21} + \frac{B_{22}}{D_{12}}\,. \tag{3.110}$$

Finally, (3.89), (3.101), and (3.110) result in

$$3 - A_{11} \leq 3 - B_{11} - D_{11} \leq B_{12}D_{21} + D_{12}B_{21} + B_{22}\,.$$

Using (3.106) and (3.109) in the last inequality, we get

$$3 - A_{11} \leq A_{12}A_{21} - B_{12}B_{21} - D_{12}D_{21} + B_{22} \leq A_{12}A_{21}\,,$$

which contradicts (3.86), because $0 < \varphi(A_{22}) \leq 1$.

The contradictions obtained in (a)–(c) prove that the problem (3.85_1), (3.85_2), (3.2) has only the trivial solution. ☐

Proof of Theorem 2.11. If $A_{12}A_{21} < 1$ then the validity of the theorem follows immediately from Theorem 2.10. Therefore, suppose in the sequel that

$$A_{12}A_{21} \geq 1. \tag{3.111}$$

According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the problem (3.85_1), (3.85_2), (3.2) has only the trivial solution.

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.85_1), (3.85_2), (3.2). Define the numbers M_i, m_i $(i = 1, 2)$ by (3.14) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.15_i) are satisfied for $i = 1, 2$. Furthermore, let the numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.17). It is clear that (3.2) guarantees

$$M_i \geq 0, \quad m_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

For the sake of clarity we shall devide the discussion into the following cases.

(a) Both functions u_1 and u_2 do not change their signs. According to Lemma 3.2, we can assume without loss of generality that

$$u_1(t) \geq 0, \quad u_2(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

(b) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \geq 0$ for $t \in [a,b]$. Obviously, one of the following items is satisfied:

(b1) $\alpha_2 < \beta_2$

(b2) $\alpha_2 > \beta_2$

(c) Both functions u_1 and u_2 change their signs. According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$.

At first we note that, in view of (2.3), the inequality (2.18) guarantees

$$\begin{aligned} A_{ii}A_{12}A_{21} &\leq \Big[A_{ii} + (1 - A_{ii})A_{3-i\,3-i}\Big]A_{12}A_{21} = \\ &= \Big(A_{11} + A_{22} - A_{11}A_{22}\Big)A_{12}A_{21} < 1 \quad \text{for} \quad i = 1,2. \end{aligned} \tag{3.112}$$

Now we are in position to discuss the cases (a)–(c).

Case (a): $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for $t \in [a,b]$. In view of the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, (3.85_2) implies $u_2'(t) \leq 0$ for $t \in [a,b]$. Therefore, $u_2 \equiv 0$ and, by virtue of the assumption $\ell_{11} \in \mathcal{P}_{ab}$, (3.85_1) arrives at $u_1'(t) \leq 0$ for $t \in [a,b]$. Consequently, $u_1 \equiv 0$ as well, which is a contradiction.

Case (b): $u_1(t) \geq 0$ for $t \in [a,b]$ and u_2 changes its sign. Obviously, $m_1 = 0$ and (3.32) is true. The integration of (3.85_1) from a to α_1, in view of (3.14), (3.15_1), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, yields (3.56).

Case (b1): $\alpha_2 < \beta_2$. The integrations of (3.85_2) from a to α_2 and from α_2 to β_2, in view of (3.14), (3.15_2), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, arrive at (3.87) and (3.88), respectively. Using (2.3), (3.56), and (3.87) in the relation (3.88), we get

$$0 < m_2 \leq M_1 D_{21} \leq M_2 A_{12} A_{21} \leq m_2 B_{22} A_{12} A_{21}\,.$$

Hence we get $1 \leq A_{22}A_{12}A_{21}$, which contradicts (3.112).

Case (b2): $\alpha_2 > \beta_2$. The integration of (3.85_2) from β_2 to α_2, on account of (3.14), (3.15_2), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, yields

$$M_2 + m_2 = -\int\limits_{\beta_2}^{\alpha_2} \ell_{21}(u_1)(s)ds - \int\limits_{\beta_2}^{\alpha_2} \ell_{22}(u_2)(s)ds \leq m_2 \int\limits_{\beta_2}^{\alpha_2} \ell_{22}(1)(s)ds \leq m_2 A_{22}\,. \tag{3.113}$$

By virtue of (2.3) and (3.32), (3.113) implies

$$0 < M_2 \leq m_2(A_{22} - 1) < 0,$$

a contradiction.

Case (c): u_1 and u_2 change their signs, $\alpha_1 < \beta_1$, and $\beta_2 < \alpha_2$. Obviously, (3.16) is true. The integrations of (3.85_1) from a to α_1 and from α_1 to β_1, in view of (3.14), (3.15_1), and the assumptions $\ell_{11}, \ell_{12} \in \mathcal{P}_{ab}$, imply (3.61) and (3.62). On the other hand, the integrations of (3.85_2) from a to β_2 and from β_2 to α_2, on account of (3.14), (3.15_2), and the assumptions $\ell_{21}, \ell_{22} \in \mathcal{P}_{ab}$, result in (3.37) and (3.38).

By virtue of (2.3) and (3.16), from the inequalities (3.61), (3.62) and (3.37), (3.38) we get

$$0 < \frac{M_1}{M_2} + \frac{M_1}{m_2}(1 - D_{11}) + \frac{m_1}{m_2} \leq A_{12} + \frac{m_1}{M_2} B_{11} \tag{3.114}$$

and

$$0 < \frac{m_2}{M_1} + \frac{M_2}{m_1} + \frac{m_2}{m_1}(1 - D_{22}) \leq A_{21} + \frac{M_2}{M_1} B_{22}\,, \tag{3.115}$$

respectively.

On the other hand, in view of (2.3), the inequalities (3.38) and (3.62) imply

$$m_1 \leq m_2 D_{12}\,, \qquad M_2 \leq m_1 D_{21}\,. \tag{3.116}$$

Combining (3.116) and (3.37), we get

$$M_2 \leq m_2 D_{12} D_{21} \leq M_1 A_{12} A_{21}^2 + M_2 A_{22} A_{12} A_{21}\,,$$

i.e.,

$$M_2\Big(1 - A_{22} A_{12} A_{21}\Big) \leq M_1 A_{12} A_{21}^2\,. \tag{3.117}$$

Furthermore, combining (3.37), (3.61), and (3.116), we obtain

$$m_1 \leq m_2 D_{12} \leq M_1 A_{12} A_{21} + M_2 A_{22} A_{12} \leq$$
$$\leq m_1 A_{11} A_{12} A_{21} + M_2 A_{12}^2 A_{21} + M_2 A_{22} A_{12}\,,$$

i.e.,

$$m_1\Big(1 - A_{11} A_{12} A_{21}\Big) \leq M_2 A_{12}\Big(A_{12} A_{21} + A_{22}\Big). \tag{3.118}$$

Now, (3.117) and (3.118) yield

$$A_{12} + \frac{m_1}{M_2} B_{11} \leq \frac{\Big(1 + A_{11} A_{22}\Big) A_{12}}{1 - A_{11} A_{12} A_{21}}\,, \quad A_{21} + \frac{M_2}{M_1} B_{22} \leq \frac{A_{21}}{1 - A_{22} A_{12} A_{21}}\,, \tag{3.119}$$

because the condition (3.112) is true

It follows from (3.114), (3.115), and (3.119) that

$$\frac{(1 + A_{11} A_{22}) A_{12} A_{21}}{(1 - A_{11} A_{12} A_{21})(1 - A_{22} A_{12} A_{21})} \geq$$
$$\geq \frac{m_2}{M_2} + \frac{M_1}{m_1} + \frac{M_1 m_2}{M_2 m_1}(1 - D_{22}) + 1 - D_{11} + \frac{M_1 M_2}{m_1 m_2}(1 - D_{11}) +$$

$$+\frac{M_1}{m_1}(1-D_{11})(1-D_{22})+\frac{m_1}{M_1}+\frac{M_2}{m_2}+1-D_{22}\,. \quad (3.120)$$

Using the condition (3.22), we get

$$\frac{M_1 m_2}{M_2 m_1}(1-D_{22})+\frac{M_1 M_2}{m_1 m_2}(1-D_{11}) \geq 2\,\frac{M_1}{m_1}\sqrt{(1-D_{11})(1-D_{22})}\,, \quad (3.121)$$

$$\frac{M_1}{m_1}+2\,\frac{M_1}{m_1}\sqrt{(1-D_{11})(1-D_{22})}+\frac{M_1}{m_1}(1-D_{11})(1-D_{22}) =$$

$$= \frac{M_1}{m_1}\left(1+\sqrt{(1-D_{11})(1-D_{22})}\right)^2, \quad (3.122)$$

$$\frac{M_1}{m_1}\left(1+\sqrt{(1-D_{11})(1-D_{22})}\right)^2+\frac{m_1}{M_1} \geq 2\Big(1+\sqrt{(1-D_{11})(1-D_{22})}\Big), \quad (3.123)$$

and

$$\frac{m_2}{M_2}+\frac{M_2}{m_2} \geq 2. \quad (3.124)$$

Now, in view of (3.121)–(3.124), (3.120) implies

$$\frac{\big(1+A_{11}A_{22}\big)A_{12}A_{21}}{\big(1-A_{11}A_{12}A_{21}\big)\big(1-A_{22}A_{12}A_{21}\big)} \geq$$

$$\geq 2+2\Big(1+\sqrt{(1-D_{11})(1-D_{22})}\Big)+1-D_{11}+1-D_{22} =$$

$$= 4+\left(\sqrt{1-D_{11}}+\sqrt{1-D_{22}}\right)^2 \geq$$

$$\geq 4+\left(\sqrt{1-A_{11}}+\sqrt{1-A_{22}}\right)^2 = \omega_0. \quad (3.125)$$

Therefore, using (3.112) and the inequality (3.111), we get

$$\Big(1+A_{11}A_{22}\Big)A_{12}A_{21} \geq$$

$$\geq \omega_0\Big[1-\big(A_{11}+A_{22}\big)A_{12}A_{21}+A_{11}A_{22}\big(A_{12}A_{21}\big)^2\Big] \geq$$

$$\geq \omega_0\Big[1-\big(A_{11}+A_{22}-A_{11}A_{22}\big)A_{12}A_{21}\Big],$$

which contradicts (2.18).

The contradictions obtained in (a)–(c) prove that the problem (3.85_1), (3.85_2), (3.2) has only the trivial solution. □

4. Counter–examples

In this section, the counter–examples are constructed verifying that the results obtained above are optimal in a certain sense.

Example 4.1. Let $\sigma_{ij} \in \{-1,1\}$, $h_{ij} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,j=1,2)$ be such that

$$\sigma_{11} = 1, \qquad \int_a^b h_{11}(s)ds \geq 1.$$

It is clear that there exists $t_0 \in\,]a,b]$ such that

$$\int_a^{t_0} h_{11}(s)ds = 1.$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by

$$\ell_{ij}(v)(t) \stackrel{\text{def}}{=} h_{ij}(t)v\big(\tau_{ij}(t)\big) \quad \text{for} \quad t \in [a,b],\ v \in C([a,b];\mathbb{R}) \quad (i,j=1,2), \tag{4.1}$$

where $\tau_{11}(t) = t_0$, $\tau_{12}(t) = a$, $\tau_{21}(t) = a$, and $\tau_{22}(t) = a$ for $t \in [a,b]$. Put

$$u(t) = \int_a^t h_{11}(s)ds \quad \text{for} \quad t \in [a,b].$$

It is easy to verify that $(u,0)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i=1,2)$.

An analogous example can be constructed for the case, where

$$\sigma_{22} = 1, \qquad \int_a^b h_{22}(s)ds \geq 1.$$

This example shows that the constant 1 on the right–hand side of the inequalities in (2.3) and (2.7) is optimal and cannot be weakened.

Example 4.2. Let $\sigma_{ij} \in \{-1,1\}$, $h_{ij} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,j=1,2)$ be such that

$$\sigma_{22} = -1, \qquad \int_a^b h_{22}(s)ds \geq 3.$$

It is clear that there exist $t_0 \in\,]a,b[$ and $t_1 \in\,]t_0,b]$ such that

$$\int_a^{t_0} h_{22}(s)ds = 1, \qquad \int_{t_0}^{t_1} h_{22}(s)ds = 2.$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{11}(t) = a$, $\tau_{12}(t) = a$, $\tau_{21}(t) = a$ for $t \in [a,b]$, and

$$\tau_{22}(t) = \begin{cases} t_1 & \text{for} \quad t \in [a,t_0[\\ t_0 & \text{for} \quad t \in [t_0,b] \end{cases}.$$

Put

$$u(t) = \begin{cases} \int\limits_a^t h_{22}(s)ds & \text{for} \quad t \in [a, t_0[\\ 1 - \int\limits_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0, b] \end{cases}.$$

It is easy to verify that $(0, u)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$.

An analogous example can be constructed for the case, where

$$\sigma_{11} = -1, \qquad \int\limits_a^b h_{11}(s)ds \geq 3.$$

This example shows that the constant 3 on the right–hand side of the inequalities in (2.7) and (2.13) is optimal and cannot be weakened.

Example 4.3. Let $\sigma_{ij} = 1$ for $i, j = 1, 2$ and let $h_{ij} \in L\big([a, b]; \mathbb{R}_+\big)$ $(i, j = 1, 2)$ be such that

$$\int\limits_a^b h_{11}(s)ds < 1, \qquad \int\limits_a^b h_{22}(s)ds < 1, \tag{4.2}$$

and

$$\int\limits_a^b h_{12}(s)ds \int\limits_a^b h_{21}(s)ds \geq \left(1 - \int\limits_a^b h_{11}(s)ds\right)\left(1 - \int\limits_a^b h_{22}(s)ds\right).$$

It is clear that there exists $t_0 \in\,]a, b]$ such that

$$\int\limits_a^{t_0} h_{12}(s)ds \int\limits_a^{t_0} h_{21}(s)ds = \left(1 - \int\limits_a^{t_0} h_{11}(s)ds\right)\left(1 - \int\limits_a^{t_0} h_{22}(s)ds\right).$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ be defined by (4.1), where $\tau_{ij}(t) = t_0$ for $t \in [a, b]$ $(i, j = 1, 2)$. Put

$$u_1(t) = \int\limits_a^t h_{11}(s)ds + \frac{1 - \int\limits_a^{t_0} h_{11}(s)ds}{\int\limits_a^{t_0} h_{12}(s)ds} \int\limits_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a, b],$$

$$u_2(t) = \int\limits_a^t h_{21}(s)ds + \frac{\int\limits_a^{t_0} h_{21}(s)ds}{1 - \int\limits_a^{t_0} h_{22}(s)ds} \int\limits_a^t h_{22}(s)ds \quad \text{for} \quad t \in [a, b].$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$.

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one.

Example 4.4. Let $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{21} = -1$, and $\sigma_{22} = 1$. Let $\alpha \in [0,1[$ and $h_{12}, h_{21} \in L\big([a,b];\mathbb{R}_+\big)$ be such that

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq 8(1-\alpha).$$

It is clear that there exist $t_0 \in\,]a,b]$ and $t_1, t_2 \in\,]a, t_0[$ such that

$$\int_a^{t_0} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = 8(1-\alpha)$$

and

$$\int_a^{t_1} h_{12}(s)ds = \frac{1}{4}\int_a^{t_0} h_{12}(s)ds, \qquad \int_a^{t_2} h_{21}(s)ds = \frac{1}{2}\int_a^{t_0} h_{21}(s)ds.$$

Furthermore, we choose $h_{11}, h_{22} \in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{11}(t) = 0 \quad \text{for} \quad t \in [a,t_1] \cup [t_0,b], \qquad h_{22}(t) = 0 \quad \text{for} \quad t \in [t_2,b]$$

and

$$\int_a^b h_{11}(s)ds = \int_a^b h_{22}(s)ds = \alpha.$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = t_0$, $\tau_{22}(t) = t_2$ for $t \in [a,b]$, and

$$\tau_{12}(t) = \begin{cases} t_0 & \text{for} \quad t \in [a,t_1[\\ t_2 & \text{for} \quad t \in [t_1,b] \end{cases}, \qquad \tau_{21}(t) = \begin{cases} t_1 & \text{for} \quad t \in [a,t_2[\\ t_0 & \text{for} \quad t \in [t_2,b] \end{cases}.$$

Put

$$u_1(t) = \begin{cases} \int_{t_2}^{t_0} h_{21}(s)ds \int_a^t h_{12}(s)ds & \text{for} \quad t \in [a,t_1[\\ 1-\alpha-2\int_{t_1}^{t} h_{11}(s)ds - \int_{t_2}^{t_0} h_{21}(s)ds \int_{t_1}^{t} h_{12}(s)ds & \text{for} \quad t \in [t_1,b] \end{cases},$$

$$u_2(t) = \begin{cases} -(1-\alpha)\int_a^t h_{21}(s)ds - \int_{t_2}^{t_0} h_{21}(s)ds \int_a^t h_{22}(s)ds & \text{for} \quad t \in [a,t_2[\\ -\int_{t_2}^{t_0} h_{21}(s)ds + 2\int_{t_2}^{t} h_{21}(s)ds & \text{for} \quad t \in [t_2,b] \end{cases}.$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.6) in Theorem 2.2 cannot be replaced by the nonstrict one provided $A_{11} = A_{22}$.

Example 4.5. Let $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{21} = 1$, $\sigma_{22} = -1$ and let $h_{ij} \in L\big([a,b];\mathbb{R}_+\big)$ $(i,j = 1,2)$ be such that

$$\int_a^b h_{11}(s)ds < 1, \quad \int_a^b h_{22}(s)ds \leq 1, \quad \int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq 1 - \int_a^b h_{11}(s)ds.$$

It is clear that there exists $t_0 \in \,]a,b]$ satisfying

$$\int_a^{t_0} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = 1 - \int_a^{t_0} h_{11}(s)ds.$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = t_0$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_0$, and $\tau_{22}(t) = a$ for $t \in [a,b]$. Put

$$u_1(t) = \int_a^t h_{11}(s)ds + \frac{1 - \int_a^{t_0} h_{11}(s)ds}{\int_a^{t_0} h_{12}(s)ds} \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a,b],$$

$$u_2(t) = \int_a^t h_{21}(s)ds \quad \text{for} \quad t \in [a,b].$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.8) in Theorem 2.3 cannot be replaced by the nonstrict one provided $A_{22} \leq 1$.

Example 4.6. Let $\sigma_{11} = 1$, $\sigma_{12} = 1$, $\sigma_{21} = 1$, $\sigma_{22} = -1$, and $h_{11}, h_{22} \in L\big([a,b];\mathbb{R}_+\big)$ be such that

$$\int_a^b h_{11}(s)ds < 1, \qquad 1 < \int_a^b h_{22}(s)ds < 3. \tag{4.3}$$

Obviously, there exists $t_0 \in \,]a,b[$ satisfying

$$\int_a^{t_0} h_{22}(s)ds = \frac{\int_a^b h_{22}(s)ds - 1}{2}. \tag{4.4}$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{21}(t) = 0 \quad \text{for} \quad t \in [t_0,b]$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(1 - \int_a^b h_{11}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2\right].$$

It is clear that there exists $t_1 \in]a,b]$ such that

$$\int_a^{t_1} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = \left(1 - \int_a^{t_1} h_{11}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2\right].$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = t_1$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and

$$\tau_{22}(t) = \begin{cases} b & \text{for} \quad t \in [a,t_0[\\ t_0 & \text{for} \quad t \in [t_0,b] \end{cases}. \tag{4.5}$$

Put

$$u_1(t) = \frac{\int_a^{t_1} h_{12}(s)ds}{1 - \int_a^{t_1} h_{11}(s)ds} \int_a^t h_{11}(s)ds + \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a,b],$$

$$u_2(t) = \begin{cases} \frac{\int_a^{t_1} h_{12}(s)ds}{1-\int_a^{t_1} h_{11}(s)ds} \int_a^t h_{21}(s)ds + \frac{\int_a^b h_{22}(s)ds-1}{2} \int_a^t h_{22}(s)ds & \text{for} \quad t \in [a,t_0[\\ 1 - \int_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0,b] \end{cases}.$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.8) in Theorem 2.3 cannot be replaced by the nonstrict one provided $A_{22} > 1$.

Example 4.7. Let $\sigma_{1i} = 1$, $\sigma_{2i} = -1$ for $i = 1,2$ and let $h_{11}, h_{22} \in L\big([a,b];\mathbb{R}_+\big)$ be such that (4.3) is true. Obviously, there exists $t_0 \in]a,b[$ satisfying

$$\int_a^{t_0} h_{22}(s)ds = 1. \tag{4.6}$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{21}(t) = 0 \quad \text{for} \quad t \in [a,t_0]$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(1 - \int_a^b h_{11}(s)ds\right)\left(3 - \int_a^b h_{22}(s)ds\right).$$

It is clear that there exists $t_1 \in]a,b]$ such that

$$\int_a^{t_1} h_{12}(s)ds \int_{t_0}^b h_{21}(s)ds = \left(1 - \int_a^{t_1} h_{11}(s)ds\right)\left(2 - \int_{t_0}^b h_{22}(s)ds\right).$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{11}(t)=t_1$, $\tau_{12}(t)=t_0$, $\tau_{21}(t)=t_1$ for $t\in[a,b]$, and τ_{22} is given by (4.5). Put

$$u_1(t)=\frac{\int\limits_a^{t_1} h_{12}(s)ds}{1-\int\limits_a^{t_1} h_{11}(s)ds}\int\limits_a^t h_{11}(s)ds+\int\limits_a^t h_{12}(s)ds \quad \text{for} \quad t\in[a,b],$$

$$u_2(t)=\begin{cases} 1-\int\limits_t^{t_0} h_{22}(s)ds & \text{for} \quad t\in[a,t_0[\\ 1-\dfrac{\int\limits_a^{t_1} h_{12}(s)ds}{1-\int\limits_a^{t_1} h_{11}(s)ds}\int\limits_{t_0}^t h_{21}(s)ds-\int\limits_{t_0}^t h_{22}(s)ds & \text{for} \quad t\in[t_0,b] \end{cases}.$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

This example shows that the strict inequality (2.10) in Theorem 2.5 cannot be replaced by the nonstrict one provided $A_{22}>1$.

Example 4.8. Let $\sigma_{ii}=-1$, $\sigma_{i\,3-i}=1$ for $i=1,2$ and let $h_{ij}\in L\big([a,b];\mathbb{R}_+\big)$ $(i=1,2)$ be such that

$$\int\limits_a^b h_{11}(s)ds\le 1, \qquad \int\limits_a^b h_{22}(s)ds\le 1, \qquad \int\limits_a^b h_{12}(s)ds\int\limits_a^b h_{21}(s)ds\ge 1.$$

It is clear that there exists $t_0\in\,]a,b]$ satisfying

$$\int\limits_a^{t_0} h_{12}(s)ds\int\limits_a^{t_0} h_{21}(s)ds=1.$$

Let the operators $\ell_{ij}\in\mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{ii}(t)=a$ and $\tau_{i\,3-i}(t)=t_0$ for $t\in[a,b]$ $(i=1,2)$. Put

$$u_1(t)=\int\limits_a^t h_{12}(s)ds, \quad u_2(t)=\int\limits_a^{t_0} h_{12}(s)ds\int\limits_a^t h_{21}(s)ds \quad \text{for} \quad t\in[a,b].$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

This example shows that the strict inequality (2.14) in Theorem 2.9 cannot be replaced by the nonstrict one provided $\max\{A_{11},A_{22}\}\le 1$.

Example 4.9. Let $\sigma_{ii}=-1$, $\sigma_{i\,3-i}=1$ for $i=1,2$ and let $h_{11},h_{22}\in L\big([a,b];\mathbb{R}_+\big)$ be such that

$$\int\limits_a^b h_{11}(s)ds\le 1, \qquad 1<\int\limits_a^b h_{22}(s)ds<3. \tag{4.7}$$

Obviously, there exists $t_0 \in \,]a,b[$ such that (4.4) is true. Furthermore, we choose $h_{12}, h_{21} \in L([a,b];\mathbb{R}_+)$ with the properties

$$h_{21}(t) = 0 \quad \text{for} \quad t \in [t_0, b]$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq 1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2 .$$

It is clear that there exists $t_1 \in \,]a,b]$ satisfying

$$\int_a^{t_1} h_{12}(s)ds \int_a^{t_0} h_{21}(s)ds = 1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2 .$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = a$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and τ_{22} is given by (4.5). Put

$$u_1(t) = \int_a^t h_{12}(s)ds \quad \text{for} \quad t \in [a,b],$$

$$u_2(t) = \begin{cases} \int_a^{t_1} h_{12}(s)ds \int_a^{t} h_{21}(s)ds + \frac{\int_a^b h_{22}(s)ds - 1}{2} \int_a^t h_{22}(s)ds & \text{for} \quad t \in [a, t_0[\\ 1 - \int_{t_0}^t h_{22}(s)ds & \text{for} \quad t \in [t_0, b] \end{cases} .$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

An analogous example can be constructed for the case, where

$$1 < \int_a^b h_{11}(s)ds < 3, \qquad \int_a^b h_{22}(s)ds \leq 1. \tag{4.8}$$

This example shows that the strict inequality (2.14) in Theorem 2.9 cannot be replaced by the nonstrict one provided that $\min\{A_{11}, A_{22}\} \leq 1$, $\max\{A_{11}, A_{22}\} > 1$, and $\omega = 1$.

Example 4.10. Let $\sigma_{ii} = -1$, $\sigma_{i\,3-i} = 1$ for $i = 1,2$ and let $h_{11}, h_{22} \in L([a,b];\mathbb{R}_+)$ be such that

$$1 < \int_a^b h_{ii}(s)ds < 3 \quad \text{for} \quad i = 1,2.$$

Obviously, there exist $t_1, t_2 \in \,]a,b[$ satisfying

$$\int_a^{t_i} h_{ii}(s)ds = \frac{\int_a^b h_{ii}(s)ds - 1}{2} \quad \text{for} \quad i = 1,2.$$

Furthermore, we choose $h_{12}, h_{21} \in L([a,b];\mathbb{R}_+)$ with the properties

$$h_{12}(t) = 0 \quad \text{for} \quad t \in [t_1, b], \qquad h_{21}(t) = 0 \quad \text{for} \quad t \in [a, t_2],$$

and

$$\int\limits_a^b h_{12}(s)ds \int\limits_a^b h_{21}(s)ds \geq$$
$$\geq \left[1 - \frac{1}{4}\left(\int\limits_a^b h_{11}(s)ds - 1\right)^2\right]\left[1 - \frac{1}{4}\left(\int\limits_a^b h_{22}(s)ds - 1\right)^2\right]$$

It is clear that there exists $\alpha \in \,]0,1]$ such that

$$\alpha \int\limits_a^{t_1} h_{12}(s)ds \int\limits_{t_2}^b h_{21}(s)ds =$$
$$= \left[1 - \frac{1}{4}\left(\int\limits_a^b h_{11}(s)ds - 1\right)^2\right]\left[1 - \frac{1}{4}\left(\int\limits_a^b h_{22}(s)ds - 1\right)^2\right]$$

Put

$$u_1(t) = \begin{cases} \frac{\int_a^b h_{11}(s)ds - 1}{2} \int\limits_a^t h_{11}(s)ds + \frac{\alpha \int_{t_2}^b h_{21}(s)ds}{1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2} \int\limits_a^t h_{12}(s)ds & \text{for} \quad t \in [a, t_1[\\ 1 - \int\limits_{t_1}^t h_{11}(s)ds & \text{for} \quad t \in [t_1, b] \end{cases},$$

$$u_2(t) = \begin{cases} -\frac{\int_{t_2}^b h_{21}(s)ds}{1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2} \int\limits_a^t h_{22}(s)ds & \text{for} \quad t \in [a, t_2[\\ \int\limits_{t_2}^t h_{21}(s)ds + \frac{\int_{t_2}^b h_{21}(s)ds \int_a^{t_2} h_{22}(s)ds}{1 - \frac{1}{4}\left(\int_a^b h_{22}(s)ds - 1\right)^2} \left(\int\limits_{t_2}^t h_{22}(s)ds - 1\right) & \text{for} \quad t \in [t_2, b] \end{cases}.$$

Since $u_2(t_2) < 0$ and $u_2(b) > 0$, there exists $t_0 \in \,]t_2, b]$ satisfying $u_2(t_0) = \alpha u_2(b)$. Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and

$$\tau_{11}(t) = \begin{cases} b & \text{for} \quad t \in [a, t_1[\\ t_1 & \text{for} \quad t \in [t_1, b] \end{cases}, \qquad \tau_{22}(t) = \begin{cases} b & \text{for} \quad t \in [a, t_2[\\ t_2 & \text{for} \quad t \in [t_2, b] \end{cases}. \tag{4.9}$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

This example shows that the strict inequality (2.14) in Theorem 2.9 cannot be replaced by the nonstrict one provided that $\min\{A_{11}, A_{22}\} > 1$ and $\omega = 1$.

Example 4.11. Let $\sigma_{11}=-1$, $\sigma_{12}=1$, $\sigma_{21}=-1$, $\sigma_{22}=-1$ and let $h_{11},h_{22}\in L\big([a,b];\mathbb{R}_+\big)$ be such that (4.7) holds. Obviously, there exists $t_0\in\,]a,b[$ such that (4.6) is satisfied. Furthermore, we choose $h_{12},h_{21}\in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{21}(t)=0 \quad \text{for} \quad t\in[a,t_0]$$

and

$$\int_a^b h_{12}(s)ds\int_a^b h_{21}(s)ds\geq 3-\int_a^b h_{22}(s)ds.$$

It is clear that there exists $t_1\in\,]a,b]$ satisfying

$$\int_a^{t_1} h_{12}(s)ds\int_{t_0}^b h_{21}(s)ds=2-\int_{t_0}^b h_{22}(s)ds.$$

Let the operators $\ell_{ij}\in\mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{11}(t)=a$, $\tau_{12}(t)=t_0$, $\tau_{21}(t)=t_1$ for $t\in[a,b]$, and τ_{22} is given by (4.5). Put

$$u_1(t)=\int_a^t h_{12}(s)ds \quad \text{for} \quad t\in[a,b],$$

$$u_2(t)=\begin{cases}\int_a^t h_{22}(s)ds & \text{for} \quad t\in[a,t_0[\\ 1-\int_a^{t_1} h_{12}(s)ds\int_{t_0}^t h_{21}(s)ds-\int_{t_0}^t h_{22}(s)ds & \text{for} \quad t\in[t_0,b]\end{cases}.$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

An analogous example can be constructed for the case, where the functions $h_{11},h_{22}\in L\big([a,b];\mathbb{R}_+\big)$ satisfy (4.8).

This example shows that the strict inequality (2.16) in Theorem 2.10 cannot be replaced by the nonstrict one provided that $\min\{A_{11},A_{22}\}\leq 1$, $\max\{A_{11},A_{22}\}>1$, and $\omega=1$.

Example 4.12. Let $\sigma_{11}=-1$, $\sigma_{12}=1$, $\sigma_{21}=-1$, $\sigma_{22}=-1$ and let $h_{11},h_{22}\in L\big([a,b];\mathbb{R}_+\big)$ be such that

$$1<\int_a^b h_{11}(s)ds\leq\int_a^b h_{22}(s)ds<3.$$

Obviously, there exist $t_1,t_2\in\,]a,b[$ satisfying

$$\int_a^{t_1} h_{11}(s)ds=\frac{\int_a^b h_{11}(s)ds-1}{2}, \qquad \int_a^{t_2} h_{22}(s)ds=1.$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{12}(t) = 0 \quad \text{for} \quad t \in [t_1, b], \qquad h_{21}(t) = 0 \quad \text{for} \quad t \in [a, t_2],$$

and

$$\int_a^b h_{12}(s)ds \int_a^b h_{21}(s)ds \geq \left(3 - \int_a^b h_{22}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)ds - 1\right)^2\right].$$

It is clear that there exist $\alpha \in]0,1]$ and $t_0 \in]a, t_2]$ such that

$$\alpha \int_a^{t_1} h_{12}(s)ds \int_{t_2}^b h_{21}(s)ds = \left(2 - \int_{t_2}^b h_{22}(s)ds\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)ds - 1\right)^2\right]$$

and

$$\int_a^{t_0} h_{22}(s)ds = \alpha.$$

Let the operators $\ell_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and τ_{11}, τ_{22} are given by (4.9). Put

$$u_1(t) = \begin{cases} \dfrac{\left(2 - \int_{t_2}^b h_{22}(s)ds\right)\left(\int_a^b h_{11}(s)ds - 1\right)}{2\int_{t_2}^b h_{21}(s)ds} \int_a^t h_{11}(s)ds + \alpha \int_a^t h_{12}(s)ds & \text{for} \quad t \in [a, t_1[\\ \dfrac{2 - \int_{t_2}^b h_{22}(s)ds}{\int_{t_2}^b h_{21}(s)ds}\left(1 - \int_{t_1}^t h_{11}(s)ds\right) & \text{for} \quad t \in [t_1, b] \end{cases},$$

$$u_2(t) = \begin{cases} \int_a^t h_{22}(s)ds & \text{for} \quad t \in [a, t_2[\\ 1 - \dfrac{\alpha \int_a^{t_1} h_{12}(s)ds}{1 - \frac{1}{4}\left(\int_a^b h_{11}(s)ds - 1\right)^2} \int_{t_2}^t h_{21}(s)ds - \int_{t_2}^t h_{22}(s)ds & \text{for} \quad t \in [t_2, b] \end{cases}.$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1,2)$.

An analogous example can be constructed for the case, where

$$1 < \int_a^b h_{22}(s)ds \leq \int_a^b h_{11}(s)ds < 3.$$

This example shows that the strict inequality (2.16) in Theorem 2.10 cannot be replaced by the nonstrict one provided that $\min\{A_{11}, A_{22}\} > 1$ and $\omega = 1$.

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References

[1] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina, Introduction to the theory of functional differential equations, Nauka, Moscow, 1991, in Russian.

[2] N. Dilnaya, A. Rontó, *Multistage iterations and solvability of linear Cauchy problems*, Math. Notes (Miskolc) **4** (2003), No. 2, 89–102.

[3] R. Hakl, E. Bravyi, A. Lomtatidze, *Optimal conditions on unique solvability of the Cauchy problem for the first order linear functional differential equations*, Czechoslovak Math. J. **52** (2002), No. 3, 513–530.

[4] R. Hakl, A. Lomtatidze, B. Půža, *New optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations*, Math. Bohem. **127** (2002), No. 4, 509–524.

[5] R. Hakl, A. Lomtatidze, B. Půža, *On a boundary value problem for first order scalar functional differential equations*, Nonlinear Anal. **53** (2003), No. 3–4, 391–405.

[6] R. Hakl, A. Lomtatidze, B. Půža, *On nonnegative solutions of first order scalar functional differential equations*, Mem. Differential Equations Math. Phys. **23** (2001), 51–84.

[7] R. Hakl, A. Lomtatidze, J. Šremr, Some boundary value problems for first order scalar functional differential equations, Folia Facult. Scien. Natur. Univ. Masar. Brunensis, Brno, 2002.

[8] R. Hakl, S. Mukhigulashvili, *On a boundary value problem for n-th order linear functional differential systems*, Georgian Math. J. **12** (2005), No. 2, 229–236.

[9] J. Hale, Theory of functional differential equations, Springer–Verlag, New York–Heidelberg–Berlin, 1977.

[10] I. Kiguradze, B. Půža, Boundary value problems for systems of linear functional differential equations, Folia Facult. Scien. Natur. Univ. Masar. Brunensis, Brno, 2003.

[11] I. Kiguradze, B. Půža, *On boundary value problems for systems of linear functional differential equations*, Czechoslovak Math. J. **47** (1997), 341–373.

[12] V. Kolmanovskii, A. Myshkis, Introduction to the theory and applications of functional differential equations, Kluwer Acad. Publ., Dordrecht–Boston–London, 1999.

[13] A. Rontó, *On the initial value problem for systems of linear differential equations with argument deviations*, Math. Notes (Miskolc) **6** (2005), No. 1, 105–127.

[14] A. N. Ronto, *Tochnye uslovija razreshimosti zadachi Koshi dlja sistem linejnych funkcional'no–differentsial'nych uravnenij pervogo porjadka, zadavaemych $(\sigma_1, \sigma_2, \ldots, \sigma_n; \tau)$–polozhitel'nymi operatorami*, Ukrain. Mat. Zh., **53** (2001), No. 1, 94–112, in Russian.

[15] Š. Schwabik, M. Tvrdý, O. Vejvoda, Differential and integral equations: boundary value problems and adjoints, Academia, Praha, 1979.

[16] J. Šremr, *On systems of linear functional differential inequalities*, Boundary Value Problems, submitted.

[17] W. Walter, Differential and integral inequalities, Springer–Verlag, Berlin, Heidelberg, New York, 1970.

Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic

E-mail address: `sremr@ipm.cz`

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