An Eigenvalue Criterion for Stability of a Steady Navier–Stokes Flow in $\mathbb{R}^3$

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Abstract. We study the resolvent equation associated with a linear operator $\mathcal{L}$ arising from the linearized equation for perturbations of a steady Navier–Stokes flow $\boldsymbol{U}^*$. We derive estimates which, together with a stability criterion from [33], show that the stability of $\boldsymbol{U}^*$ (in the L^2–norm) depends only on the position of the eigenvalues of $\mathcal{L}$, regardless the presence of the essential spectrum.

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1. Introduction

Assume that $\Omega \subset \mathbb{R}^3$ is an exterior domain and $\boldsymbol{U}^*$ is a steady solution of the Navier–Stokes system

$$\left.\begin{array}{ll} \partial_t \boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V} \;=\; -\nabla P + \nu\Delta\boldsymbol{V} + \boldsymbol{F} & \text{in } \Omega\times(0,+\infty),\\ \operatorname{div}\boldsymbol{V} \;=\; 0 & \text{in } \Omega\times(0,+\infty) \end{array}\right\} \tag{1.1}$$

with the boundary conditions

$$\left.\begin{array}{ll} \boldsymbol{V} \;=\; \boldsymbol{0} & \text{in } \partial\Omega\times(0,+\infty),\\ \boldsymbol{V}(\boldsymbol{x},t) \;\to\; (\tau,0,0) & \text{for } |\boldsymbol{x}|\to+\infty \end{array}\right\} \tag{1.2}$$

where $(\tau,0,0)$ is a constant velocity at infinity. The solution $\boldsymbol{U}^*$ can be written in the form $\boldsymbol{U}^* = \boldsymbol{U} + (\tau,0,0)$ where

$$\left.\begin{array}{ll} \tau\,\partial_1\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} \;=\; -\nabla P + \nu\Delta\boldsymbol{U} + \boldsymbol{F} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{U} \;=\; 0 & \text{in } \Omega,\\ \boldsymbol{U} \;=\; (-\tau,0,0) & \text{in } \partial\Omega,\\ \boldsymbol{U}(\boldsymbol{x}) \;\to\; \boldsymbol{0} & \text{for } |\boldsymbol{x}|\to+\infty. \end{array}\right\} \tag{1.3}$$

The problem of stability of solution $\boldsymbol{U}^*$ to the problem (1.1) has so far attracted much attention; see e.g. J. G. Heywood [18], [19], [20], K. Masuda [28], P. Maremonti [27], G. P. Galdi and S. Rionero [12], G. P. Galdi and M. Padula [13], W. Borchers and T. Miyakawa [4], [5], H. Kozono and T. Ogawa [24], H. Kozono and M. Yamazaki [25], [26], G. P. Galdi, J. G. Heywood and Y. Shibata [14], T. Miyakawa [30] and Y. Shibata [35]. Most of the results in these references are based on smallness assumptions on $\boldsymbol{U}$. However, as explained in [31], [32], one would also like to find a criterion related to the spectrum of a suitable linear operator, similar to the situation in a bounded domain (see D. H. Sattinger [34]) or in abstract differential equations (see e.g. H. Kielhöfer [21], [22]). Recently J. Neustupa [33] came rather close to such a criterion. The solution $\boldsymbol{U}^*$ is supposed to be such that $\nabla\boldsymbol{U}^* \equiv \nabla\boldsymbol{U} \in L^{3/2}(\Omega)^9 \cap L^3(\Omega)^9$ in [33]. Then the main result from [33] can be stated as follows:

Denote by $\mathcal{P}_2$ the usual Helmholtz projection in $L^2(\Omega)^3$. Define

$$\mathcal{L}\boldsymbol{v} \;:=\; \nu\mathcal{P}_2\,\Delta\boldsymbol{v} - \tau\,\mathcal{P}_2\partial_1\boldsymbol{v} + \mathcal{P}_2\mathfrak{B}\boldsymbol{v}, \tag{1.4}$$

where

$$\mathfrak{B}\boldsymbol{v} \;:=\; -(\boldsymbol{U}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{U}, \tag{1.5}$$

for $\boldsymbol{v} \in D(\mathcal{L}) := H_2 \cap W_0^{1,2}(\Omega)^3 \cap W^{2,2}(\Omega)^3$. (The closed subspace H_2 of $L^2(\Omega)^3$, which contains the divergence–free vector–functions, is defined in Section 2. Note that $\nu\mathcal{P}_2\,\Delta$, with the domain $D(\mathcal{L})$, is known as the Stokes operator.) Define the nonlinear operator $\mathcal{N}$ by the equation

$$\mathcal{N}\boldsymbol{v} \;:=\; -\mathcal{P}_2(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \tag{1.6}$$

for $\boldsymbol{v} \in D(\mathcal{L})$. Obviously, writing the solutions of (1.1) in the form $\boldsymbol{V} = \boldsymbol{U}^* + \boldsymbol{u} = (1,0,0) + \boldsymbol{U} + \boldsymbol{u}$, the perturbations $\boldsymbol{u}$ satisfy the operator equation

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \;=\; \mathcal{L}\boldsymbol{u} \,+\, \mathcal{N}\boldsymbol{u}. \tag{1.7}$$

Denote by $\mathfrak{B}_{\rm sym}$ the symmetric part of $\mathfrak{B}$. (Hence $\mathfrak{B}_{\rm sym}\boldsymbol{v} = -\boldsymbol{v}\cdot(\nabla\boldsymbol{U})_{\rm sym}$.) It is shown in [33] that the space H_2' generated by the eigenfunctions of $\mathcal{A}+\xi\,\mathcal{P}_2\mathfrak{B}_{\rm sym}$ (for a fixed $\xi > 0$) associated with positive eigenvalues is finite dimensional. Suppose that

(C1) *there exists a function $\varphi \in L^1(0,+\infty) \cap L^2(0,+\infty)$ such that*

$$\big\|\nabla \mathrm{e}^{\mathcal{L}t}\boldsymbol{\phi}\big\|_{2;\,\Omega_R} \;\le\; \varphi(t)\,\|\boldsymbol{\phi}\|_2 \qquad \textit{for all } \boldsymbol{\phi} \in H_2' \textit{ and } t > 0. \tag{1.8}$$

The exact notation of the norms is explained in Section 2. Ω_R denotes the set $\{\boldsymbol{x} \in \Omega;\ |\boldsymbol{x}| < R\}$ and we suppose that $R > 0$ is so large that

$$\|\nabla\boldsymbol{U}\|_{3/2;\,\Omega-\Omega_R} \;\le\; \tfrac{1}{8}. \tag{1.9}$$

Then J. Neustupa [33] could show that given $\boldsymbol{u}_0 \in H_2 \cap W_0^{1,2}(\Omega)$ with $\|\boldsymbol{u}_0\|_{1,2}$ sufficiently small, the equation (1.7) with the initial condition $\boldsymbol{u}(0) = \boldsymbol{u}_0$ has a strong solution $\boldsymbol{u}$ on the time interval $[0, +\infty)$, such that $\|\boldsymbol{u}(t)\|_{1,2}$ remains small for all $t > 0$ and $\|\nabla \boldsymbol{u}(t)\|_2 \to 0$ as $t \to +\infty$. It means that solution $\boldsymbol{U}^*$ of problem (1.1) is stable. J. Neustupa considers condition (C1) to be a substitute for the usual assumption

(C2) *there exists* $\delta > 0$ *such that* $\forall\ \lambda \in \mathrm{Sp}(\mathcal{L})\ :\ \mathit{Re}\,\lambda < -\delta$

where $\mathrm{Sp}(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$. (The assumption (C2) can never be satisfied in our situation because the essential spectrum $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$ touches the imaginary axis at point 0 from the left, independently of the concrete form of function $\boldsymbol{U}^*$; see [3] and [10].) The norm $\|\nabla \mathrm{e}^{\mathcal{L}t}\boldsymbol{\phi}\|_{2;\,\Omega_R}$ in (C1) can be alternatively replaced by $\|\mathrm{e}^{\mathcal{L}t}\boldsymbol{\phi}\|_{2;\,\Omega_R}$ and all the conclusions of [33] remail valid.

In the presented work, we assume that $\Omega = \mathbb{R}^3$. In this case, $D(\mathcal{L}) = H_2 \cap W^{2,2}(\Omega)^3$ and $\mathcal{P}_2\Delta\boldsymbol{v} = \Delta\boldsymbol{v}$, $\mathcal{P}_2\partial_1\boldsymbol{v} = \partial_1\boldsymbol{v}$ for $\boldsymbol{v} \in D(\mathcal{L})$. Since the viscosity coefficient ν plays no important role in our considerations, we also assume that $\nu = 1$. Thus, operator $\mathcal{L}$ can be simplified: $\mathcal{L}\boldsymbol{v} = \Delta\boldsymbol{v} - \tau\,\partial_1\boldsymbol{v} + \mathcal{P}_2\mathfrak{B}\boldsymbol{v}$. We show that at least in this case the essential spectrum of $\mathcal{L}$ does not play the decisive role in the stability criterion, namely that (C1) follows from the assumption (A1) that 0 is almost in the resolvent of $\mathcal{L}$, in the same sense as 0 is almost in the resolvent of respectively the Stokes and the Oseen operator (see Section 5 for the precise formulation) and from the assumption

(A2) *All eigenvalues of* $\mathcal{L}$ *have negative real parts.*

Our main result is stated in Theorem 25 at the end of the paper.

2. Notation and some auxiliary results

- For $M \subset \mathbb{R}^3$, we put $M^c := \mathbb{R}^3 - M$.
- We write B_R for the open ball with center at the origin and radius $R > 0$. It will be convenient to use the notation $B_0 := \emptyset$.
- The length $\alpha_1 + \alpha_2 + \alpha_3$ of a multiindex $\alpha = [\alpha_1, \alpha_2, \alpha_3] \in \mathbb{N}_0^3$ is denoted by $|\alpha|$.
- All our function spaces are to be understood as spaces of complex–valued functions. Let $p \in [1, \infty]$ and $M \subset \mathbb{R}^3$ be a measurable set. Then we denote by $\|\,.\,\|_{p;\,M}$ the norm in $L^p(M)$. If $M = \mathbb{R}^3$ then we use the simplified notation: $\|\,.\,\|_p$. In addition, we use the convention that $\|f\|_{p;\,M} = +\infty$ for any measurable function f from M into $\mathbb{C}$ such that $f \notin L^p(M)$. This means conversely that any measurable function $f\,:\,M \mapsto \mathbb{C}$ is in $L^p(M)$ if and only if $\|f\|_{p;\,M} < +\infty$.
- For measurable functions $f,\,g\,:\,\mathbb{R}^3 \mapsto \mathbb{C}$ with $\int_{\mathbb{R}^3} |f(\boldsymbol{x}-\boldsymbol{y})\,g(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} < +\infty$ for a.a. $\boldsymbol{x} \in \mathbb{R}^3$ we define the convolution $f * g$ in the obvious way. For functions $f\,:\,\mathbb{R}^3 \mapsto \mathbb{C}$, $\boldsymbol{g} = (g_1, g_2, g_3)\,:\,\mathbb{R}^3 \mapsto \mathbb{C}^3$, under analogous assumptions, we put $f * \boldsymbol{g} := \big(f * g_1,\, f * g_2,\, f * g_3\big)$.

- For $p \in [1, +\infty)$, $m \in \mathbb{N}$, $M \subset \mathbb{R}^3$ open, $W^{m,p}(M)$ denotes the usual Sobolev space of order m and exponent p. We write $\| \, . \, \|_{m,p;\, M}$ for the standard norm of this space. If $M = \mathbb{R}^3$ then we use the simplified notation $\| \, . \, \|_{m,p}$. The space $W^{m,p}_{loc}(M)$ is defined in the usual way.
- The spaces of vector–valued or tensor–valued functions are e.g. denoted by $L^p(M)^3$, $W^{m,p}(B)^3$ or $L^p(M)^9$, $W^{m,p}(B)^9$. The norms in these spaces are denoted in the same way as the norms in $L^p(M)$ and $W^{m,p}(B)$. Vector–valued functions are denoted by boldface letters.
- The space $C_0^\infty(\mathbb{R}^3)$ is defined in the standard way. We further denote by $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3)$ in the norm $\|\nabla \, . \, \|_2$. The dual space is denoted by $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)$. The norm of a bounded linear functional $\ell \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)$ is, as usually,
$$\|\ell\|_{-1,2} \, := \, \sup\big\{ |\ell(v)|/\|\nabla v\|_2 \, ; \; v \in \mathfrak{D}^{1,2}(\mathbb{R}^3), \; v \neq 0 \big\}.$$
Note that due to the density of $C_0^\infty(\mathbb{R}^3)$ in $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$, we can consider only $v \in C_0^\infty(\mathbb{R}^3)$ instead of $v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ in the set on the right hand side.
- Recall that each function $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ belongs to $L^6(\mathbb{R}^3)$ and the particular form of the Sobolev inequality (see [15, p. 59]) says that
$$\|u\|_6 \, \leq \, \frac{2}{\sqrt{3}} \, \|\nabla u\|_2. \tag{2.10}$$
- The dual space to $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ is denoted by $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. The norm in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ is denoted in the same way as the norm in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)$: $\| \, . \, \|_{-1,2}$.
- We denote by $C_{0,\sigma}^\infty(\mathbb{R}^3)$ the linear space of all vector–functions $\boldsymbol{\phi} \in C_0^\infty(\mathbb{R}^3)^3$ such that $\operatorname{div} \boldsymbol{\phi} = 0$.
- For $q \in (1, +\infty)$, let $H_q(\mathbb{R}^3)$ denote the closure of $C_{0,\sigma}^\infty(\mathbb{R}^3)$ in $L^q(\mathbb{R}^3)^3$. $H_q(\mathbb{R}^3)$ is the space of so called solenoidal vector–functions in $L^q(\mathbb{R}^3)^3$.
- While E denotes Newton's potential (Theorem 3), the symbols E^ρ and Γ_ϱ, are introduced in Definition 1 in Section 3, respectively in Corollary 2 in Section 3.
- In Sections 4–7, we shall also use the symbols $\mathfrak{A}$ (defined in Theorem 16), $\mathfrak{G}$ (defined by (5.7)), $\mathfrak{H}$ (defined in Theorem 16), $\mathfrak{I}_Y$ and $\mathfrak{I}$ (defined in Lemma 11, respectively at the beginning of Section 6), $\mathfrak{K}$ (introduced by Theorem 20), $\mathfrak{M}$ (defined in Lemma 14), $\mathfrak{N}$ (defined after equation (7.8)), $\mathfrak{S}_R$ (introduced in the proof of Theorem 18) and $\mathfrak{V}$ (defined in the proof of Corollary 3).

Lemma 1. *Let $\boldsymbol{f} \in L^1_{loc}(\mathbb{R}^3)^3$ such that*
$$\gamma_f \, := \, \sup\bigg\{ \frac{1}{\|\nabla \boldsymbol{v}\|_2} \bigg| \int_{\mathbb{R}^3} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \bigg| \, ; \; \boldsymbol{v} \in C_0^\infty(\mathbb{R}^3)^3, \; \boldsymbol{v} \neq \boldsymbol{0} \bigg\} \, < \, \infty. \tag{2.11}$$
Then the mapping $\ell_f : \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3 \to \mathbb{C}$ defined by the equation $\ell_f(\boldsymbol{v}) = \int_{\mathbb{R}^3} \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x}$ belongs to $\mathfrak{D}^{-1,2}(\mathbb{R}^3)^3$ and $\|\ell_f\|_{-1,2} = \gamma_f$.

The lemma is an obvious consequence of the density of $C_0^\infty(\mathbb{R}^3)^3$ in $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$. In what follows, we will always write $\boldsymbol{f}$ instead of ℓ_f if $\boldsymbol{f} \in L^1_{loc}(\mathbb{R}^3)^3$ satisfies (2.11). In this sense, the intersection $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$ is meaningful. We define

$$\|\boldsymbol{u}\|_* := \|\boldsymbol{u}\|_{-1,2} + \|\boldsymbol{u}\|_2 \qquad \text{for } \boldsymbol{u} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3. \tag{2.12}$$

Then the pair $\{\, \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3,\ \|\,.\,\|_* \,\}$ is a Banach space.

Let us now recall some results on the function spaces introduced above. We begin with well–known L^p–inequalities which hold for functions defined a.e. in $\mathbb{R}^n$. Since only the case $n = 3$ will be of interest in what follows, we confine ourselves to this case.

Theorem 1. (Young's inequality for integrals) [2, Corollary 2.25] *Let $p, q, r \in [1, \infty]$ with $1/p = 1/q + 1/r - 1$. Let $f,\ g : \mathbb{R}^3 \mapsto \mathbb{C}$ be measurable functions. Then $\|\, |f| * |g| \,\|_p \le \|f\|_q \cdot \|g\|_r$.*

This means in particular that in the case $f \in L^q(\mathbb{R}^3)$, $g \in L^r(\mathbb{R}^3)$, the integral $\int_{\mathbb{R}^3} |f(\boldsymbol{x} - \boldsymbol{y})\, g(\boldsymbol{y})|\, \mathrm{d}\boldsymbol{y}$ is finite for a.e. $\boldsymbol{x} \in \mathbb{R}^3$ and $\|f * g\|_p \le \|f\|_q \cdot \|g\|_r$.

Theorem 2. (Hardy–Littlewood–Sobolev inequality) [37, pp. 118–121] *Let $p, q \in (1, +\infty)$, $\alpha \in (0, 3)$ with $1/p = 1/q - \alpha/3$. Then there is $C = C(p, q) > 0$ such that*

$$\left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |\boldsymbol{x} - \boldsymbol{y}|^{-3+\alpha} \cdot |f(\boldsymbol{y})|\, \mathrm{d}\boldsymbol{y} \right)^p \mathrm{d}\boldsymbol{x} \right)^{1/p} \le C\, \|f\|_q$$

for $f : \mathbb{R}^3 \mapsto \mathbb{C}$ measurable.

Next we present some further properties of the space $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$.

Lemma 2. [15, p. 385; Lemma VII.4.3] *The space $C_0^\infty(\mathbb{R}^3)^3$ is dense in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ and in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$.*

As a consequence of (2.10), we obtain the next lemma:

Lemma 3. *$L^{6/5}(\mathbb{R}^3)^3 \subset \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ and there is $C > 0$ such that $\|f\|_{-1,2} \le C\, \|f\|_{6/5}$ for $f \in L^{6/5}(\mathbb{R}^3)^3$.*

Next we recall some properties of Newton's potential.

Theorem 3. *Put $E(\boldsymbol{z}) := (4\pi |\boldsymbol{z}|)^{-1}$ for $\boldsymbol{z} \in \mathbb{R}^3 - \{\boldsymbol{0}\}$. Let $\Phi \in C_0^\infty(\mathbb{R}^3)$, $\alpha \in \mathbb{N}_0^3$, $l \in \{1; 2; 3\}$. Then $E * \Phi \in C^\infty(\mathbb{R}^3)$,*

$$\begin{aligned} \partial^\alpha(E * \Phi) &= E * \partial^\alpha \Phi, \qquad \partial_l(E * \Phi) = (\partial_l E) * \Phi, \\ -\Delta(E * \Phi) &= \Phi. \end{aligned}$$

Given $p \in (1, \frac{3}{2})$, $q \in (1,3)$, $r \in (1,+\infty)$, *there exist positive constants* $c_1(p)$, $c_2(q)$, $c_3(r)$ *such that*

$$\begin{aligned} \|E * \Phi\|_{(1/p-2/3)^{-1}} &\le c_1(p)\, \|\Phi\|_p, \\ \|\partial_l(E * \Phi)\|_{(1/q-1/3)^{-1}} &\le c_2(q)\, \|\Phi\|_q, \\ \|\partial_m\partial_l(E * \Phi)\|_r &\le c_3(r)\, \|\Phi\|_r \end{aligned}$$

for $1 \le l,\, m \le 3$.

The proof of this theorem is well known. In fact, the first part follows from Lebesgue's theorem on dominated convergence; the estimate of $\partial_l(E * \Phi)$ is a consequence of Theorem 2, and the estimate of the second derivatives of $E * \Phi$ may be deduced from Calderon–Zygmund's inequality.

Lemma 4. *There exist* c_4 *and* $c_5 > 0$ *such that to any* $w \in C_0^\infty(\mathbb{R}^3)$ *one can find* $\boldsymbol{g} \in C^\infty(\mathbb{R}^3)^3$ *such that* $\operatorname{div} \boldsymbol{g} = w$, $\boldsymbol{g} \in W^{1,q}(\mathbb{R}^3)^3$ *for all* $q \in (\frac{3}{2}, +\infty)$ *and*

$$c_4\, \|w\|_{-1,2} \;\le\; \|\boldsymbol{g}\|_2 \;\le\; c_5\, \|w\|_{-1,2}.$$

Proof. Following [15, p. 391–392], we put $g_l := -\partial_l(E * w)$ for $l \in \{1;\, 2;\, 3\}$. Then the statement follows from Theorem 3. Note that, in particular, $E * w \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ by [15]. □

Now we turn our attention to the space $H_q(\mathbb{R}^3)$.

Theorem 4. *Let* $q \in (1,+\infty)$. *Then, for any* $\boldsymbol{f} \in L^q(\mathbb{R}^3)^3$, *there exists a unique function* $\mathcal{P}_q \boldsymbol{f} \in H_q(\mathbb{R}^3)$ *and a function* $G_q \boldsymbol{f} \in W^{1,1}_{loc}(\mathbb{R}^3)$, *unique up to an additive constant, such that* $\nabla G_q \boldsymbol{f} \in L^q(\mathbb{R}^3)^3$ *and*

$$\mathcal{P}_q \boldsymbol{f} + \nabla G_q \boldsymbol{f} \;=\; \boldsymbol{f}.$$

This defines a linear mapping $\mathcal{P}_q :\, L^q(\mathbb{R}^3)^3 \mapsto H_q(\mathbb{R}^3)$. *There exists* $c_6(q) > 0$ *such that*

$$\|\mathcal{P}_q \boldsymbol{f}\|_q \;\le\; c_6(q)\, \|\boldsymbol{f}\|_q$$

for $\boldsymbol{f} \in L^q(\mathbb{R}^3)^3$.

The proof follows from [15, Section III.1].

Theorem 5. *Let* $q \in (1,+\infty)$. *Then* $\mathcal{P}_q \big|_{L^q(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3} = \mathcal{P}_2 \big|_{L^q(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3}$.

Proof. Let $\mathcal{P}'_q :\, L^{q'}(\mathbb{R}^3)^3 \mapsto L^{q'}(\mathbb{R}^3)^3$ denote the adjoint operator to $\mathcal{P}_q$. Then $\mathcal{P}'_q = \mathcal{P}_{q'}$; compare with [15, Exercise III.1.6] and [11]. Let $\boldsymbol{u} \in L^{q'}(\mathbb{R}^3)^3$. Since $\boldsymbol{\phi} = \mathcal{P}_2 \boldsymbol{\phi} = \mathcal{P}_{q'} \boldsymbol{\phi}$ for all $\boldsymbol{\phi} \in C^\infty_{0,\sigma}(\mathbb{R}^3)$, we have

$$0 \;=\; \int_{\mathbb{R}^3} \boldsymbol{u} \cdot (\boldsymbol{\phi} - \boldsymbol{\phi})\, \mathrm{d}\boldsymbol{x} \;=\; \int_{\mathbb{R}^3} \boldsymbol{u} \cdot (\mathcal{P}_2 \boldsymbol{\phi} - \mathcal{P}_{q'} \boldsymbol{\phi})\, \mathrm{d}\boldsymbol{x} \;=\; \int_{\mathbb{R}^3} (\mathcal{P}_2 \boldsymbol{u} - \mathcal{P}_q \boldsymbol{u}) \cdot \boldsymbol{\phi}\, \mathrm{d}\boldsymbol{x}.$$

It follows by [15, Lemma III.1.1] that there exists $g \in W^{1,1}_{loc}(\mathbb{R}^3)$ such that $(\mathcal{P}_q - \mathcal{P}_2)(\boldsymbol{u}) = \nabla g$. This implies that g is a distributional solution of the Laplace equation in $\mathbb{R}^3$. This observation and Liouville's theorem yield $\mathcal{P}_2\boldsymbol{u} = \mathcal{P}_q\boldsymbol{u}$. □

In view of Theorem 5, we will always write only $\mathcal{P}$ instead of $\mathcal{P}_q$.

Theorem 6. *Let $\boldsymbol{f} \in L^2(\mathbb{R}^3)^3$. Then $\boldsymbol{f} \in H_2(\mathbb{R}^3)$ if and only if $\int_{\mathbb{R}^3} \boldsymbol{f} \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$.*

The intersection $W^{1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ can be characterized as the space of all functions $\boldsymbol{g}$ from $W^{1,2}(\mathbb{R}^3)$ such that $\operatorname{div}\boldsymbol{g} = 0$ a.e. in $\mathbb{R}^3$.

Proof. For the first statement of the lemma, we refer to [36, Lemma II.2.5.4]. The second statement follows from the first one. □

In the next two theorems, we state some well–known results on the Oseen system and the Stokes resolvent problem, respectively.

Theorem 7. *For $\ell \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$, there is a unique function $\boldsymbol{u} \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ such that $\operatorname{div}\boldsymbol{u} = 0$ and*

$$\int_{\mathbb{R}^3} \big(\nabla\boldsymbol{u} \cdot \nabla\boldsymbol{\phi} + \tau\,\partial_1\boldsymbol{u} \cdot \boldsymbol{\phi}\big)\, \mathrm{d}\boldsymbol{x} \;=\; \ell(\boldsymbol{\phi})$$

for $\boldsymbol{\phi} \in C_{0,\sigma}^\infty(\mathbb{R}^3)^3$.

The proof follows from [15, Theorem VII.1.2, VII.2.1, II.5.1, II.6.1].

We note that only the uniqueness statement in Theorem 7 will be needed in the following. Nevertheless, Theorem 7 is a motivation for assumption (A1) in Section 5, pertaining to resolution of the perturbed Oseen problem (5.2).

Theorem 8. *Let $\sigma \in (0, +\infty)$. We define*

$$\begin{aligned} g_1(r) \;&:=\; \mathrm{e}^{-r} + r^{-2}\,(\mathrm{e}^{-r} + r\,\mathrm{e}^{-r} - 1), \\ g_2(r) \;&:=\; \mathrm{e}^{-r} + 3r^{-2}\,(\mathrm{e}^{-r} + r\,\mathrm{e}^{-r} - 1), \\ F^{(\sigma)}_{jk}(\boldsymbol{z}) \;&:=\; \frac{1}{4\pi\,|\boldsymbol{z}|}\Big(\delta_{jk}\,g_1(\sigma^{1/2}\,|\boldsymbol{z}|) - z_j\,z_k\,g_2(\sigma^{1/2}\,|\boldsymbol{z}|)\Big) \end{aligned}$$

for $r,\ \sigma \in \mathbb{C} - \{0\}$, $\boldsymbol{z} \equiv (z_1, z_2, z_3) \in \mathbb{R}^3 - \{0\}$ and $j,\ k \in \{1;\,2;\,3\}$. (It means that the functions $F^{(\sigma)}_{jk}$ represent the velocity part of a fundamental solution of the Stokes resolvent problem (2.13).) Let $s \in (1, +\infty)$, $\boldsymbol{g} \equiv (g_1, g_2, g_3) \in L^s(\mathbb{R}^3)^3$. Define $w_j(\boldsymbol{g}) := \sum_{k=1}^3 F^{(\sigma)}_{jk} * g_k$ for $j = 1,\ 2,\ 3$.

Then $\boldsymbol{w}(\boldsymbol{g}) \equiv \big(w_1(\boldsymbol{g}),\, w_2(\boldsymbol{g}),\, w_3(\boldsymbol{g})\big) \in W^{2,s}(\mathbb{R}^3)^3$ and there is $\varrho(\boldsymbol{g}) \in W^{1,1}_{loc}(\mathbb{R}^3)$ such that $\nabla\varrho(\boldsymbol{g}) \in L^s(\mathbb{R}^3)^3$ and the pair $\big(\boldsymbol{w}(\boldsymbol{g}),\, \varrho(\boldsymbol{g})\big)$ solves the Stokes resolvent system

$$-\Delta\boldsymbol{u} + \sigma\,\boldsymbol{u} + \nabla\pi \;=\; \boldsymbol{g}, \quad \operatorname{div}\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3. \tag{2.13}$$

If $(\boldsymbol{u}, \pi) \in W^{2,s}(\mathbb{R}^3)^3 \times W^{1,1}_{loc}(\mathbb{R}^3)$ *with* $\nabla\pi \in L^s(\mathbb{R}^3)^3$ *is another solution of (2.13) then* $\boldsymbol{w}(\boldsymbol{g}) = \boldsymbol{u}$. *There is* $c_7(s,\sigma) > 0$ *such that*

$$\|\boldsymbol{w}(\boldsymbol{g})\|_{2,s} \leq c_7(s,\sigma)\,\|\boldsymbol{g}\|_s \quad \textit{for } \boldsymbol{g} \in L^s(\mathbb{R}^3)^3.$$

The proof follows from [29] and [6, Theorem 1.3, Lemma 1.1].

In addition to the assumptions on the solution $\boldsymbol{U}*$ made in Section 1, we shall further suppose that there exists $\epsilon_0 > 0$ such that

$$\nabla\boldsymbol{U}^* \equiv \nabla\boldsymbol{U} \in L^s(\Omega)^9 \qquad \forall\, s \in \left(\tfrac{4}{3},\, 3+\epsilon_0\right). \tag{2.14}$$

Note that a steady solution $\boldsymbol{U}^* = (1,0,0) + \boldsymbol{U}$ of (1.1), (1.2) with these properties exists under the assumption that $\boldsymbol{F} \in L^q(\mathbb{R}^3)^3$ for $q \in (1, q_0]$, with some $q_0 > 3$; see [16, Section IX.7]. Using the Sobolev inequality (see e.g. [1, p. 104] or [15, p. 31]) and (2.14), we can deduce that

$$\boldsymbol{U} \in L^a(\mathbb{R}^3)^3 \qquad \forall\, a \in \left(\tfrac{12}{5},\, +\infty\right). \tag{2.15}$$

Furthermore, we can deduce from [1, Corollary 5.16, p. 106] that $\boldsymbol{U} \in L^\infty(\mathbb{R}^3 - \overline{B_R})^3$ for each $R > 0$. If we restrict ourselves e.g. to $R > 1$ then we can observe that the domains $\mathbb{R}^3 - \overline{B_R}$ satisfy an interior cone condition specified by a single cone having a fixed height and vertex angle, independent of R. Hence

$$\|\boldsymbol{U}\|_{\infty;\,\mathbb{R}^3 - \overline{B_R}} \leq C(\epsilon_0)\,\|\boldsymbol{U}\|_{1,3+\epsilon_0;\,\mathbb{R}^3 - \overline{B_R}}$$

where the constant $C(\epsilon_0)$ does not depend on R. Since $\boldsymbol{U} \in W^{1,3+\epsilon_0}(\mathbb{R}^3)^9$, we obtain

$$\|\boldsymbol{U}\|_{\infty;\,\mathbb{R}^3 - \overline{B_R}} \longrightarrow 0 \qquad \text{for } R \to +\infty. \tag{2.16}$$

Notation of constants. Generic constants in our estimates are denoted by the capital letter C. If we need more generic constants in one formula then we use indices. The generic constants implicitly depend on certain quantities which may vary from section to section, but they are always listed at the beginning of each section. If these constants also depend on some additional quantities, like e.g. $\gamma_1, \ldots, \gamma_n$, then they are denoted by $C(\gamma_1, \ldots, \gamma_n)$.

3. The scalar Oseen equation in $\mathbb{R}^3$

In this section, we consider the scalar Oseen equation with the resolvent term

$$-\Delta v + \tau\,\partial_1 v + \lambda\, v = \Phi \qquad \text{in } \mathbb{R}^3. \tag{3.1}$$

The results we are going to derive will later be used in order to solve a perturbed vector Oseen equation with the resolvent term in the whole space $\mathbb{R}^3$ (Section 5), and to obtain estimates of the solutions (Section 6).

The generic constants in this section may depend on τ. The dependence on any other quantity will be indicated explicitly, as mentioned at the end of Section 2.

Definition 1. *Put*

$$\begin{aligned} s(\boldsymbol{z}) &:= \tau\,(|\boldsymbol{z}| - z_1), \\ E^{(0)}(\boldsymbol{z}) &:= \frac{\mathrm{e}^{-s(z)/2}}{4\pi\,|\boldsymbol{z}|}, \\ E^{(\lambda)}(\boldsymbol{z}) &:= \frac{1}{4\pi\,|\boldsymbol{z}|}\,\mathrm{e}^{-\sqrt{\lambda+(\tau/2)^2}\,|z|+\tau\,z_1/2} \end{aligned}$$

for $\boldsymbol{z} \in \mathbb{R}^3 - \{\mathbf{0}\}$ *and* $\lambda \in \mathbb{C} - \{0\}$ *such that* $\mathrm{Re}\,\lambda \geq 0$.

Note that throughout the paper, we denote by letter λ nonzero complex numbers. Whenever we admit the value zero, we will use letter ϱ.

We will now establish some estimates of convolutions of $E^{(\lambda)}$. We begin by stating an observation for which we refer to [9, Lemma 4.3].

Lemma 5. *Let* $\beta \in (1, +\infty)$. *Then*

$$\int_{\partial B_r} \big(1 + s(\boldsymbol{x})\big)^{-\beta}\,\mathrm{d}S_x \;\leq\; C(\beta)\,r$$

for $r \in (0, +\infty)$.

The next lemma was proved in [8] (see [8, Lemma 4.8]).

Lemma 6. $\big(1 + s(\boldsymbol{x} - \boldsymbol{y})\big)^{-1} \;\leq\; C\,(1 + |\boldsymbol{y}|)\,\big(1 + \tau\,s(\boldsymbol{x})\big)^{-1}$ *for* $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^3$.

Now we can derive pointwise estimates of the fundamental solution $E^{(\lambda)}$.

Theorem 9. *Let* μ, $\gamma \in (0, +\infty)$. *Then*

$$\begin{aligned} |\partial_z^\alpha E^{(\lambda)}(\boldsymbol{z})| \;\;&\leq\;\; C_1(\mu,\gamma)\,|\lambda|^{-2\gamma}\,\Big(|\boldsymbol{z}|^{-\gamma-1-|\alpha|/2} \\ &\qquad + |\boldsymbol{z}|^{-\gamma-1-|\alpha|}\Big)\,\big(1 + s(\boldsymbol{z})\big)^{-\mu}\,\mathrm{e}^{-C_2\,|\lambda|^2\,|z|} \end{aligned} \tag{3.2}$$

for $\boldsymbol{z} \in \mathbb{R}^3 - \{\mathbf{0}\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \leq 2$ *and* $\lambda \in \mathbb{C} - \{0\}$ *with* $\mathrm{Re}\,\lambda \geq 0$ *and* $|\lambda| \leq (\tau/2)^2$. *Moreover,*

$$|\partial_z^\alpha E^{(\lambda)}(\boldsymbol{z})| \;\leq\; C(\mu)\,\Big(|\boldsymbol{z}|^{-1-|\alpha|/2} + |\boldsymbol{z}|^{-1-|\alpha|}\Big)\,\big(1 + s(\boldsymbol{z})\big)^{-\mu} \tag{3.3}$$

for $\boldsymbol{z}$ *and* α *as in (3.2) and* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}\,\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$.

Proof. Take λ as in (3.2). Abbreviate, for a while, $\kappa := \tau/2$ and note that $|\lambda\,\kappa^{-2}| \leq 1$. We find that

$$\mathrm{Re}\,[\,(\lambda + \kappa^2)^{1/2} - \kappa\,] \;=\; \kappa\;\mathrm{Re}\;\frac{\lambda\,\kappa^{-2}}{(\lambda\,\kappa^{-2} + 1)^{1/2} + 1}$$

$$= \quad \kappa \, \frac{\operatorname{Re}(\lambda\,\kappa^{-2})\left(1+\operatorname{Re}(\lambda\,\kappa^{-2}+1)^{1/2}\right)-\operatorname{Im}(\lambda\,\kappa^{-2})\operatorname{Im}\overline{(\lambda\,\kappa^{-2}+1)^{1/2}}}{|\lambda\,\kappa^{-2}+1|+2\operatorname{Re}(\lambda\,\kappa^{-2}+1)^{1/2}+1},$$

$$|\lambda\,\kappa^{-2}+1| \;\le\; |\lambda\,\kappa^{-2}|+1 \;\le\; 2,$$

$$\operatorname{Im}(\lambda\,\kappa^{-2})\;\operatorname{Im}\overline{(\lambda\,\kappa^{-2}+1)^{1/2}} \;\le\; 0.$$

(The inequalities on the last two lines follow from $|\lambda\,\kappa^{-2}| \le 1$.) Hence

$$\begin{aligned} &\operatorname{Re}\left[(\lambda+\kappa^2)^{1/2}-\kappa\right] \\ &\quad\ge\; \frac{\kappa}{6}\left[\operatorname{Re}(\lambda\,\kappa^{-2})+|\operatorname{Im}(\lambda\,\kappa^{-2})|\;|\operatorname{Im}(\lambda\,\kappa^{-2}+1)^{1/2}|\right]. \end{aligned} \tag{3.4}$$

There exists $\varphi \in [-\pi/2,\, \pi/2]$ such that $\lambda\,\kappa^{-2}+1 = |\lambda\,\kappa^{-2}+1|\,\mathrm{e}^{\mathrm{i}\varphi}$. Since $\operatorname{Re}\lambda \ge 0$ and consequently $|\lambda\,\kappa^{-2}+1| \ge 1$, we also have

$$\begin{aligned} &|\operatorname{Im}(\lambda\,\kappa^{-2}+1)^{1/2}| \;=\; |\sin(\varphi/2)|\;|\lambda\,\kappa^{-2}+1|^{1/2} \;\ge\; |\sin(\varphi/2)| \\ &\quad=\; \left[(1-\cos\varphi)/2\right]^{1/2} \\ &\quad=\; \left[\frac{|\lambda\,\kappa^{-2}+1|-\operatorname{Re}(\lambda\,\kappa^{-2}+1)}{2\,|\lambda\,\kappa^{-2}+1|}\right]^{-1/2} \\ &\quad\ge\; \frac{1}{\sqrt{2}}\;\left[|\lambda\,\kappa^{-2}+1|-\operatorname{Re}(\lambda\,\kappa^{-2}+1)\right]^{1/2} \\ &\quad=\; \frac{1}{\sqrt{2}}\;\frac{|\operatorname{Im}(\lambda\,\kappa^{-2})|}{\left[|\lambda\,\kappa^{-2}+1|+\operatorname{Re}(\lambda\,\kappa^{2}+1)\right]^{1/2}} \;\ge\; \frac{1}{2}\,|\operatorname{Im}(\lambda\,\kappa^{-2})|. \end{aligned}$$

This estimate and (3.4) yield

$$\operatorname{Re}\left[(\lambda+\kappa^2)^{1/2}-\kappa\right] \;\ge\; \frac{\kappa}{6}\left[\operatorname{Re}(\lambda\,\kappa^{-2})+\frac{1}{2}\left(\operatorname{Im}(\lambda\,\kappa^{-2})\right)^2\right] \;\ge\; C\,\frac{|\lambda|^2}{\kappa^3}. \tag{3.5}$$

Obviously

$$|(\lambda+\kappa^2)^{1/2}-\kappa| \;\le\; \frac{|\lambda|\,\kappa^{-1}}{|(\lambda\,\kappa^{-2}+1)^{1/2}+1|} \;\le\; \frac{|\lambda|}{\kappa}. \tag{3.6}$$

In particular, $|(\lambda+\kappa^2)^{1/2}| \le 2\,\kappa$.

Now let $z \in \mathbb{R}^3 \setminus \{0\}$. We abbreviate $b(\lambda,z) := -(\lambda+\kappa^2)^{1/2}\,|z| + \kappa\,z_1$. Then we get from (3.5) that

$$|\mathrm{e}^{b(\lambda,z)}| \;\le\; \mathrm{e}^{-c_8\,|\lambda|^2\,|z|}\;\mathrm{e}^{-\kappa\,(|z|-z_1)} \tag{3.7}$$

with some constant $c_8 > 0$ depending only on τ. Thus,

$$|E^{(\lambda)}(\boldsymbol{z})| \;\le\; C\,|\boldsymbol{z}|^{-1}\;\mathrm{e}^{-c_8\,|\lambda|^2\,|z|}\;\mathrm{e}^{-\kappa\,(|z|-z_1)}. \tag{3.8}$$

Let $l, m \in \{1; 2; 3\}$. Then

$$\partial_l E^{(\lambda)}(\boldsymbol{z}) \;=\; (4\pi)^{-1}\,\big[-z_l\,|\boldsymbol{z}|^{-3} + |\boldsymbol{z}|^{-1}\,\partial_{z_l} b(\lambda,\boldsymbol{z})\big]\,\mathrm{e}^{b(\lambda,z)}, \tag{3.9}$$

$$\begin{aligned}\partial_m\partial_l E^{(\lambda)}(\boldsymbol{z}) \;=\; & (4\pi)^{-1}\,\big[-\delta_{lm}\,|\boldsymbol{z}|^{-3} + 3\,z_l\,z_m\,|\boldsymbol{z}|^{-5} \\ & - z_m\,|\boldsymbol{z}|^{-3}\,\partial_{z_l} b(\lambda,\boldsymbol{z}) + |\boldsymbol{z}|^{-1}\,\partial_{z_l}\,\partial_{z_m} b(\lambda,\boldsymbol{z}) - z_l\,|\boldsymbol{z}|^{-3}\,\partial_{z_m} b(\lambda,\boldsymbol{z}) \\ & + |\boldsymbol{z}|^{-1}\,\partial_{z_l} b(\lambda,\boldsymbol{z})\,\partial_{z_m} b(\lambda,\boldsymbol{z})\big]\,\mathrm{e}^{b(\lambda,z)}.\end{aligned} \tag{3.10}$$

However, using (3.6), we have

$$\begin{aligned}|\partial_{z_\nu} b(\lambda,\boldsymbol{z})| \;&=\; \big|\big(-(\lambda+\kappa^2)^{1/2}+\kappa\big)\,z_\nu\,|\boldsymbol{z}|^{-1} + \kappa\,\partial_{z_\nu}(-|\boldsymbol{z}|+z_1)\,\big| \\ &\le\; C\,\big(|\lambda| + (|\boldsymbol{z}|-z_1)^{1/2}\,|\boldsymbol{z}|^{-1/2}\big)\end{aligned}$$

for $\nu \in \{1; 2; 3\}$. Now we may conclude with (3.7) and (3.9) that

$$\begin{aligned}&|\partial_l E^{(\lambda)}(\boldsymbol{z})| \\ &\le\; C\,\big(|\boldsymbol{z}|^{-2} + |\boldsymbol{z}|^{-1}\,|\lambda| + |\boldsymbol{z}|^{-3/2}\,(|\boldsymbol{z}|-z_1)^{1/2}\big)\,\mathrm{e}^{-c_8\,|\lambda|\,|z|}\,\mathrm{e}^{-\kappa\,(|z|-z_1)} \\ &\le\; C\,(|\boldsymbol{z}|^{-2} + |\boldsymbol{z}|^{-3/2})\,\mathrm{e}^{-c_8\,|\lambda|^2\,|z|/2}\,\mathrm{e}^{-\kappa\,(|z|-z_1)/2}.\end{aligned} \tag{3.11}$$

Similarly, due to (3.10), (3.7) and because $|\partial_{z_l}\partial_{z_m} b(\lambda,\boldsymbol{z})| \le C\,|\boldsymbol{z}|^{-1}$, we get

$$\begin{aligned}&|\partial_l\partial_m E^{(\lambda)}(\boldsymbol{z})| \\ &\le\; C\,\big[\,|\boldsymbol{z}|^{-3} + \big(|\lambda| + (|\boldsymbol{z}|-z_1)^{1/2}\,|\boldsymbol{z}|^{-1/2}\big)\,|\boldsymbol{z}|^{-2} \\ &\qquad + |\boldsymbol{z}|^{-2} + \big(|\lambda| + (|\boldsymbol{z}|-z_1)^{1/2}\,|\boldsymbol{z}|^{-1/2}\big)^2\,|\boldsymbol{z}|^{-1}\big]\,\mathrm{e}^{-c_8\,|\lambda|^2\,|z|}\,\mathrm{e}^{-\kappa\,(|z|-z_1)} \\ &\le\; C\,(|\boldsymbol{z}|^{-3} + |\boldsymbol{z}|^{-2})\,\mathrm{e}^{-c_8\,|\lambda|^2\,|z|/2}\,\mathrm{e}^{-\kappa\,(|z|-z_1)/2}.\end{aligned} \tag{3.12}$$

Recalling the abbreviation $s(\boldsymbol{x}) = \tau\,(|\boldsymbol{x}| - x_1)$ from Definition 1, we observe in the case $s(\boldsymbol{x}) \ge 1$ that $s(\boldsymbol{x})^{-1} \le 2\,\big(1+s(\boldsymbol{x})\big)^{-1}$. If $s(\boldsymbol{x}) < 1$, we get $1 \le 2\,\big(1+s(\boldsymbol{x})\big)^{-1}$. Thus we find in the first case that

$$\mathrm{e}^{-\kappa\,(|z|-z_1)/2} \;\le\; C(\mu)\,s(\boldsymbol{x})^{-\mu} \;\le\; C(\mu)\,\big(1+s(\boldsymbol{x})\big)^{-\mu}$$

and in the second case, $1 \le C(\mu)\,\big(1+s(\boldsymbol{x})\big)^{-\mu}$, hence in any case

$$\mathrm{e}^{-\kappa\,(|z|-z_1)/2} \;\le\; C(\mu)\,\big(1+s(\boldsymbol{x})\big)^{-\mu}. \tag{3.13}$$

Moreover,

$$\mathrm{e}^{-c_8\,|\lambda|^2\,|z|/2} \;\le\; C(\gamma)\,|\lambda|^{-2\gamma}\,|\boldsymbol{z}|^{-\gamma}\,\mathrm{e}^{-c_8\,|\lambda|^2\,|z|/4}. \tag{3.14}$$

Inequality (3.2) follows from (3.8), (3.11), (3.12), (3.13) and (3.14). The estimate in (3.3) may be shown by similar, but somewhat simpler arguments. □

We exploit the preceding theorem to obtain L^p-estimates of convolutions of $E^{(\varrho)}$.

Theorem 10. *Let $p, q \in [1,2]$ with $p \geq q$. Further suppose that $p < 2$ or $q > 1$. Then*

$$\| \, |E^{(\lambda)}| * |f| \, \|_p \; \leq \; C(p,q) \cdot |\lambda|^{2-4\,(1-1/q+1/p)} \cdot \|f\|_q \tag{3.15}$$

for $f \in L^q(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$.

Let $q \in [1,2)$ and $p \in \big((1/q-1/2)^{-1}, \infty]$ if $q \geq 3/2$ or $p \in \big((1/q-1/2)^{-1}, (1/q-2/3)^{-1} \big)$ if $q < 3/2$. Then, for $f \in L^q(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$,

$$\| \, |E^{(\varrho)}| * |f| \, \|_p \; \leq \; C(p,q) \cdot \|f\|_q. \tag{3.16}$$

Let $q \in [1,3]$ and $p \in \big((1/q-1/4)^{-1}, (1/q-1/3)^{-1} \big)$. Then

$$\| \, |\partial_l E^{(\varrho)}| * |f| \, \|_p \; \leq \; C(p,q) \cdot \|f\|_q \tag{3.17}$$

for $l \in \{1, 2, 3\}$ and for f, ϱ as in (3.16).

Finally,

$$\| \, |E^{(\varrho)}| * |f| \, \|_6 + \| \, |\partial_l E^{(\varrho)}| * |f| \, \|_2 \leq C \cdot \|f\|_{6/5} \tag{3.18}$$

for $l \in \{1, 2, 3\}$, $f \in L^{6/5}(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$.

Proof. In the situation of (3.15), put $r := (1-1/q+1/p)^{-1}$. Note that since $p \geq q$ and $q > 1$ or $p < 2$, we have $r \in [1,2)$. Using (3.2) with $\alpha = 0$, $\mu = 2/r$, $\gamma = 0$, and referring to Lemma 5, we get

$$\begin{aligned}
\int_{\mathbb{R}^3} |E^{(\lambda)}(\boldsymbol{x})|^r \, \mathrm{d}\boldsymbol{x} \; &\leq \; C \int_{\mathbb{R}^3} |\boldsymbol{x}|^{-r} \, \big(1 + s(\boldsymbol{x})\big)^{-2} \, \mathrm{e}^{-c_8\,|\lambda|^2\,|x|} \, \mathrm{d}\boldsymbol{x} \\
&\leq \; C \left(\int_{B_1} |\boldsymbol{x}|^{-r} \, \mathrm{d}\boldsymbol{x} + \int_1^\infty \alpha^{-r+1} \, \mathrm{e}^{-c_8\,|\lambda|^2\,\alpha} \, \mathrm{d}\alpha \right) \\
&\leq \; C \left(1 + |\lambda|^{-4+2r} \int_{|\lambda|^2}^\infty t^{-r+1} \, \mathrm{e}^{-c_8 t} \, \mathrm{d}t \right) \\
&\leq \; C \left(1 + |\lambda|^{-4+2r} \int_0^\infty t^{-r+1} \, \mathrm{e}^{-c_8 t} \, \mathrm{d}t \right) \\
&\leq \; C \, (1 + |\lambda|^{-4+2r}) \; \leq \; C \, |\lambda|^{-4+2r},
\end{aligned} \tag{3.19}$$

where the last and last but one inequality hold because $r < 2$. Now inequality (3.15) follows from Theorem 1.

In the situation of (3.16) and (3.17), we also put $r := (1-1/q+1/p)^{-1}$. The exponents p and q are chosen in such a way that $r \in (2,3)$ under the assumptions of (3.16), and $r \in (4/3, 3/2)$ under those of (3.17). Further observe that for $\boldsymbol{z} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$,

$$\left.\begin{aligned}
|E^{(\varrho)}(\boldsymbol{z})| \; &\leq \; C \, |\boldsymbol{z}|^{-1}, \\
|\partial_l E^{(\varrho)}(\boldsymbol{z})| \; &\leq \; C \, \big[\, |\boldsymbol{z}|^{-2} + |\boldsymbol{z}|^{-3/2} \, \big(1 + s(\boldsymbol{z})\big)^{-3/2} \, \big].
\end{aligned}\right\} \tag{3.20}$$

Via estimates similar to (3.19), inequality (3.3) and Lemma 5 imply that

$$\int_{\mathbb{R}^3} |E^{(\varrho)}(\boldsymbol{x})|^r \, \mathrm{d}\boldsymbol{x} \;\le\; C(p,q)$$

in the case of (3.16), and

$$\int_{\mathbb{R}^3} |\partial_l E^{(\varrho)}(\boldsymbol{x})|^r \, \mathrm{d}\boldsymbol{x} \;\le\; C(p,q)$$

in the situation of (3.17). Now we obtain (3.16) and (3.17) applying again Theorem 1. Finally, as concerns (3.18), we refer to the estimates in (3.20) once more, which allows us to apply Theorem 2 to $E^{(\varrho)} * f$ and $(\chi_{B_1} \cdot \partial_l E^{(\varrho)}) * f$, and Theorem 1 with $p = 2$, $q = 6/5$, $r = 3/2$ as well as Lemma 5 to $(\chi_{B_1^c} \cdot \partial_l E^{(\varrho)}) * f$. Inequality (3.18) then follows. □

Theorem 11. *Let* $\varrho \in \mathbb{C}$ *with* $\operatorname{Re}\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. *Take* $\Phi \in C_0^\infty(\mathbb{R}^3)$, *and put* $u := E^{(\varrho)} * \Phi$. *Then* $u \in C^\infty(\mathbb{R}^3)$, *$u$ verifies (3.1), and*

$$\left.\begin{aligned} \partial^\alpha u \;&=\; E^{(\varrho)} * \partial^\alpha \Phi \qquad && \text{for } \alpha \in \mathbb{N}_0^3, \\ \partial_l u \;&=\; (\partial_l E^{(\varrho)}) * \Phi && \text{for } 1 \le l \le 3. \end{aligned}\right\} \tag{3.21}$$

Let $q \in (1,\infty)$ *and* $R \in (0,\infty)$. *Then*

$$\|\partial_l \partial_m u\|_{q;\,B_R} \;\le\; C(q,R)\, \|\Phi\|_q. \tag{3.22}$$

Proof. Theorem 9 yields that for any $S > 0$,

$$|\partial^\beta E^{(\varrho)}(\boldsymbol{z})| \;\le\; C(S)\, |\boldsymbol{z}|^{-1-|\beta|} \quad \text{for } \boldsymbol{z} \in B_S \setminus \{\mathbf{0}\},\ \beta \in \mathbb{N}_0^3 \text{ with } |\beta| \le 1. \tag{3.23}$$

In particular, we have $E^{(\varrho)} \in L^1_{loc}(\mathbb{R}^3)$, and we may conclude that $u \in C^\infty(\mathbb{R}^3)$,

$$\partial^\alpha u(\boldsymbol{x}) \;=\; \int_{\mathbb{R}^3} E^{(\varrho)}(\boldsymbol{y})\, \partial^\alpha \Phi(\boldsymbol{x}-\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3,\ \alpha \in \mathbb{N}_0^3.$$

This proves the first equation in (3.21). Let $R_0 > 0$ with $\operatorname{supp}(\Phi) \subset B_{R_0}$, and take $l \in \{1, 2, 3\}$, $\boldsymbol{x} \in \mathbb{R}^3$. It follows from the first equation in (3.21) that

$$\partial_l u(\boldsymbol{x}) \;=\; \lim_{\epsilon \downarrow 0} \int_{B_{R_0+|x|} \setminus B_\epsilon(x)} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\, \partial_l \Phi(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}. \tag{3.24}$$

However, we can integrate by parts in (3.24) for $\epsilon > 0$, say, smaller than $(R_0+|\boldsymbol{x}|)/2$. Since in view of (3.23), we have

$$\int_{\partial B_\epsilon(x)} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\, \Phi(\boldsymbol{y})\, \frac{x_l - y_l}{\epsilon}\, \mathrm{d}S_y \;\longrightarrow\; 0 \quad \text{for } \epsilon \downarrow 0, \tag{3.25}$$

we thus obtain the second equation in (3.21). Returning to the proof of the claim that u verifies (3.1), we observe that by (3.9), (3.10), for any $S > 0$,

$$|\partial^\alpha (E^{(\varrho)} - E)(\boldsymbol{z})| \;\leq\; C(S)\,|\boldsymbol{z}|^{-|\alpha|} \tag{3.26}$$

for $\boldsymbol{z} \in B_S \setminus \{\boldsymbol{0}\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 2$, where function E was introduced in Theorem 3. Estimate (3.26) with $\alpha \in \mathbb{N}_0^3$, such that $|\alpha| = 1$, yields that

$$\int_{\partial B_\epsilon(x)} \partial_l (E^{(\varrho)} - E)(\boldsymbol{x}-\boldsymbol{y})\; \Phi(\boldsymbol{y})\; \frac{x_l - y_l}{\epsilon}\; \mathrm{d}S_y \;\longrightarrow\; 0 \quad (\epsilon \downarrow 0) \quad \text{for } 1 \leq l \leq 3. \tag{3.27}$$

Since the function Φ is Lipschitz continuous, we obtain

$$\int_{\partial B_\epsilon(x)} \partial_l E(\boldsymbol{x}-\boldsymbol{y})\; \big(\, \Phi(\boldsymbol{y}) - \Phi(\boldsymbol{x}) \,\big)\; \frac{x_l - y_l}{\epsilon}\; \mathrm{d}S_y \;\longrightarrow\; 0 \quad (\epsilon \downarrow 0) \quad \text{for } 1 \leq l \leq 3. \tag{3.28}$$

We further note that for $1 \leq l \leq 3$,

$$\left.\begin{aligned} &\int_{\partial B_\epsilon(x)} \partial_l E(\boldsymbol{x}-\boldsymbol{y})\; \frac{x_l - y_l}{\epsilon}\; \mathrm{d}S_y \;=\; -\frac{1}{3} \\ &-\; \Delta E^{(\varrho)} + \tau\, \partial_1 E^{(\varrho)} + \varrho\, E^{(\varrho)} \;=\; 0 \end{aligned}\right\} \tag{3.29}$$

$$-\,\Delta u + \tau\, \partial_1 u + \varrho\, u \;=\; E^{(\varrho)} * (-\Delta \Phi + \tau\, \partial_1 \Phi + \varrho\, \Phi) \tag{3.30}$$

where the last equation follows from (3.21). After expressing the right–hand side of (3.30) as a limit like in (3.24), we integrate by parts and afterwards apply (3.25), (3.27)–(3.30). It follows that u satisfies (3.1).

This leaves us to establish (3.22). So, let $l, m \in \{1,\, 2,\, 3\}$. Theorem 1 with $r = 1$ and (3.26) yield

$$\big\|\, \big(\, \chi_{(0,2R)}\; \partial_l \partial_m (E^{(\varrho)} - E) \,\big) * \Phi \,\big\|_q \;\leq\; C(q,R)\; \|\Phi\|_q. \tag{3.31}$$

By Theorem 9, we have

$$|\partial_l \partial_m (E^{(\varrho)} - E)(\boldsymbol{z})| \;\leq\; C\, \big[\, |\boldsymbol{z}|^{-3} + |\boldsymbol{z}|^{-2} \big(1 + s(\boldsymbol{z})\big)^{-2} \,\big] \quad \text{for } \boldsymbol{z} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}.$$

Thus, Hölder's inequality and Lemma 5 now imply that

$$\big|\, \big[\, \big(\chi_{(2R,\infty)}\; \partial_l \partial_m (\, E^{(\varrho)} - E)\big) * \Phi \,\big](\boldsymbol{x}) \,\big| \;\leq\; C(q)\; \|\Phi\|_q \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3.$$

Hence

$$\big\|\, \big(\, \chi_{(2R,\infty)}\; \partial_l \partial_m (E^{(\varrho)} - E) \,\big) * \Phi \,\big\|_{2;\, B_R} \;\leq\; C(q,R)\; \|\Phi\|_q. \tag{3.32}$$

Combining (3.31), (3.32) and the last inequality in Theorem 3, we obtain (3.22). □

By the density argument, we may deduce from Theorem 10 and 11:

Corollary 1. *Let $q \in (1,2)$, $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$. Let $f \in L^q(\mathbb{R}^3)$, and put $u := E^{(\varrho)} * f$. Then $u \in W^{2,q}_{loc}(\mathbb{R}^3)$, $\partial_l u = (\partial_l E^{(\varrho)}) * f$ $(1 \leq l \leq 3)$, u satisfies (3.1), and*

$$\|\partial_l \partial_m u\|_{q;\,B_R} \;\leq\; C(q,R)\,\|f\|_q \quad \text{for } 1 \leq l, m \leq 3.$$

Moreover,

$$\partial_l (E^{(\varrho)} * h) \;=\; E^{(\varrho)} * \partial_l h \quad \text{for } l \in \{1,\,2,\,3\} \text{ and } h \in W^{1,q}(\mathbb{R}^3). \tag{3.33}$$

Due to Corollary 1, we need not distinguish between $\partial_l(E^{(\varrho)} * f)$ and $(\partial_l E^{(\varrho)}) * f$ for $f \in L^q(\mathbb{R}^3)$. Therefore we may write $\partial_l E^{(\varrho)} * f$ instead of $\partial_l(E^{(\varrho)} * f)$ or $(\partial_l E^{(\varrho)}) * f$.

We can use some of the preceding results in order to prove the uniqueness of solution of the scalar Oseen equation.

Theorem 12. *Let $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$. Suppose that $u \in W^{2,1}_{loc}(\mathbb{R}^3)$ satisfies the equation $-\Delta u + \tau\,\partial_1 u + \varrho\, u = 0$ and that $u\,|_{B_{R_0^c}} \in L^p(B_{R_0^c})$, $\nabla u\,|_{B_{R_0^c}} \in L^{\overline{p}}(B_{R_0^c})^3$ for some $R_0 \in (0,\infty)$ and some p, $\overline{p} \in (1,\infty)$. Then $u = 0$.*

Proof. Let $\Phi \in C_0^\infty(\mathbb{R}^3)$. Put $\overline{\boldsymbol{x}} := (-x_1, x_2, x_3)$, $\overline{\Phi}(\boldsymbol{x}) := \Phi(\overline{\boldsymbol{x}})$ and $w(\boldsymbol{x}) := (E^{(\varrho)} * \overline{\Phi})(\overline{\boldsymbol{x}})$ for $\boldsymbol{x} \in \mathbb{R}^3$. Then we know by Theorem 11 that $w \in C^\infty(\mathbb{R}^3)$ and that the equation $-\Delta w - \tau\,\partial_1 w + \varrho\, w = \Phi$ is satisfied.

Let $\overline{R} \in [R_0,\infty)$ with $\operatorname{supp}(\overline{\Phi}) \subset B_{\overline{R}}$. Note that $|\overline{\boldsymbol{x}} - \boldsymbol{y}| \geq |\boldsymbol{x}|/2$ for $\boldsymbol{x} \in B^c_{2\overline{R}}$, $\boldsymbol{y} \in B_{\overline{R}}$. Thus, by referring to Lemma 6 and Theorem 9, we get for $\boldsymbol{x} \in B^c_{2R}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$:

$$|\partial^\alpha w(\boldsymbol{x})| \;\leq\; C(\overline{R})\,\|\Phi\|_1\,|\boldsymbol{x}|^{-1-|\alpha|/2}\,\big(1 + s(\boldsymbol{x})\big)^{-1-|\alpha|/2}. \tag{3.34}$$

Moreover, due to our assumptions on $u\,|_{B_{R_0^c}}$ and $\nabla u\,|_{B_{R_0^c}}$, we may choose a sequence $\{R_n\}$ in $[\overline{R},\infty)$ such that $R_n \to \infty$ and the sequences $\{\|u\|_{p;\,\partial B_{R_n}}\}$ and $\{\|\nabla u\|_{\overline{p};\,\partial B_{R_n}}\}$ are bounded. However, by Hölder's inequality, (3.34) and Lemma 5, we obtain

$$\begin{aligned} &\int_{\partial B_{R_n}} \big(\,\big|u\,(\partial_l w)\big| + |uw| + \big|(\partial_l u)\,w\big|\big)\,\mathrm{d}S_x \\ &\quad\leq\; C(\overline{R})\,\|\Phi\|_1\,\big(\,\|u\|_{p;\,\partial B_{R_n}} + \|\nabla u\|_{\overline{p};\,\partial B_{R_n}}\,\big)\cdot R_n^{-\epsilon} \end{aligned} \tag{3.35}$$

for $n \in \mathbb{N}$, with some $\epsilon = \epsilon(p,\overline{p}) > 0$. Note that the right–hand side of (3.35) tends to zero for $n \to \infty$. We further find that

$$\int_{\mathbb{R}^3} u\,\Phi\,\mathrm{d}\boldsymbol{x} \;=\; \int_{B_{\overline{R}}} u\,\Phi\,\mathrm{d}\boldsymbol{x} \;=\; \lim_{n\to\infty}\int_{B_{R_n}} u\,(-\Delta w - \tau\,\partial_1 w + \varrho\, w)\,\mathrm{d}\boldsymbol{x}. \tag{3.36}$$

Integrating by parts on the right–hand side, we obtain an integral over B_{R_n} which vanishes because $-\Delta u + \tau\,\partial_1 u + \varrho\, u = 0$. Moreover, we obtain surface integrals on ∂B_{R_n} which tend to zero for $n \to \infty$ due to (3.35). Since Φ was chosen arbitrarily in $C_0^\infty(\mathbb{R}^3)$, we may conclude that $u = 0$. □

We further note

Theorem 13. *Let $f \in L^2(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \setminus \{\mathbf{0}\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$. Then $E^{(\lambda)} * f \in W^{2,2}(\mathbb{R}^3)$ and $\|\partial_l \partial_m (E^{(\lambda)} * f)\|_2 \leq \|f\|_2$ for $1 \leq l, m \leq 3$.*

Let $g \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in (1,2)$ (so that $E^{(0)} * g \in W^{2,p}_{loc}(\mathbb{R}^3)$ by Corollary 1). Then $\|\partial_l \partial_m (E^{(\lambda)} * g)\|_2 \leq \|g\|_2$ for $1 \leq l, m \leq 3$ and $\|\partial_1 E^{(\lambda)} * g)\|_2 \leq \|g\|_2$.

Proof. We know by inequality (3.15) that $E^{(\lambda)} * f \in L^2(\mathbb{R}^3)$. Denoting by $\hat{g}$ the Fourier transform of a function $g \in L^1(\mathbb{R}^3)$ (which means that $\hat{g}(\boldsymbol{\xi}) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\,\xi\cdot y}\, g(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$), we get $\hat{E}^{(\varrho)}(\boldsymbol{\xi}) = (2\pi)^{-3/2}\, (\varrho + |\boldsymbol{\xi}|^2 + \tau\,\mathrm{i}\,\xi_1)^{-1}$ for $\boldsymbol{\xi} \in \mathbb{R}^3$; compare with [23, p. 19–20]. Thus, if $\Phi \in C_0^\infty(\mathbb{R}^3)$, we get $\|\partial_l \partial_m (E^{(\lambda)} * \Phi)\|_2 \leq \|\Phi\|_2$ for $1 \leq l, m \leq 3$ by Plancherel's theorem. If $\operatorname{Im}\lambda = 0$, we further get $\|\partial_1 (E^{(\lambda)} * \Phi)\|_2 \leq \|\Phi\|_2$. Now the first part of the theorem may be shown by the density argument and, as concerns derivatives of order 1, by interpolation. The second part follows by a continuity argument with respect to λ. □

The following lemma is a consequence of Theorem 13.

Lemma 7. *The inequalities $\|\nabla E^{(\varrho)} * w\|_2 \leq C\, \|w\|_{-1,2}$ and $\|E^{(\varrho)} * w\|_p \leq C(p)\, \|w\|_{-1,2}$ hold for $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$, $p \in (4,6)$, $w \in C_0^\infty(\mathbb{R}^3)$.*

Moreover, if $\varrho = 0$ then the estimate $\|\partial_1 E^{(\varrho)} * w\|_{-1,2} \leq C\, \|w\|_{-1,2}$ holds for $w \in C_0^\infty(\mathbb{R}^3)$.

Proof. Let $w \in C_0^\infty(\mathbb{R}^3)$, and choose $\boldsymbol{g} = (g_1, g_2, g_3) := \boldsymbol{g}(w)$ as in Lemma 4. Then we find for $k \in \{1, 2, 3\}$, using (3.33), that

$$\|\partial_k E^{(\varrho)} * w\|_2 \;=\; \Big\| \sum_{l=1}^{3} \partial_k \partial_l (E^{(\varrho)} * g_l) \Big\|_2 \;\leq\; C\, \|\boldsymbol{g}\|_2 \;\leq\; C\, \|w\|_{-1,2}, \tag{3.37}$$

where we applied Theorem 13 in the last but one inequality and Lemma 4 in the last one. Moreover, referring to (3.33), inequality (3.17) with $q = 2$, and finally to Lemma 4, we find for $p \in (4,6)$

$$\|E^{(\varrho)} * w\|_p \;=\; \Big\| \sum_{l=1}^{3} (\partial_l E^{(\varrho)} * g_l) \Big\|_p \;\leq\; C(p)\, \|\boldsymbol{g}\|_2 \;\leq\; C(p)\, \|w\|_{-1,2}.$$

If $\varrho = 0$, we can prove the last inequality in Lemma 7 by an estimate as in (3.37), again based on (3.33), Theorem 13 and Lemma 4. □

By the density argument, we may now define convolutions of $E^{(\varrho)}$ with $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. The details are stated in the next corollary:

Corollary 2. *Let $\varrho \in \mathbb{C}$ with $\operatorname{Re}\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$. Then there is a unique linear mapping $\Gamma_\varrho : \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \mapsto \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ such that*

$$\Gamma_\varrho(\boldsymbol{w}) \;=\; E^{(\varrho)} * \boldsymbol{w} \qquad \text{for } \boldsymbol{w} \in C_0^\infty(\mathbb{R}^3)^3, \tag{3.38}$$

$$\|\nabla\Gamma_\varrho(\boldsymbol{w})\|_2 \;\leq\; C\,\|\boldsymbol{w}\|_{-1,2} \qquad \text{for } \boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3. \tag{3.39}$$

Moreover, operator Γ_ϱ satisfies the inequality

$$\|\Gamma_\varrho(\boldsymbol{w})\|_p \;\leq\; C(p)\,\|\boldsymbol{w}\|_{-1,2} \quad \text{for } \boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3,\; p \in (4,6], \tag{3.40}$$

and in the case $\varrho = 0$, $\partial_1\Gamma_\varrho(\boldsymbol{w})$ belongs to $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$.

If $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$ and $1 \leq l, m \leq 3$ then

$$\left.\begin{array}{l} \partial_l\Gamma_\varrho(\boldsymbol{w}) = (\partial_l E^{(\varrho)}) * \boldsymbol{w} \\ \Gamma_\varrho(\boldsymbol{w}) \in W^{2,1}_{loc}(\mathbb{R}^3)^3 \\ \partial_l\partial_m\Gamma_\varrho(\boldsymbol{w}) \in L^2(\mathbb{R}^3)^3 \end{array}\right\} \tag{3.41}$$

$$-\Delta\Gamma_\varrho(\boldsymbol{w}) + \tau\,\partial_1\Gamma_\varrho(\boldsymbol{w}) + \varrho\,\Gamma_\varrho(\boldsymbol{w}) \;=\; \boldsymbol{w}. \tag{3.42}$$

Furthermore, if $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ then $\operatorname{div}\Gamma_\varrho(\boldsymbol{w}) = 0$.

Finally, if $\boldsymbol{w} \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, or if $\varrho \neq 0$ and $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$, we have $\Gamma_\varrho(\boldsymbol{w}) = E^{(\varrho)} * \boldsymbol{w}$.

Proof. Let $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. By Lemma 2, there is a sequence $\{\boldsymbol{w}_n\}$ in $C_0^\infty(\mathbb{R}^3)^3$ with $\|\boldsymbol{w}_n - \boldsymbol{w}\|_{-1,2} \to 0$. Thus, by (3.18) and Lemma 3, the sequence $(E^{(\varrho)} * \boldsymbol{w}_n)$ converges in $L^6(\mathbb{R}^3)^3$, and Lemma 7 yields that the sequence $(\nabla E^{(\varrho)} * \boldsymbol{w}_n)$ converges in $L^2(\mathbb{R}^3)^3$. In the case $\varrho = 0$, Lemma 7 further yields the convergence of $\{\partial_1 E^{(\varrho)} * \boldsymbol{w}_n\}$ in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. These references additionally imply that the respective limit functions do not depend on the choice of the sequence $\{w_n\}$ such that $\boldsymbol{w}_n \to \boldsymbol{w}$ in $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. Thus, the linear operator $\Gamma_\varrho : \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \mapsto \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, verifying the first relation in (3.38), can be defined in an obvious way, and this operator satisfies the second relation in (3.38) as well as (3.40) with $q = 6$. Furthermore, due to Lemma 7, this operator fulfills (3.40) with $q \in (4, 6)$, and it also satisfies the inclusion $\Gamma_\varrho(\boldsymbol{w}) \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ if $\varrho = 0$.

Let $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$. Then we know by Lemma 2 that there is a sequence $\{\boldsymbol{w}_n\}$ in $C_0^\infty(\mathbb{R}^3)^3$ such that $\|\boldsymbol{w}_n - \boldsymbol{w}\|_* \to 0$. Inequality (3.17) and the relation $\|\boldsymbol{w}_n - \boldsymbol{w}\|_2 \to 0$ imply that

$$\|\partial_l E^{(\varrho)} * \boldsymbol{w}_n - (\partial_l E^{(\varrho)}) * \boldsymbol{w}\|_p \to 0 \;\; (n \to \infty) \quad \text{for } l \in \{1,\,2,\,3\},\; p \in (4,6).$$

On the other hand, since $\|\boldsymbol{w}_n - \boldsymbol{w}\|_{-1,2} \to 0$, we may conclude with (3.39) that

$$\|\partial_l E^{(\varrho)} * \boldsymbol{w}_n - \partial_l\Gamma_\varrho(\boldsymbol{w})\|_2 \to 0 \;\; (n \to \infty) \quad \text{for } l \in \{1,\,2,\,3\}.$$

We have thus proved the first identity in (3.41). The other statements in (3.41), as well as validity of equation (3.42), follow from Theorem 13, (3.39), (3.40) (the convergence of $(E^{(\varrho)} * \boldsymbol{w}_n)$ in $L^6(\mathbb{R}^3)^3$), and Theorem 11.

If $w \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, we may choose a sequence $\{\boldsymbol{\varphi}_n\}$ in $C_0^\infty(\mathbb{R}^3)^3$ with $\|\boldsymbol{\varphi}_n - \boldsymbol{w}\|_2 \to 0$ and $\operatorname{div} \boldsymbol{\varphi}_n = 0$ for $n \in \mathbb{N}$. Then $\partial_l \Gamma_\varrho(\boldsymbol{w}) = (\partial_l E^{(\varrho)}) * \boldsymbol{w}$ by (3.41), $\|(\partial_l E^{(\varrho)}) * (\boldsymbol{\varphi}_n - \boldsymbol{w})\|_p \to 0$ $(n \to \infty)$ for $p \in (4, 6)$, $1 \le l \le 3$ by (3.17), and $(\partial_l E^{(\varrho)}) * \boldsymbol{\varphi}_n = E^{(\varrho)} * \partial_l \boldsymbol{\varphi}_n$ for $n \in \mathbb{N}$, $1 \le l \le 3$ by (3.21). In this way we obtain that $\operatorname{div} \Gamma_\varrho(\boldsymbol{w}) = 0$.

If $\boldsymbol{w} \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, we have $\boldsymbol{w} \in L^2(\mathbb{R}^3)^3 \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ by Lemma 3, hence $\partial_l \Gamma_\varrho(\boldsymbol{w}) = \partial_l E^{(\varrho)} * \boldsymbol{w}$ by (3.41). On the other hand, the functions $\Gamma_\varrho(\boldsymbol{w})$ and and $E^{(\varrho)} * \boldsymbol{w}$ belong to $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, as follows from the definition of Γ_ϱ and (3.18), respectively. Now the inequality (2.10) implies $\Gamma_\varrho(\boldsymbol{w}) = E^{(\varrho)} * \boldsymbol{w}$. If $\varrho \neq 0$ and $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$, the preceding equation follows from (3.15). □

4. Estimates of operators $\mathfrak{B}$, $\mathfrak{B}_{\mathrm{sym}}$ and $\mathcal{P}\mathfrak{B}$, $\mathcal{P}\mathfrak{B}_{\mathrm{sym}}$

In this section, any generic constant may depend on τ and $\boldsymbol{U}$. Other quantities entering into these constants will be indicated explicitly.

The operators $\mathfrak{B}$ and $\mathfrak{B}_{\mathrm{sym}}$ were defined in Section 1 in the domain of $\mathcal{L}$. These operators can be naturally extended to the space $W^{1,1}_{loc}(\mathbb{R}^3)^3$.

Lemma 8. *If $q \in [6/5, 2]$ and $\boldsymbol{v} \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ then $\mathfrak{B}\boldsymbol{v} \in L^q(\mathbb{R}^3)^3$ and*

$$\|\mathcal{P}\mathfrak{B}\boldsymbol{v}\|_q \le C(q)\, \|\mathfrak{B}\boldsymbol{v}\|_q \le C(q)\, \|\nabla \boldsymbol{v}\|_2.$$

In particular, $\mathcal{P}\mathfrak{B}\boldsymbol{v} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$.

Proof. Take q, $\boldsymbol{v}$ as in the lemma. Then $(1/q - \frac{1}{6})^{-1} \in [\frac{3}{2}, 3]$ and $(1/q - \frac{1}{2})^{-1} \in [3, \infty]$, so that $\|\nabla \boldsymbol{U}\|_{(1/q-1/6)^{-1}} < \infty$ and $\|\boldsymbol{U}\|_{(1/q-1/2)^{-1}} < \infty$ by (2.14), (2.15) and (2.16). Thus, due to inequality (2.10), we have

$$\|\mathfrak{B}\boldsymbol{v}\|_q \le C\left(\|\nabla \boldsymbol{U}\|_{(1/q-1/6)^{-1}} \|\boldsymbol{v}\|_6 + \|\boldsymbol{U}\|_{(1/q-1/2)^{-1}} \|\nabla \boldsymbol{v}\|_2 \right) \le C(q)\, \|\nabla \boldsymbol{v}\|_2.$$

Now the first part of the lemma follows from Theorem 4. The last statement is a consequence of Lemma 3 and the fact that $\mathcal{P}$ maps $L^2(\mathbb{R}^3)^3$ into $H_2(\mathbb{R}^3)$ and $L^{6/5}(\mathbb{R}^3)^3$ into $H_{6/5}(\mathbb{R}^3)$ (see Theorem 5). □

Lemma 9. *Let $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$, $q \in (1, 2)$ and $\varrho \in \mathbb{C}$ with $\operatorname{Re} \varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. Then $\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big) \in L^2(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ and $\mathcal{P}\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big) \in H_2(\mathbb{R}^3) \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$. Moreover, if $\Phi \in L^q(\mathbb{R}^3)^3$ then $\mathfrak{B}(E^{(\varrho)} * \Phi) \in L^q(\mathbb{R}^3)^3$ and*

$$\|\mathcal{P}\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big)\|_* \le C\, \|\boldsymbol{w}\|_{-1,2}, \tag{4.1}$$

$$\|\mathcal{P}\mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \le C(q)\, \|\Phi\|_q. \tag{4.2}$$

Furthermore, there exist non–increasing functions D_1, $D_2^{(q)}$: $[0,\infty) \mapsto (0,\infty)$ *depending on* τ, $\boldsymbol{U}$, *and in the case of* $D_2^{(q)}$ *also on* q, *such that* $D_1(R) \to 0$, $D_2^{(q)}(R) \to 0$ *for* $R \to \infty$, *and*

$$\big\| \mathcal{P}\big[\, \chi_{B_R^c}\, \mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big)\,\big]\,\big\|_* \;\le\; D_1(R)\, \|\boldsymbol{w}\|_{-1,2}, \tag{4.3}$$

$$\big\| \mathcal{P}\big[\, \chi_{B_R^c}\, \mathfrak{B}(E^{(\varrho)} * \Phi)\,\big]\,\big\|_q \;\le\; D_2^{(q)}(R)\, \|\Phi\|_q \tag{4.4}$$

for $R \in (0,\infty)$, $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ *and* $\Phi \in L^q(\mathbb{R}^3)^3$.

Proof. Recall that the norm $\|\,.\,\|_*$ was defined in (2.12). Let $\widetilde{q} \in \{\frac{6}{5},\, q,\, 2\}$. Then $\widetilde{q} \le 2$, so $(1/\widetilde{q} - \frac{1}{3})^{-1} \le 6$. Moreover we have $\widetilde{q} > 1$, hence $(1/\widetilde{q} - \frac{3}{4})^{-1} > 4$ in the case $\widetilde{q} < \frac{4}{3}$. Obviously $(1/\widetilde{q} - \frac{1}{3})^{-1} < (1/\widetilde{q} - \frac{3}{4})^{-1}$ in that latter case. Thus we may choose $\overline{p} \in (4,6]$ with $(1/\widetilde{q} - \frac{1}{3})^{-1} \le \overline{p}$, and with $\overline{p} < (1/\widetilde{q} - \frac{3}{4})^{-1}$ in the case $\widetilde{q} < \frac{4}{3}$. As a consequence, $(1/\widetilde{q} - 1/\overline{p})^{-1} \in (\frac{4}{3},\, 3]$, so $\|\nabla U\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} < \infty$ by (2.14). Moreover $(1/\widetilde{q} - \frac{1}{2})^{-1} \in (2,\infty]$, hence $\|U\|_{(1/\widetilde{q}-\frac{1}{2})^{-1}} < \infty$ by (2.15) and (2.16). In addition, we get for $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$, $R \in [0,\infty)$,

$$\begin{aligned}
&\|\chi_{B_R^c}\, \mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big)\|_{\widetilde{q}} \\
&\quad\le\; C\,\big(\, \|\nabla \boldsymbol{U}\|_{(1/\widetilde{q}-1/\overline{p})^{-1};\, B_R^c}\, \|\Gamma_\varrho(\boldsymbol{w})\|_{\overline{p}} + \|\boldsymbol{U}\|_{(1/\widetilde{q}-1/2)^{-1};\, B_R^c}\, \|\nabla \Gamma_\varrho(\boldsymbol{w})\|_2\,\big) \\
&\quad\le\; C\,\big(\, \|\nabla \boldsymbol{U}\|_{(1/\widetilde{q}-1/\overline{p})^{-1};\, B_R^c} + \|\boldsymbol{U}\|_{(1/\widetilde{q}-1/2)^{-1};\, B_R^c}\big)\, \|\boldsymbol{w}\|_{-1,2},
\end{aligned} \tag{4.5}$$

where the last inequality follows from (3.39), (3.40) and the fact that $\overline{p} \in (4,6]$. Now we may conclude that $\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big) \in L^{\widetilde{q}}(\mathbb{R}^3)^3$ for $\widetilde{q} \in \{\frac{6}{5},\, q,\, 2\}$, $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$, so the inclusion $\mathcal{P}\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big) \in H_2(\mathbb{R}^3) \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ now follows from Theorem 4 and Lemma 3. The latter references, inequality (4.5) with $\widetilde{q} \in \{\frac{6}{5},\, 2\}$, as well as the inequalities $\|\nabla \boldsymbol{U}\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} < \infty$, $\|\boldsymbol{U}\|_{(1/\widetilde{q}-1/2)^{-1}} < \infty$ (see above) and $\|\boldsymbol{U}\,|\,B_R^c\|_\infty \to 0$ $(R \to \infty)$ (see Corollary (2.15) and (2.16)) yield that the first inequality in (4.1) is valid ($R = 0$ in (4.5)), and that there is a function D_1 with the properties stated in the lemma. Next take

$$\begin{aligned}
&\overline{r} \in \big(\,(1/q - \tfrac{1}{2})^{-1},\, \infty) && \text{if } q \ge \tfrac{3}{2}, \\
&\overline{r} \in \big(\,(1/q - \tfrac{1}{2})^{-1},\, (1/q - \tfrac{2}{3})^{-1}\,\big) && \text{if } q < \tfrac{3}{2}
\end{aligned}$$

and choose $r_0 \in \big(\,(1/q - \frac{1}{4})^{-1},\, (1/q - \frac{1}{3})^{-1}\,\big)$. Then $(1/q - 1/\overline{r})^{-1} \in (\frac{3}{2},\, 3)$, $(1/q - 1/r_0)^{-1} \in (3,\infty)$, and

$$\begin{aligned}
\|\chi_{B_R^c}\, \mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \;\le\; C\,\big(\, &\|\nabla \boldsymbol{U}\|_{(1/q-1/\overline{r})^{-1};\, B_R^c}\, \|E^{(\varrho)} * \Phi\|_{\overline{r}} \\
&+ \|\boldsymbol{U}\|_{(1/q-1/r_0)^{-1};\, B_R^c}\, \|\nabla E^{(\varrho)} * \Phi\|_{r_0}\,\big)
\end{aligned} \tag{4.6}$$

for $\Phi \in L^q(\mathbb{R}^3)^3$. Now the second estimate in (4.2) as well as inequality (4.4) follow from (4.6), (2.14)–(2.16), (3.16) and (3.17). □

The ensuing theorem is a key technical result of our theory. It will allow us to solve the resolvent problem (5.9) related to the perturbed Oseen system (5.2), under the assumption that the resolvent parameter λ is small (Theorem 19), and to establish resolvent estimates for small λ (Theorem 21).

Theorem 14. *Let $q \in (1,2)$. Then there exist functions D_3, $D_4^{(q)} : (0,\infty) \mapsto (0,\infty)$ depending on τ, $\boldsymbol{U}$, and in the case of $D_2^{(q)}$ also on q, such that*

$$\|\mathcal{P}\mathfrak{B}\big(\Gamma_\lambda(\boldsymbol{w})\big) - \mathcal{P}\mathfrak{B}\big(\Gamma_0(\boldsymbol{w})\big)\|_* \\ \leq \left\{ 2\, D_1(R) + D_3(R) \left[\frac{1}{\sqrt{\widetilde{R}}} - \frac{1}{2}\ln\Big(1 - \frac{1}{\widetilde{R}}\Big)\right] + D_3(\widetilde{R})\, |\lambda|^{1/3} \right\} \|\boldsymbol{w}\|_* \quad (4.7)$$

for $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$, $R \in (0,\infty)$, $\widetilde{R} \in [2R+1, \infty)$, $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$ and

$$\|\mathcal{P}\mathfrak{B}(E^{(\lambda)} * \Phi) - \mathcal{P}\mathfrak{B}(E^{(0)} * \Phi)\|_q \\ \leq \Big(2D_2^{(q)}(R) + D_4^{(q)}(R)\, \widetilde{R}^{-1+2/q'} + D_4^{(q)}(\widetilde{R})\, |\lambda|^{1/3} \Big) \|\Phi\|_q \quad (4.8)$$

for λ, R, $\widetilde{R}$ as in (4.7), and for $\Phi \in L^q(\mathbb{R}^3)^3$. (The functions D_1 and $D_2^{(q)}$ were introduced in Lemma 9.)

Proof. Let $\psi \in C^\infty(\mathbb{R})$ with $\psi|_{(-\infty,-1]} = 0$, $0 \leq \psi \leq 1$, $\psi|_{[0,\infty)} = 1$. For $R \in (1,\infty)$, $\boldsymbol{x} \in \mathbb{R}^3$, put $\psi_R(\boldsymbol{x}) := \psi(|\boldsymbol{x}| - R)$, so that $\psi_R \in C^\infty(\mathbb{R}^3)$ with $\psi_R|_{B_{R-1}} = 0$, $\psi|_{B_R^c} = 1$, and $|\nabla\psi_R(\boldsymbol{x})| \leq C$ for $\boldsymbol{x} \in \mathbb{R}^3$. Note that the upper bound of $|\nabla\psi_R(\boldsymbol{x})|$ is independent of R.

Take λ, R, $\widetilde{R}$ as in the theorem, and suppose that $\boldsymbol{g} \in C_0^\infty(\mathbb{R}^3)^3$. Then

$$\mathcal{P}\mathfrak{B}(E^{(\lambda)} * \boldsymbol{g}) - \mathcal{P}\mathfrak{B}(E^{(0)} * \boldsymbol{g}) = \sum_{i=1}^{3} \mathfrak{N}_i, \quad (4.9)$$

where $\mathfrak{N}_1 := \mathcal{P}\big(\chi_{B_R^c}\, \mathfrak{B}(E^{(\lambda)} * \boldsymbol{g} - E^{(0)} * \boldsymbol{g})\big)$ and

$$\mathfrak{N}_2 := \mathcal{P}\big[\chi_{B_R}\, \mathfrak{B}\big(E^{(\lambda)} * (\chi_{B_{\widetilde{R}}^c}\, \boldsymbol{g}) - E^{(0)} * (\chi_{B_{\widetilde{R}}^c}\, \boldsymbol{g})\big)\big],$$
$$\mathfrak{N}_3 := \mathcal{P}\big[\chi_{B_R}\, \mathfrak{B}\big(E^{(\lambda)} * (\chi_{B_{\widetilde{R}}}\, \boldsymbol{g}) - E^{(0)} * (\chi_{B_{\widetilde{R}}}\, \boldsymbol{g})\big)\big].$$

Let us abbreviate $\boldsymbol{u}^{(\lambda)} := E^{(\lambda)} * (\chi_{B_{\widetilde{R}}}\, \boldsymbol{g})$, $\boldsymbol{u}^{(0)} := E^{(0)} * (\chi_{B_{\widetilde{R}}}\, \boldsymbol{g})$. By (3.16), (3.17), Theorem 11 and 12, we have

$$\boldsymbol{u}^{(\lambda)} - \boldsymbol{u}^{(0)} = -E^{(0)} * \lambda\, \boldsymbol{u}^{(\lambda)}. \quad (4.10)$$

Take $\gamma \in (\frac{3}{2}, 2)$, for example $\gamma = \frac{7}{4}$, and set $s := \big[(1/\gamma - \frac{1}{4})^{-1} + (1/\gamma - \frac{1}{3})^{-1}\big]/2$. Then we get from (4.10) and (3.15)–(3.17) that

$$\|\boldsymbol{u}^{(\lambda)} - \boldsymbol{u}^{(0)}\|_\infty + \|\nabla(\boldsymbol{u}^{(\lambda)} - \boldsymbol{u}^{(0)})\|_s = \|E^{(0)} * \lambda\, \boldsymbol{u}^{(\lambda)}\|_\infty + \|\nabla E^{(0)} * \lambda\, \boldsymbol{u}^{(\lambda)}\|_s$$

$$\leq \quad C\,|\lambda|\,\|\boldsymbol{u}^{(\lambda)}\|_{\gamma} \;\leq\; C\,|\lambda|^{3-4/\gamma}\,\|\boldsymbol{g}\|_{1;\,B_{\widetilde{R}}} \;\leq\; C\,|\lambda|^{1/3}\,\|\boldsymbol{g}\|_{1,;\,B_{\widetilde{R}}}$$

where the last inequality holds because $\gamma > \frac{3}{2}$ and $|\lambda| \leq (\tau/2)^2$. Thus, due to Theorem 4, for $\widetilde{q} \in \{\frac{6}{5},\, q,\, 2\}$, we have

$$\begin{aligned}
\|\mathfrak{N}_3\|_{\widetilde{q}} \quad &\leq \quad C\,\Big(\|\nabla \boldsymbol{U}\|_{\widetilde{q};\,B_R}\,\|\boldsymbol{u}^{(\lambda)} - \boldsymbol{u}^{(0)}\|_{\infty} \\
&\qquad + \|\boldsymbol{U}\|_{(1/\widetilde{q}-1/s)^{-1};\,B_R}\,\|\nabla(\boldsymbol{u}^{(\lambda)} - \boldsymbol{u}^{(0)})\|_{s}\Big) \\
&\leq \quad C\,\big(\,R^{3/\widetilde{q}-1}\,\|\nabla \boldsymbol{U}\|_3 + R^{3/\widetilde{q}-3/s}\,\|\boldsymbol{U}\|_{\infty}\,\big)\,|\lambda|^{1/3}\,\|\boldsymbol{g}\|_{1;\,B_{\widetilde{R}}}.
\end{aligned}$$

We can conclude, using Lemma 3, that

$$\|\mathfrak{N}_3\|_{*} \quad \leq \quad C(\widetilde{R})\,|\lambda|^{1/3}\,\|\boldsymbol{g}\|_2, \tag{4.11}$$

$$\|\mathfrak{N}_3\|_{q} \quad \leq \quad C(q,\widetilde{R})\,|\lambda|^{1/3}\,\|\boldsymbol{g}\|_q. \tag{4.12}$$

As an immediate consequence of (4.3), (4.4), we get

$$\|\mathfrak{N}_1\|_{*} \quad \leq \quad 2D_1(R)\,\|\boldsymbol{g}\|_{-1,2}, \tag{4.13}$$

$$\|\mathfrak{N}_1\|_{q} \quad \leq \quad 2D_2^{(q)}(R)\,\|\boldsymbol{g}\|_q. \tag{4.14}$$

Let us now turn our attention to the estimate of $\mathfrak{N}_2$. At the beginning, we observe that for $\widetilde{q} \in \{q,\, \frac{6}{5},\, 2\}$, we have

$$\begin{aligned}
\|\mathfrak{N}_2\|_{\widetilde{q}} \quad &\leq \quad C\,\big(\|\nabla \boldsymbol{U}\|_{\widetilde{q};\,B_R} + \|U\|_{\widetilde{q};\,B_R}\big) \\
&\qquad \cdot \sum_{\varrho\in\{0,\,\lambda\}} \Big(\big\|\,\big[\,E^{(\varrho)} * (\chi_{B^c_{\widetilde{R}}}\,\boldsymbol{g})\,\big]\,\big\|_{\infty;\,B_R} + \big\|\,\big[\,\nabla E^{(\varrho)} * (\chi_{B^c_{\widetilde{R}}}\,\boldsymbol{g})\,\big]\,\big\|_{\infty;\,B_R}\Big) \\
&\leq \quad C(\widetilde{q},R)\,\big(\|\nabla \boldsymbol{U}\|_2 + \|\boldsymbol{U}\|_3\big) \\
&\qquad \cdot \sum_{\varrho\in\{0,\,\lambda\}}\,\sum_{j=1}^{3}\Bigg(\sup_{x\in B_R}\,\bigg|\int_{B^c_{\widetilde{R}}} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\,g_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg| \\
&\qquad + \sum_{l=1}^{3}\,\sup_{x\in B_R}\,\bigg|\int_{B^c_{\widetilde{R}}} \partial_l E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\,g_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg|\Bigg).
\end{aligned} \tag{4.15}$$

Now let $\boldsymbol{x} \in B_R$, $j \in \{1,\, 2,\, 3\}$. Then

$$\begin{aligned}
&\bigg|\int_{B^c_{\widetilde{R}}} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\,g_j(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg| \\
&\leq \quad \bigg|\int_{\mathbb{R}^3} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\,(\psi_R\,g_j)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg| + \bigg|\int_{B_{\widetilde{R}}\setminus B_{\widetilde{R}-1}} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\,(\psi_R\,g_j)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\bigg|
\end{aligned}$$

$$\leq \left| \int_{\mathbb{R}^3} \boldsymbol{v}_{\widetilde{R}}\, g_j \,\mathrm{d}\boldsymbol{y} \right| + \left(\int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y} \right)^{1/2} \|\boldsymbol{g}\|_{2;\, B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}}, \quad (4.16)$$

where $\boldsymbol{v}_{\widetilde{R}}(\boldsymbol{y}) := E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\ \psi_{\widetilde{R}}(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathbb{R}^3$. Note that $\psi_{\widetilde{R}}|_{B_{\widetilde{R}-1}} = 0$. This latter observation, (3.3), Lemma 5, 6 and the estimates

$$\begin{aligned} |\boldsymbol{x}-\boldsymbol{y}| \ &\geq\ \frac{|\boldsymbol{y}|}{2} + \frac{|\boldsymbol{y}|}{2} - |\boldsymbol{x}| \ \geq\ \frac{|\boldsymbol{y}|}{2} + (\widetilde{R}-1)/2 - |\boldsymbol{x}| \\ &\geq\ \frac{|\boldsymbol{y}|}{2} + R - |\boldsymbol{x}| \ \geq\ \frac{|\boldsymbol{y}|}{2} \end{aligned} \quad (4.17)$$

for $\boldsymbol{y} \in B^c_{\widetilde{R}-1}$ imply that $\boldsymbol{v}_{\widetilde{R}} \in C^\infty(\mathbb{R}^3) \cap \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$. Thus

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \boldsymbol{v}_{\widetilde{R}}\, g_j \,\mathrm{d}\boldsymbol{y} \right| \ &\leq\ \|\boldsymbol{g}\|_{-1,2} \left(\int_{B^c_{\widetilde{R}}} |\nabla_y \big(E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\ \psi_{\widetilde{R}}(\boldsymbol{y}) \big)|^2 \,\mathrm{d}\boldsymbol{y} \right)^{1/2} \\ &\leq\ C\, \|\boldsymbol{g}\|_{-1,2} \left(\int_{B^c_{\widetilde{R}-1}} |\boldsymbol{x}-\boldsymbol{y}|^{-3} \big(1 + s(\boldsymbol{x}-\boldsymbol{y}) \big)^{-3} \,\mathrm{d}\boldsymbol{y} \right. \\ &\qquad \left. + \int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |\boldsymbol{x}-\boldsymbol{y}|^{-2} \big(1 + s(\boldsymbol{x}-\boldsymbol{y}) \big)^{-2} \,\mathrm{d}\boldsymbol{y} \right)^{1/2} \\ &\leq\ C(R)\, \|\boldsymbol{g}\|_{-1,2} \left(\int_{B^c_{\widetilde{R}-1}} |\boldsymbol{y}|^{-3} \big(1 + s(\boldsymbol{y}) \big)^{-3} \,\mathrm{d}\boldsymbol{y} \right. \\ &\qquad \left. + \int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |\boldsymbol{y}|^{-2} \big(1 + s(\boldsymbol{y}) \big)^{-2} \,\mathrm{d}\boldsymbol{y} \right)^{1/2}, \end{aligned} \quad (4.18)$$

where the last inequality follows from (4.17) and Lemma 6. Now we apply Lemma 5 to obtain

$$\left| \int_{\mathbb{R}^3} \boldsymbol{v}_{\widetilde{R}}\, g_j \,\mathrm{d}\boldsymbol{y} \right| \ \leq\ C(R)\, \|\boldsymbol{g}\|_{-1,2} \left[\frac{1}{\sqrt{\widetilde{R}}} + \frac{1}{2} \ln\!\left(\frac{\widetilde{R}}{\widetilde{R}-1} \right) \right]. \quad (4.19)$$

Again using (3.3), (4.17), Lemma 6 and 5, we find that

$$\left(\int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})|^2 \,\mathrm{d}\boldsymbol{y} \right)^{1/2} \ \leq\ \frac{C(R)}{2} \ln\!\left(\frac{\widetilde{R}}{\widetilde{R}-1} \right). \quad (4.20)$$

Combining (4.16), (4.19) and (4.20), we get

$$\left| \int_{B^c_{\widetilde{R}}} E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\, g_j(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \right| \ \leq\ C(R)\, \|\boldsymbol{g}\|_* \left[\frac{1}{\sqrt{\widetilde{R}}} + \frac{1}{2} \ln\!\left(\frac{\widetilde{R}}{\widetilde{R}-1} \right) \right]. \quad (4.21)$$

A similar reasoning, albeit somewhat simpler because $\nabla E^{(\varrho)}$ decays more rapidly that $E^{(\varrho)}$, allows us to conclude that

$$\Big| \int_{B^c_{\widetilde{R}}} \partial_l E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\, g_j(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \Big| \le C(R)\, \frac{\|\boldsymbol{g}\|_*}{\sqrt{\widetilde{R}}} \qquad (1 \le l \le 3). \tag{4.22}$$

Here and in (4.21), $\boldsymbol{x}$ was an arbitrary point from B_R. Thus, we may conclude from (4.15), (4.21) and (4.22), for $\widetilde{q} \in \{\frac{6}{5}, 2\}$:

$$\|\mathfrak{N}_2\|_{\widetilde{q}} \le C(\widetilde{q}, R)\, \|\boldsymbol{g}\|_* \left[\frac{1}{\sqrt{\widetilde{R}}} + \frac{1}{2} \ln\Big(\frac{\widetilde{R}}{\widetilde{R}-1} \Big) \right], \tag{4.23}$$

hence by Lemma 3,

$$\|\mathfrak{N}_2\|_{-1,2} \le C(R)\, \|\boldsymbol{g}\|_* \left[\frac{1}{\sqrt{\widetilde{R}}} + \frac{1}{2} \ln\Big(\frac{\widetilde{R}}{\widetilde{R}-1} \Big) \right]. \tag{4.24}$$

Since $q < 2$ (hence $q' > 2$), we get by a much simpler computation, based on Hölder's inequality, (3.3), (4.17), Lemma 5 and 6, that

$$\begin{aligned} \Big| \int_{B^c_{\widetilde{R}}} \partial^\alpha E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})\, g_j(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \Big| &\le \Big(\int_{B^c_{\widetilde{R}}} |\partial^\alpha E^{(\varrho)}(\boldsymbol{x}-\boldsymbol{y})|^{q'}\, \mathrm{d}\boldsymbol{y} \Big)^{1/q'} \|\boldsymbol{g}\|_q \\ &\le C(q)\, \widetilde{R}^{-1-|\alpha|/2+2/q'}\, \|\boldsymbol{g}\|_q \end{aligned}$$

for $\boldsymbol{x} \in B_R$, $1 \le j \le 3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Now we again refer to (4.15), to obtain

$$\|\mathfrak{N}_2\|_q \le C(q, R)\, \|\boldsymbol{g}\|_q\, \widetilde{R}^{-1+2/q'}. \tag{4.25}$$

Next, using a density argument based on Lemma 2 and the first inequality in (4.1), we may deduce (4.7) from (4.9), (4.11), (4.13), (4.23) and (4.24). Finally the estimate in (4.8) follows from the second estimate in (4.2) (the density argument), (4.9), (4.12), (4.14) and (4.25). □

Lemma 10. *Let $q \in [1, \frac{6}{5}]$. Then $\|\mathfrak{B}_{\mathrm{sym}}(\Phi)\|_q \le C(q)\, \|\Phi\|_2$ for $\Phi \in L^2(\mathbb{R}^3)^3$. In particular, $\mathcal{P}\mathfrak{B}_{\mathrm{sym}}(\Phi)$ is well defined for $\Phi \in L^2(\mathbb{R}^3)^3$. Moreover $\mathfrak{B}_{\mathrm{sym}}(\Phi) \in L^2(\mathbb{R}^3)^3$ for $\Phi \in W^{2,2}(\mathbb{R}^3)^3$.*

Proof. Since $(1/q-1/2)^{-1} \in (2,3]$, we have by (2.14) that $\|\nabla \boldsymbol{U}\|_{(1/q-1/2)^{-1}} < \infty$. Thus, for $\Phi \in L^2(\mathbb{R}^3)^3$,

$$\|\mathfrak{B}_{\mathrm{sym}}(\Phi)\|_q \le C\, \|\nabla \boldsymbol{U}\|_{(1/q-1/2)^{-1}}\, \|\Phi\|_2 \le C(q)\, \|\Phi\|_2.$$

If $\Phi \in W^{2,2}(\mathbb{R}^3)^3$, the standard Sobolev inequality yields $\|\Phi\|_\infty \le C\, \|\Phi\|_{2,2}$, so $\|\mathfrak{B}_{\mathrm{sym}}(\Phi)\|_2 \le C\, \|\nabla \boldsymbol{U}\|_2\, \|\Phi\|_\infty < \infty$. □

Theorem 15. *Let $\xi \in \mathbb{R}$. Let σ be a positive eigenvalue of the operator $\Delta + \xi\, \mathcal{P}\mathfrak{B}_{\rm sym}$ and $\boldsymbol{f}$ be a corresponding eigenfunction. (I.e. $\boldsymbol{f}$ belongs to space H_2' – see Section 1.) Then $\boldsymbol{f} \in W^{2,s}(\mathbb{R}^3)^3$ for $s \in [1,\, \frac{6}{5}]$ and*

$$\|\nabla \boldsymbol{f}\|_2 \,\le\, C(\xi)\, \|\boldsymbol{f}\|_2, \qquad \|\boldsymbol{f}\|_{2,s} \,\le\, C(\xi, s, \sigma)\, \|\boldsymbol{f}\|_2. \tag{4.26}$$

In particular $\boldsymbol{f} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$.

Proof. A simple variational argument yields $\|\nabla \boldsymbol{f}\|_2^2 + \sigma\, \|\boldsymbol{f}\|_2^2 = \int_{\mathbb{R}^3} \boldsymbol{g} \cdot \overline{\boldsymbol{f}}\; \mathrm{d}\boldsymbol{x}$, where $\boldsymbol{g} := \xi\, \mathcal{P}\mathfrak{B}_{\rm sym} \boldsymbol{f}$. It follows from Lemma 10, (2.10) and Theorem 4 that

$$\|\nabla \boldsymbol{f}\|_2^2 + \sigma\, \|\boldsymbol{f}\|_2^2 \,\le\, |\xi|\, \|\mathcal{P}\mathfrak{B}_{\rm sym} \boldsymbol{f}\|_{6/5}\, \|\boldsymbol{f}\|_6 \,\le\, C(\xi)\, \|\boldsymbol{f}\|_2\, \|\nabla \boldsymbol{f}\|_2.$$

This implies that $\|\nabla \boldsymbol{f}\|_2 \le C(\xi)\, \|\boldsymbol{f}\|_2$.

We know due to Lemma 10 that $\boldsymbol{g} \in H_2(\mathbb{R}^3)$. Moreover $-\Delta \boldsymbol{f} + \sigma \boldsymbol{f} = \boldsymbol{g}$ and $\operatorname{div} \boldsymbol{f} = 0$ in $\mathbb{R}^3$, where the equation $\operatorname{div} \boldsymbol{f} = 0$ follows from Theorem 6. In this situation, we may conclude by means of Theorem 8 that

$$\boldsymbol{f} \;=\; \Big(\sum_{k=1}^{3} F_{jk}^{(\sigma)} * g_k \Big)_{1 \le j \le 3}$$

with $F_{jk}^{(\sigma)}$ introduced in that reference. However, $\boldsymbol{g} \in L^s(\mathbb{R}^3)^3$ for $s \in [1,\, \frac{6}{5}]$ according to Lemma 10, so Theorem 8 implies that $\boldsymbol{f} \in W^{2,s}(\mathbb{R}^3)^3$ and $\|\boldsymbol{f}\|_{2,s} \le C(s, \sigma)\, \|\boldsymbol{g}\|_s$ for $s \in (1,\, \frac{6}{5}]$. The second inequality in (4.26) now follows from Lemma 10. Since the case $s = \frac{6}{5}$ is admitted, Lemma 3 yields $\boldsymbol{f} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$. □

5. Solving the perturbed Oseen system (5.2) and the related resolvent problem (5.9).

In this section, we use the same convention on generic constants as in Section 4.

Let us start with a simple result from operator theory, which the reader can easily verify.

Lemma 11. *Let X, Y be vector spaces, $A : X \mapsto Y$ a linear and bijective operator and $B : X \mapsto Y$ a linear operator. Let $\mathfrak{I}_Y$ denote the identical mapping of Y onto itself. Then the operator $\mathfrak{I}_Y + B \circ A^{-1} : Y \mapsto Y$ is bijective if and only if $A + B : X \mapsto Y$ has the same property. If one (and hence both) of these statements is true, we have*

$$(A + B)^{-1} \,=\, A^{-1} \circ (\mathfrak{I}_Y + B \circ A^{-1})^{-1},$$
$$(\mathfrak{I}_Y + B \circ A^{-1})^{-1} \,=\, A \circ (A + B)^{-1}.$$

In the following, the role of A will be played by the operator $-\Delta + \tau\,\partial_1$, set up in a suitable function space, whereas B will correspond to $-\mathcal{P}\mathfrak{B}$. A suitable function space is given by

Theorem 16. *Let $\mathfrak{H}$ denote the space of all functions $\boldsymbol{v} \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3 \cap W^{2,1}_{loc}(\mathbb{R}^3)^3$ such that $\partial_l\partial_m \boldsymbol{v} \in L^2(\mathbb{R}^3)^3$ for $l, m \in \{1, 2, 3\}$, $\partial_1 \boldsymbol{v} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ and $\operatorname{div} \boldsymbol{v} = 0$. Define $\mathfrak{A}(\boldsymbol{v}) := -\Delta\boldsymbol{v} + \tau\,\partial_1\boldsymbol{v}$ for $\boldsymbol{v} \in \mathfrak{H}$. Then $\mathfrak{A} : \mathfrak{H} \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ is linear and bijective, with $\mathfrak{A}^{-1} = \Gamma_0\,|_{\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)}$.*

Proof. Obviously $\mathfrak{A}(\boldsymbol{v}) \in L^2(\mathbb{R}^3)^3$ and $\int_{\mathbb{R}^3} \mathfrak{A}(\boldsymbol{v}) \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x} = 0$ for $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3)^3$ and $\boldsymbol{v} \in \mathfrak{H}$. Thus, Theorem 6 yields $\mathfrak{A}(\boldsymbol{v}) \in H_2(\mathbb{R}^3)$ for $\boldsymbol{v} \in \mathfrak{H}$. It is obvious that $\Delta\boldsymbol{v} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$, so $\mathfrak{A}(\boldsymbol{v}) \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ ($\boldsymbol{v} \in \mathfrak{H}$). Therefore $\mathfrak{A} : \mathfrak{H} \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ is well defined. We know by Corollary 2 that $\Gamma_0(\boldsymbol{w}) \in \mathfrak{H}$ and $\mathfrak{A}\big(\Gamma_0(\boldsymbol{w})\big) = \boldsymbol{w}$ for $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$. This shows that $\mathfrak{A}$ is onto. Theorem 12 implies that $\mathfrak{A}$ is one–to–one. □

We further suppose that the following assumption (A1) is satisfied:

(A1) *For any $\ell \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ there is a unique function $\boldsymbol{u} \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ such that* $\operatorname{div} \boldsymbol{u} = 0$ *and*

$$\int_{\mathbb{R}^3} \big(\nabla\boldsymbol{u} \cdot \nabla\boldsymbol{v} + \tau\,\partial_1\boldsymbol{u} \cdot \boldsymbol{v} - \mathcal{P}\mathfrak{B}\boldsymbol{u} \cdot \boldsymbol{v} \big)\,\mathrm{d}\boldsymbol{x} \;=\; \ell(\boldsymbol{v}) \tag{5.1}$$

for all $\boldsymbol{v} \in C_0^\infty(\mathbb{R}^3)^3$ with $\operatorname{div} \boldsymbol{v} = 0$.

This means: we assume that the perturbed Oseen system

$$-\Delta\boldsymbol{u} + \tau\,\partial_1 u - \mathfrak{B}\boldsymbol{u} + \nabla\pi = \boldsymbol{g}, \quad \operatorname{div} \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^{\boldsymbol{3}} \tag{5.2}$$

admits a unique weak solution in the same way as the Oseen system does (compare with Theorem 7). We will now solve a version of (5.2) in which the pressure is eliminated.

Theorem 17. *The relation $\mathcal{P}\mathfrak{B}\boldsymbol{v} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ holds for $\boldsymbol{v} \in \mathfrak{H}$.*

Define $\widetilde{\mathfrak{A}} : \mathfrak{H} \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ by $\widetilde{\mathfrak{A}}(\boldsymbol{v}) := \mathfrak{A}(\boldsymbol{v}) - \mathcal{P}\mathfrak{B}\boldsymbol{v}$ for $\boldsymbol{v} \in \mathfrak{H}$. Then $\widetilde{\mathfrak{A}}$ is well defined, linear and bijective.

Proof. Since $\mathfrak{H} \subset \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, the first claim of the theorem holds according to Lemma 8. In view of Theorem 16, we may conclude that the operator $\widetilde{\mathfrak{A}} : \mathfrak{H} \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ is well defined. As an easy consequence of the uniqueness statement in (A1), we obtain that $\widetilde{\mathfrak{A}}$ is one–to–one. This leaves us to show that $\widetilde{\mathfrak{A}}$ is onto. To that end, take $\Phi \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, and let $\boldsymbol{u} \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ be the solution of (5.1) with ℓ given by

$$\ell(\boldsymbol{\varphi}) \;:=\; \int_{\mathbb{R}^3} \Phi \cdot \boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \qquad \text{for } \boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3)^3. \tag{5.3}$$

Then

$$\int_{\mathbb{R}^3} \big(\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\varphi} + \tau \, \partial_1 \boldsymbol{u} \cdot \boldsymbol{\varphi} \big) \, \mathrm{d}\boldsymbol{x} \;=\; \int_{\mathbb{R}^3} \boldsymbol{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} \quad \text{for } \boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3)^3, \tag{5.4}$$

with $\boldsymbol{f} := \Phi + \mathcal{P}\mathfrak{B}\boldsymbol{u}$. By the first statement of Theorem 17, we have $\boldsymbol{f} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)^3$. Due to this and Theorem 16, we know that there is $\widetilde{\boldsymbol{v}} \in \mathfrak{H}$ with $\mathfrak{A}(\widetilde{\boldsymbol{v}}) = \boldsymbol{f}$. Now (5.4) and the uniqueness result in Theorem 7 yield $\boldsymbol{u} = \widetilde{\boldsymbol{v}}$, hence $\boldsymbol{u} \in \mathfrak{H}$. At this point we may deduce from (5.4) and the definition of $\boldsymbol{f}$ that $\widetilde{\mathfrak{A}}(\boldsymbol{u}) = \Phi$. This proves that $\widetilde{\mathfrak{A}}$ is onto. □

Corollary 3. *The mapping* $\widetilde{Z}_0 : \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3) \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, *with* $\widetilde{Z}_0(\boldsymbol{w}) := \boldsymbol{w} - \mathcal{P}\mathfrak{B}\big(\Gamma_0(\boldsymbol{w})\big)$ *for* $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, *is well defined, linear, bijective and bounded.*

Proof. The operators $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$, from Theorem 16 and 17, respectively, are bijective, so we get by Lemma 11 and the first statement in Theorem 17 that the operator $\mathfrak{V}$ from the space $\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ into itself, with

$$\mathfrak{V}(\boldsymbol{w}) := \boldsymbol{w} - \mathcal{P}\mathfrak{B}\big(\mathfrak{A}^{-1}(\boldsymbol{w})\big) \quad (\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)),$$

is bijective. Since $\mathfrak{A}^{-1} = \Gamma_0 \,|_{\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)}$ by Theorem 16, we see that $\mathfrak{V} = \widetilde{Z}_0$, hence $\widetilde{Z}_0$ is bijective. The boundedness of $\widetilde{Z}_0$ follows from (4.1). □

Theorem 18. *Let* $q \in (1,2)$, *and define an operator* $Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ *by* $Z_0^{(q)}(\Phi) := \Phi - \mathcal{P}\mathfrak{B}(E^{(0)} * \Phi)$ *for* $\Phi \in L^q(\mathbb{R}^3)^3$. *Then* $Z_0^{(q)}$ *is well defined, linear, bounded and bijective.*

Proof. We know from Lemma 9 that $Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ is well defined and bounded. The claim that $Z_0^{(q)}$ is bijective will be proved by reducing it to the fact that $\widetilde{Z}_0$ is one–to–one (Corollary 3). To this end take $R \in (0,\infty)$ and define $\mathfrak{S}_R : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ by $\mathfrak{S}_R(\Phi) := \mathcal{P}\big(\chi_{B_R} \cdot \mathfrak{B}(E^{(0)} * \Phi)\big)$ for $\Phi \in L^q(\mathbb{R}^3)^3$. In order to show that $\mathfrak{S}_R$ is compact, we take a bounded sequence (Φ_n) in $L^q(\mathbb{R}^3)^3$. Then we may deduce from (3.16), (3.17) and Corollary 1 that the sequence $\big((E^{(0)} * \Phi_n)\,|_{B_R}\big)_{n\geq 1}$ is bounded in $W^{2,q}(B_R)^3$. On the other hand, let $\epsilon \in (0,1)$. Then, for $\Phi \in L^q(\mathbb{R}^3)^3$,

$$\begin{aligned} \|\mathfrak{S}_R(\Phi)\|_q \;\leq\; C\,\big(& \|\nabla \boldsymbol{U}\|_{a(\epsilon);\,B_R}\, \|(E^{(0)} * \Phi)\|_{b(\epsilon);\,B_R} \\ & + \|\boldsymbol{U}\|_{3+\epsilon;\,B_R}\, \|(\nabla E^{(0)} * \Phi)\|_{(\,1/q - 1/(3+\epsilon)\,)^{-1};\,B_R}\,\big), \end{aligned} \tag{5.5}$$

where $a(\epsilon) := q + \epsilon$, $b(\epsilon) := \big(1/q - 1/(q+\epsilon)\big)$ in the case $q \geq \frac{3}{2}$, and $a(\epsilon) := \frac{3}{2} + \epsilon$,

$b(\epsilon) := \big(1/q - 1/(\frac{3}{2}+\epsilon)\big)^{-1}$ if $q < \frac{3}{2}$. However,

$$\left(\frac{1}{q} - \frac{1}{\frac{3}{2}+\epsilon}\right)^{-1} < \frac{3q}{3-2q} \qquad \text{if } q < \frac{3}{2},$$
$$\left(\frac{1}{q} - \frac{1}{3+\epsilon}\right)^{-1} < \frac{3q}{3-q}.$$

Since the sequence $\big((E^{(0)} * \Phi_n)_{B_R}\big)_{n\geq 1}$ is bounded in $W^{2,q}(B_R)^3$, as noted above, we may now apply the standard theory of compact imbeddings in Sobolev spaces. This theory implies that there is a subsequence $\{\widetilde{\Phi}\}_n$ of $\{\Phi_n\}$ such that the sequence $\big\{\big((E^{(0)}*\widetilde{\Phi}_n)\,|_{B_R}\big\}_{n\geq 1}$ converges in $L^{\left(1/q-1/(q+\epsilon)\right)^{-1}}(B_R)^3$ (the case $q \geq \frac{3}{2}$), or in $L^{\left(1/q-1/(3/2+\epsilon)\right)^{-1}}(B_R)^3$ (the case $q < \frac{3}{2}$), respectively, and the sequence $\big\{(\nabla E^{(0)} * \widetilde{\Phi}_n)\,|_{B_R}\big\}_{n\geq 1}$ is convergent in $L^{\left(1/q-1/(3+\epsilon)\right)^{-1}}(B_R)^3$. In view of (5.5) and (2.14)–(2.16), we may conclude the sequence $\big\{\mathfrak{S}_R(\Phi_n)\big\}_{n\geq 1}$ converges in $L^q(\mathbb{R}^3)^3$. Thus we have shown that the operator $\mathfrak{S}_R : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ is compact. This is true for any $R > 0$. We further note that by (4.4), we may choose $R \in (0,\infty)$ so large that

$$\|\mathcal{P}\big(\chi_{B_R^c}\,\mathfrak{B}(E^{(0)} * \Phi)\big)\|_q \leq (2\tau)^{-1}\,\|\Phi\|_q \quad \text{for } \Phi \in L^q(\mathbb{R}^3)^3. \tag{5.6}$$

Let us now fix such a value R. Then the operator

$$\mathfrak{G}_R : L^q(\mathbb{R}^3)^3 \ni \Phi \mapsto \Phi - \mathcal{P}\big(\chi_{B_R^c}\,\mathfrak{B}(E^{(0)} * \Phi)\big) \in L^q(\mathbb{R}^3)^3 \tag{5.7}$$

is one–to–one. Moreover, a simple fixed point argument based on (5.6) yields that $\mathfrak{G}_R$ is onto. Thus $\mathfrak{G}_R$ is linear and bijective, in particular with Fredholm's index zero. Since $\mathfrak{S}_R$ is compact and $Z_0^{(q)} = \mathfrak{G}_R + \mathfrak{S}_R$, we may conclude that $Z_q^{(0)}$ is Fredholm with index zero.

Let us now show that $Z_0^{(q)}$ is one–to–one. To this end, take $\Phi \in L^q(\mathbb{R}^3)^3$ with $Z_0^{(q)}(\Phi) = 0$. Let $(1/q - \frac{1}{4})^{-1} < \overline{p} < (1/q - \frac{1}{3})^{-1}$ with $\overline{p} > \frac{3}{2}$,

$$p \in \big((1/q - \tfrac{1}{2})^{-1},\, \infty\big) \qquad \text{if } q \geq \tfrac{3}{2},$$
$$p \in \big((1/q - \tfrac{1}{2})^{-1},\, (1/q - \tfrac{2}{3})^{-1}\big) \qquad \text{else.}$$

Since $\overline{p} < (1/q - \frac{1}{3})^{-1}$, we have $\frac{2}{3} - 1/\overline{p} < \frac{1}{2}$. Obviously $(\frac{2}{3} + 1/p)^{-1} < \frac{3}{2}$. Thus we may choose

$$\gamma_0 \in (1,\, \tfrac{3}{2}) \cap \big((\tfrac{2}{3} + 1/p)^{-1},\, \tfrac{3}{2}\big) \quad \text{with } 1/\gamma_0 - 1/\overline{p} < \tfrac{1}{2}.$$

Then $\gamma_0 < \frac{3}{2} < \overline{p}$, so the last relation implies $(1/\gamma_0 - 1/\overline{p})^{-1} > 2$, hence $\|\boldsymbol{U}\|_{(1/\gamma_0 - 1/\overline{p})^{-1}} < \infty$ by (2.15), (2.16). Since $\gamma_0 > (\frac{2}{3} + 1/p)^{-1}$, we further have $(1/\gamma_0 - 1/p)^{-1} > \frac{3}{2}$.

Now suppose that $q \geq \frac{3}{2}$. Then $p > (1/q - \frac{1}{2})^{-1} > 3$. On the other hand, $1/\gamma_0 - \frac{1}{3} > \frac{1}{3}$, so we may conclude that $(1/\gamma_0 - 1/p)^{-1} < 3$. It follows from (2.14) that $\|\nabla \boldsymbol{U}\|_{(1/\gamma_0 - 1/p)^{-1}} < \infty$. Now we get from (3.16), (3.17),

$$\begin{aligned} \mathfrak{B}(E^{(0)} * \Phi)\|_{\gamma_0} \;&\leq\; C\big(\,\|\nabla \boldsymbol{U}\|_{(1/\gamma_0-1/p)^{-1}}\, \|E^{(0)} * \Phi\|_p \\ &\quad + \|\boldsymbol{U}\|_{(1/\gamma_0-1/\overline{p})^{-1}}\, \|\nabla E^{(0)} * \Phi\|_{\overline{p}}\big) \;\leq\; C(p,\overline{p})\, \|\Phi\|_q. \end{aligned} \tag{5.8}$$

Since $Z_0^{(q)}(\Phi) = 0$, we may conclude with Theorem 4 that $\Phi \in L^{\gamma_0}(\mathbb{R}^3)^3$. Thus there is always some $q_1 \in (1, \frac{3}{2})$ with $\Phi \in L^{q_1}(\mathbb{R}^3)^3$.

Let $\overline{p}_1 \in \big((1/q_1 - \frac{1}{4})^{-1}, (1/q_1 - \frac{1}{3})^{-1}\big)$. Since $q_1 < \frac{3}{2}$, we have $(1/q_1 - \frac{1}{2})^{-1} < 6$, so we may choose $p_1 \in \big((1/q_1 - \frac{1}{2})^{-1}, (1/q_1 - \frac{2}{3})^{-1}\big)$ with $p_1 < 6$. Then $\big(\frac{5}{6} - 1/\overline{p}_1\big)^{-1} > 2$, $\big(\frac{5}{6} - 1/p_1\big)^{-1} \in (\frac{3}{2}, 3)$, so that by (2.14)–(2.16), $\|\boldsymbol{U}\|_{(5/6-1/\overline{p}_1)^{-1}} < \infty$ and $\|\nabla \boldsymbol{U}\|_{(5/6-1/p_1)^{-1}} < \infty$. As a consequence, by an estimate as in (5.8), and by referring again to (3.15), (3.16), we get $\|\mathfrak{B}(E^{(0)} * \Phi)\|_{6/5} \leq C(\overline{p}_1, p_1, q_1)\, \|\Phi\|_{q_1}$. Thus we have found that $\mathfrak{B}(E^{(0)} * \Phi) \in L^{6/5}(\mathbb{R}^3)^3$. In view of Theorem 4 and the assumption $Z_0^{(q)}(\Phi) = 0$, we thus arrive at the relation $\Phi \in L^{6/5}(\mathbb{R}^3)^3$. Now inequality (3.18) yields $E^{(0)} * \Phi \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, hence $\mathcal{P}\mathfrak{B}(E^{(0)} * \Phi) \in H_2(\mathbb{R}^3) \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$ by Lemma 8. Since $Z_0^{(q)}(\Phi) = 0$, we thus obtain $\Phi \in H_2(\mathbb{R}^3) \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, so Corollary 2 implies $Z_0^{(q)}(\Phi) = \widetilde{Z}_0(\Phi)$. Therefore $\widetilde{Z}_0(\Phi) = 0$, and we may conclude with Corollary 3 that $\Phi = 0$. This proves that $Z_0^{(q)}$ is one–to–one. However, a Fredholm operator with index zero which in addition is one–to–one is bijective, so the proof of Theorem 18 is completed. □

Corollary 4. *Let $q \in (1,2)$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$. Then the operators*

$$\begin{aligned} \widetilde{Z}_\lambda \quad &: \quad \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3) \ni \Phi \\ &\qquad \longmapsto \Phi - \mathcal{P}\mathfrak{B}(E^{(\lambda)} * \Phi) \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3), \\ Z^{(q)} \quad &: \quad L^q(\mathbb{R}^3)^3 \ni \Phi \mapsto \Phi - \mathcal{P}\mathfrak{B}(E^{(\lambda)} * \Phi) \in L^q(\mathbb{R}^3)^3 \end{aligned}$$

are well defined, linear and bounded. If $\boldsymbol{\psi}$, $\boldsymbol{g} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ with $\widetilde{Z}_\lambda(\boldsymbol{\psi}) = \boldsymbol{g}$, and if we set $\boldsymbol{u} := E^{(\lambda)} * \boldsymbol{\psi}$, then $\boldsymbol{u} \in W^{2,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3$ and

$$-\Delta \boldsymbol{u} + \tau\, \partial_1 \boldsymbol{u} + \lambda\, \boldsymbol{u} - \mathcal{P}\mathfrak{B}\boldsymbol{u} \;=\; \boldsymbol{g}, \quad \operatorname{div} \boldsymbol{u} = 0. \tag{5.9}$$

If, in addition, $\boldsymbol{g} \in L^q(\mathbb{R}^3)^3$ then the relations $\boldsymbol{\psi} \in L^q(\mathbb{R}^3)^3$, $Z_\lambda^{(q)}(\boldsymbol{\psi}) = \boldsymbol{g}$ hold.

Proof. The corollary follows from Lemma 9, Theorem 13 and Corollary 2. In particular, the last statement is a consequence of the inclusion $\mathcal{P}\mathfrak{B}\big(\Gamma_\varrho(\boldsymbol{w})\big) \in H_2(\mathbb{R}^3) \cap \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ (for $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3$) in Lemma 9. □

Theorem 19. *There exists* $\epsilon_1 \in (0, (\tau/2)^2]$*, depending on* $\boldsymbol{U}$*, such that* $\widetilde{Z}_\lambda : \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3) \mapsto \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ *is bijective and* $\|\Phi\|_* \leq C\,\|\widetilde{Z}_\lambda(\Phi)\|_*$ *for* $\Phi \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$*,* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\operatorname{Re}\lambda \geq 0$*,* $|\lambda| \leq \epsilon_1$*.*

Let $q \in (1, 2)$*. Then there exists* $\epsilon_2(q) \in (0, (\tau/2)^2]$*, depending on* $\boldsymbol{U}$ *and* q*, such that the operator* $Z_\lambda^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ *is bijective and* $\|\Phi\|_q \leq C(q)\,\|Z_\lambda^{(q)}(\Phi)\|_q$ *for* $\Phi \in L^q(\mathbb{R}^3)^3$*,* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\operatorname{Re}\lambda \geq 0$*,* $|\lambda| \leq \epsilon_2(q)$*.*

Proof. By Corollary 3 and the open mapping theorem, there exists $C_0 > 0$ such that

$$\|\boldsymbol{w}\|_* \leq C_0\,\|\widetilde{Z}_0(\boldsymbol{w})\|_* \quad \text{for } \boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3). \tag{5.10}$$

Similarly, by Theorem 18, there is $\widetilde{C}_0(q) > 0$ such that

$$\|\Phi\|_q \leq \widetilde{C}_0(q)\,\|Z_0^{(q)}(\Phi)\|_q \quad \text{for } \Phi \in L^q(\mathbb{R}^3)^3. \tag{5.11}$$

Now, in view of (4.7), we may choose $R > 0$ so large that $2D_1(R) \leq (12\,C_0\,\tau)^{-1}$, with $D_1(R)$ from Lemma 9. Since $\ln(1/(1 - 1/\widetilde{R})) \to 0$ ($\widetilde{R} \to \infty$), we may fix some $\widetilde{R} \in [2R + 1, \infty)$ such that

$$D_3(R)\left[\frac{1}{\sqrt{\widetilde{R}}} + \frac{1}{2}\ln\!\left(\frac{1}{1 - 1/\widetilde{R}}\right)\right] \leq \frac{1}{12\,C_0\,\tau},$$

where the constant $D_3(R)$ was introduced in Theorem 14. Finally we choose $\epsilon_1 \in (0, (\tau/2)^2]$ so small that $D_3(\widetilde{R}) \cdot \epsilon_1^{1/3} \leq (12 \cdot C_0 \cdot \tau)^{-1}$. Then it follows from (4.7) and the last statement of Corollary 2 that

$$\|\widetilde{Z}_\lambda(\boldsymbol{w}) - \widetilde{Z}_0(\boldsymbol{w})\|_* \leq \frac{1}{4\,C_0}\,\|\boldsymbol{w}\|_* \tag{5.12}$$

for $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq \epsilon_1$. Thus, we may deduce with (5.10) and a simple shoestring argument that $\|\boldsymbol{w}\|_* \leq 2C_0\,\|\widetilde{Z}_\lambda(\boldsymbol{w})\|_*$.

Let $\boldsymbol{g} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, and put $\Phi_0 := (\widetilde{Z}_0)^{-1}(\boldsymbol{g})$, $\Phi_{n+1} := (\widetilde{Z}_0)^{-1}\big(\boldsymbol{g} - (\widetilde{Z}_\lambda - \widetilde{Z}_0)(\Phi_n)\big)$ for $n \in \mathbb{N}_0$. Then (5.10) and (5.12) yield the convergence of the sequence $\{\Phi_n\}$ in the norm $\|\,.\,\|_*$. The limit function Φ verifies the equation $\widetilde{Z}_\lambda(\Phi) = \boldsymbol{g}$. This proves that $\widetilde{Z}_\lambda$ is bijective. An analogous argument based on (5.11) and (4.8) implies existence of $\epsilon_2(q) \in (0, (\tau/2)^2]$ with the desired properties. □

6. Resolvent estimates for the perturbed Oseen system (5.2)

In the rest of this article, we write $\mathfrak{I}$ for the identical mapping of $H_2(\mathbb{R}^3)$ onto itself. Put $D(\mathcal{L}) := H_2(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3)^3$. Since $W^{2,2}(\mathbb{R}^3)^3 \subset \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ and because of

Lemma 8, the term $\mathcal{P}\mathfrak{B}\boldsymbol{v}$ is well defined and belongs to $H_2(\mathbb{R}^3)$ for $\boldsymbol{v} \in D(\mathcal{L})$, so we may define

$$\mathcal{L}\boldsymbol{v} := \Delta\boldsymbol{v} - \tau\,\partial_1\boldsymbol{v} + \mathcal{P}\mathfrak{B}\boldsymbol{v} \quad (\boldsymbol{v} \in D(\mathcal{L})). \tag{6.1}$$

Then $\mathcal{L} : D(\mathcal{L}) \mapsto H_2(\mathbb{R}^3)$ is linear and densely defined in $H_2(\mathbb{R}^3)$. We will use the usual notation $\varrho(\mathcal{L})$ for the resolvent set of $\mathcal{L}$.

Note that if $\boldsymbol{g} \in L^2(\mathbb{R}^3)^3$, $\boldsymbol{u} \in D(\mathcal{L})$ with $\mathcal{L}\boldsymbol{u} = \mathcal{P}\boldsymbol{g}$, Theorem 4 yields some $\pi \in W^{1,1}_{loc}(\mathbb{R}^3)$ such that the pair $(\boldsymbol{u}, \pi)$ solves the perturbed Oseen problem (5.2). Thus, estimates of the operator $(\lambda\mathfrak{I} - \mathcal{L})^{-1}$, for $\lambda \in \varrho(\mathcal{L})$, correspond to estimates of solutions of the resolvent problem (5.9).

The ensuing theorem is due to [3], [10], [17, Theorem 1.3.2].

Theorem 20. *There is at most a countable set* $\mathfrak{K} \subset \mathbb{C}$ *such that* $\mathrm{Sp}(\mathcal{L}) \smallsetminus \mathfrak{K} \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \leq -(\mathrm{Im}\,\lambda)^2/\tau^2\}$. *Set* $\mathfrak{K}$ *consists of eigenvalues of operator* $\mathcal{L}$.

There exist $a \in (0, \infty)$ *and* $\vartheta \in (\pi/2, \pi)$ *such that*

$$S_{\vartheta,a} := \big\{\lambda \in \mathbb{C} \smallsetminus \{a\};\ |\arg(\lambda - a)| \leq \vartheta\big\} \subset \varrho(\mathcal{L}).$$

From now, **we assume that operator $\mathcal{L}$ satisfies condition (A2)** – see Section 1 for its formulation. It means, in particular, that $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \mathfrak{K}$. Note that by (A2) and Theorem 20, we have

$$\big\{\lambda \in \mathbb{C} \smallsetminus \{0\};\ \mathrm{Re}\,\lambda \geq 0\big\} \cup \big(\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < 0\} \cap S_{\vartheta,a}\big) \subset \varrho(\mathcal{L}). \tag{6.2}$$

In this section and in Section 7, we write C for constants which may depend on τ, $\boldsymbol{U}$, a or ϑ. As usual in this article, if such a constant depends on additional quantities γ_1, ..., γ_n, for some $n \in \mathbb{N}$, we denote it by $C(\gamma_1, ..., \gamma_n)$.

Lemma 12. *Let* $\lambda \in \mathbb{C} \smallsetminus \{0\}$ *with* $\mathrm{Re}\,\lambda > 0$ *and* $|\lambda| \leq \epsilon_1$, *with* ϵ_1 *from Theorem 19. Take* $\boldsymbol{g} \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$. *Then* $(\lambda\mathfrak{I} - \mathcal{L})^{-1}\boldsymbol{g} = E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{g})$ *and* $(\lambda\,\mathfrak{I} - \mathcal{L})^{-1}\boldsymbol{g} \in \mathfrak{D} \cap D(\mathcal{L})$.

Proof. We have to compare $\boldsymbol{u} := (\lambda\mathfrak{I} - \mathcal{L})^{-1}\boldsymbol{g}$ and $\widetilde{\boldsymbol{u}} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{g})$. Corollary 4 and Theorem 6 yield that $\widetilde{\boldsymbol{u}} \in D(\mathcal{L})$ and $(\lambda\mathfrak{I} - \mathcal{L})(\widetilde{\boldsymbol{u}}) = \boldsymbol{g}$. Since $\lambda \in \varrho(\mathcal{L})$ by (6.2), it follows that $\boldsymbol{u} = \widetilde{\boldsymbol{u}}$. Observing that $D(\mathcal{L}) \subset \mathfrak{D}^{1,2}_0(\mathbb{R}^3)^3$ and $\boldsymbol{u} = (1/\lambda) \cdot (\mathcal{L}\boldsymbol{u} + \boldsymbol{g})$, and recalling Lemma 8, we obtain $\boldsymbol{u} \in \mathfrak{D}^{-1,2}_0(\mathbb{R}^3)^3$. □

The next theorem is the crucial element of our theory; it states resolvent estimates for the perturbed Oseen system (5.2), under the assumption that the resolvent parameter λ has a small modulus and non–negative real part.

Theorem 21. *The inequality* $\|\nabla(\lambda\mathfrak{I} - \mathcal{L})^{-1}\boldsymbol{g}\|_2 \leq C\,\|\boldsymbol{g}\|_*$ *holds for* $\boldsymbol{g} \in \mathfrak{D}^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \smallsetminus \{0\}$ *with* $\mathrm{Re}\,\lambda \geq 0$, $|\lambda| \leq \epsilon_1$, *where* ϵ_1 *was introduced in Theorem 19.*

Let $s \in (1, \frac{6}{5}]$, $\delta \in (0,1]$. *Then there is* $\epsilon_3(s,\delta) \in (0,\epsilon_1]$, *also depending on* τ *and* $\boldsymbol{U}$, *such that for* $\boldsymbol{f} \in L^s(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, $R \in (0,\infty)$, $\lambda \in \mathbb{C}\setminus\{0\}$ *with* $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq \epsilon_3(s,\delta)$, *the ensuing estimates hold:*

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-2}\boldsymbol{f}\|_{2;\,B_R} + \big\|\nabla\big[\,(\overline{\lambda}\,\mathfrak{I}-\mathcal{L})^{-1}\circ(\lambda\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\big]\,\big\|_{2;\,B_R} \\ \leq \quad C(s,\delta,R)\,|\lambda|^{-4(1-1/s)-\delta}\,\|\boldsymbol{f}\|_s, \tag{6.3}$$

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_{2;\,B_R} \;\leq\; C(s,\delta,R)\,|\lambda|^{-2-4(1-1/s)-\delta}\,\|\boldsymbol{f}\|_s. \tag{6.4}$$

Proof. Take $\boldsymbol{g} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ and $\lambda \in \mathbb{C}\setminus\{0\}$ with $\operatorname{Re}\lambda \geq 0$, $|\lambda| \leq \epsilon_1$. Then we get by Lemma 12, Corollary 2 and Theorem 19 that

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}\|_2 \;=\; \|E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{g})\|_2 \;\leq\; C\,\|(\widetilde{Z}_\lambda)^{-1}(\boldsymbol{g})\|_{-1,2} \;\leq\; C\,\|\boldsymbol{g}\|_*.$$

This proves the first claim of the theorem. Now let $\boldsymbol{f} \in L^s(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, and take λ as before. Then $\boldsymbol{f} \in L^{6/5}(\mathbb{R}^3)^3$ by interpolation, hence $\boldsymbol{f} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$. Thus we may define $\boldsymbol{u}^{(1)} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{f})$, and obtain $\boldsymbol{u}^{(1)} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3) \cap W^{2,2}(\mathbb{R}^3)^3$ by Lemma 12. Repeating this argument, we put $\boldsymbol{u}^{(i+1)} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{u}^{(i)})$ for $i \in \{1,\,2\}$, $\boldsymbol{w} := E^{(\overline{\lambda})} * (\widetilde{Z}_\lambda)^{-1}(\boldsymbol{u}^{(1)})$, and obtain $\boldsymbol{u}^{(2)}$, $\boldsymbol{u}^{(3)}$, $\boldsymbol{w} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap W^{2,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$. Moreover Lemma 12 yields

$$\boldsymbol{u}^{(i)} = (\lambda\mathfrak{I}-\mathcal{L})^{-i}\boldsymbol{f}, \quad \boldsymbol{w} = (\overline{\lambda}\,\mathfrak{I}-\mathcal{L})^{-1}\circ(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}. \tag{6.5}$$

Take $q \in [\frac{4}{3},\,2)$, and suppose that $|\lambda| \leq \min\{\,\epsilon_2(r);\; r \in \{s,q\}\,\}$, with $\epsilon_2(q)$, $\epsilon_2(s)$ from Theorem 19. Put $p := \big((1/q-\frac{1}{4})^{-1} + (1/q-\frac{1}{3})^{-1}\big)$. Since $q \geq \frac{4}{3}$, we have $p \geq 2$. (Actually only values of q close to 2 are of interest because it is them who lead to values of δ close to 0 in (6.3) and (6.4), as will be seen below.)

Since $\boldsymbol{f} \in \mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)^3$, as explained above, we have $(\widetilde{Z}_\lambda)^{-1}(\boldsymbol{f}) = (Z_\lambda^{(s)})^{-1}(\boldsymbol{f})$ by Corollary 4. In addition, inequality (3.15) implies $\boldsymbol{u}^{(1)} \in L^s(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$. Again referring to Corollary 4 and (3.15), we may conclude that $\boldsymbol{u}^{(2)} \in L^q(\mathbb{R}^3)^3$,

$$\left.\begin{aligned} &(\widetilde{Z}_\lambda)^{-1}(\boldsymbol{u}^{(1)}) = (Z_\lambda^{(r)})^{-1}(\boldsymbol{u}^{(1)}) \qquad \text{for } r \in \{q,s\},\\ &(\widetilde{Z}_\lambda)^{-1}(\boldsymbol{u}^{(2)}) = (Z_\lambda^{(q)})^{-1}(\boldsymbol{u}^{(2)}). \end{aligned}\right\} \tag{6.6}$$

Recalling (6.5), and applying (3.17), (6.6), Theorem 19 and (3.15), we find that

$$\begin{aligned} &\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-2}\boldsymbol{f}\|_{2;\,B_R} \;\leq\; \|\nabla\boldsymbol{u}^{(2)}\|_{2;\,B_R} \;\leq\; C(R)\,\|\nabla\boldsymbol{u}^{(2)}\|_p \\ &\quad\leq\quad C(R,p,q)\,\|(Z_\lambda^{(q)})^{-1}(\boldsymbol{u}^{(1)})\|_q \;\leq\; C(R,p,q)\,\|\boldsymbol{u}^{(1)}\|_q \\ &\quad\leq\quad C(R,p,q,s)\,|\lambda|^{2-4(1-1/s+1/q)}\;\|(Z_\lambda^{(s)})^{-1}(\boldsymbol{f})\|_s \\ &\quad\leq\quad C(R,p,q,s)\,|\lambda|^{2-4(1-1/s+1/q)}\;\|\boldsymbol{f}\|_s \end{aligned}$$

$$= \quad C(R,p,q,s)\,|\lambda|^{-4(1-1/s)-\delta}\,\|\boldsymbol{f}\|_s, \tag{6.7}$$

with $\delta := 4(1/q-1/2)$. An analogous argument, starting with (6.5), yields

$$\|(\overline{\lambda}\cdot\mathfrak{I}-\mathcal{L})^{-1}\circ(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\|_2 \,\le\, C(R,p,q,s)\,|\lambda|^{-4(1-1/s)-\delta}\,\|\boldsymbol{f}\|_s.$$

Thus, since q may be chosen arbitrarily in $[\frac{4}{3},\,2)$, we have proved (6.3). In order to estimate $(\lambda\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}$, we again proceed as in (6.7), but with $\boldsymbol{f}$ replaced by $\boldsymbol{u}^{(1)}$. Note that $\boldsymbol{f}$ may in fact be replaced by $\boldsymbol{u}^{(1)}$ since $\boldsymbol{u}^{(1)}\in L^s(\mathbb{R}^3)^3\cap\mathfrak{D}_0^{-1,2}(\mathbb{R}^3)^3\cap H_2(\mathbb{R}^3)$, as explained above, and because of (6.6). In this way we arrive at the inequality

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_{2;\,B_R} \,\le\, C(R,p,q,s)\,|\lambda|^{-4(1-1/s)-\delta}\,\|\boldsymbol{u}^{(1)}\|_s. \tag{6.8}$$

However, by (3.15) and Theorem 19, we have

$$\|\boldsymbol{u}^{(1)}\|_s \,\le\, C(s)\,|\lambda|^{-2}\cdot\|(Z_\lambda^{(s)})^{-1}(\boldsymbol{f})\|_s \,\le\, C(s)\,|\lambda|^{-2}\,\|\boldsymbol{f}\|_s. \tag{6.9}$$

By combining (6.8) and (6.9), we arrive at (6.4). □

Corollary 5. *The inequality*

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\|_2 \,\le\, C(\xi,\sigma)\,\|\boldsymbol{f}\|_2 \tag{6.10}$$

holds for ξ, σ, $\boldsymbol{f}$ *as in Theorem 15 and for* $\lambda\in\mathbb{C}\setminus\{0\}$ *with* $\operatorname{Re}\lambda\ge 0$, $|\lambda|\le\epsilon_1$.

Let $\delta\in(0,1]$. *Then there is* $\epsilon_4(\delta)\in(0,\epsilon_1]$, *also depending on* τ *and* $\boldsymbol{U}$, *such that*

$$\begin{aligned}&\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-2}\boldsymbol{f}\|_{2;\,B_R}+\big\|\nabla\big[\,(\overline{\lambda}\,\mathfrak{I}-\mathcal{L})^{-1}\circ(\lambda\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\big]\,\big\|_{2;\,B_R}\\ &\quad\le\quad C(\xi,\sigma,\delta,R)\,|\lambda|^{-\delta}\,\|\boldsymbol{f}\|_2,\end{aligned} \tag{6.11}$$

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_{2;\,B_R} \,\le\, C(\xi,\sigma,\delta,R)\,|\lambda|^{-2-\delta}\,\|\boldsymbol{f}\|_2 \tag{6.12}$$

for $R\in(0,\infty)$, $\lambda\in\mathbb{C}\setminus\{0\}$ *with* $\operatorname{Re}\lambda\ge 0$, $|\lambda|\le\epsilon_4(\delta)$, *and for* σ, ξ, $\boldsymbol{f}$ *as in Theorem 15.*

Proof. Take σ, ξ, $\boldsymbol{f}$ as in Theorem 15. Then $\|\boldsymbol{f}\|_* \le \|\boldsymbol{f}\|_2 + C\,\|\boldsymbol{f}\|_{6/5} \le C(\xi,\sigma)\,\|\boldsymbol{f}\|_2$ by Lemma 3 and (4.26). Inequality (6.10) now follows with the first statement of Theorem 21. Let $\delta\in(0,1]$, and put $s:=1/(1-\delta/8)$. Then $s\in(1,\,\frac{6}{5})$, so $\|\boldsymbol{f}\|_s\le C(\delta,\xi,\sigma)\,\|\boldsymbol{f}\|_2$ by (4.26), and $-4(1-1/s)-\delta/2=-\delta$. From these observations and inequalities (6.3) and (6.4) with δ replaced by $\delta/2$, we obtain (6.11) and (6.12), respectively. □

Lemma 13. *Let* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda\ge 0$, *and* $\boldsymbol{g}\in H_2(\mathbb{R}^3)$. *Then*

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}\|_2 \,\le\, C\,(\|\boldsymbol{g}\|_2+\|(\lambda\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{g})\|_2).$$

Proof. Put $\boldsymbol{u} := (\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}$. Then $\boldsymbol{u} \in D(\mathcal{L})$ and $-\Delta\boldsymbol{u}+\lambda\,\boldsymbol{u} = \boldsymbol{g}-\tau\,\partial_1\boldsymbol{u}+\mathcal{P}\mathfrak{B}\boldsymbol{u}$, so that

$$\operatorname{Re}\int_{\mathbb{R}^3}(-\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}+\lambda\,|\boldsymbol{u}|^2)\,\mathrm{d}\boldsymbol{x} \;=\; \operatorname{Re}\int_{\mathbb{R}^3}\big(\boldsymbol{g}-\tau\,\partial_1\boldsymbol{u}+\mathcal{P}\mathfrak{B}\boldsymbol{u}\big)\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}. \tag{6.13}$$

But $\int_{\mathbb{R}^3}-\Delta\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x} = \|\nabla\boldsymbol{u}\|_2^2$, $\operatorname{Re}\int_{\mathbb{R}^3}\partial_1\boldsymbol{u}\cdot\overline{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}=0$, so we deduce from (6.13) and Lemma 8 that

$$\|\nabla\boldsymbol{u}\|_2^2+\operatorname{Re}\lambda\,\|\boldsymbol{u}\|_2^2 \;\le\; \|\boldsymbol{g}\|_2\,\|\boldsymbol{u}\|_2+C\,\|\nabla\boldsymbol{u}\|_2\,\|\boldsymbol{u}\|_2.$$

Since $\operatorname{Re}\lambda\ge 0$, the lemma now follows by a simple shoestring argument. □

Lemma 14. *Let $\gamma_1,\ \gamma_2\in(0,\infty)$ with $\gamma_1<\gamma_2$. Put $\mathfrak{M}_{\gamma_1,\gamma_2} := \{\lambda\in\mathbb{C};\ \operatorname{Re}\lambda\ge 0,\ \gamma_1\le|\lambda|\le\gamma_2\}$. Then $\mathfrak{M}_{\gamma_1,\gamma_2}\subset\varrho(\mathcal{L})$ (see (6.2)) and*

$$\|(\lambda\mathfrak{I}-\mathcal{L})^{-1}(\Phi)\|_2+\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\Phi\|_2 \;\le\; C(\gamma_1,\gamma_2)\,\|\Phi\|_2 \quad \textit{for } \Phi\in H_2(\mathbb{R}^3).$$

Proof. Recall that $\varrho(\mathcal{L})$ is an open set in $\mathbb{C}$, and the mapping $\varrho(\mathcal{L})\ni\lambda\mapsto(\lambda\mathfrak{I}-\mathcal{L})^{-1}$ is holomorphic, in particular continuous, with respect to the operator norm of linear bounded operators from $H_2(\mathbb{R}^3)$ into $H_2(\mathbb{R}^3)$. Thus, an elementary argument involving finite coverings of $\mathfrak{M}_{\gamma_1,\gamma_2}$ and Neumann series of operators yields that $\|(\lambda\mathfrak{I}-\mathcal{L})^{-1}(\Phi)\|_2 \le C(\gamma_1,\gamma_2)\,\|\Phi\|_2$. Now the lemma follows from Lemma 13. □

Theorem 22. *There is a constant $C_1>0$ depending on τ, $\boldsymbol{U}$, ϑ and a such that*

1) the estimate

$$|\lambda|\;\|(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}\|_2 \;\le\; C\,\|\boldsymbol{g}\|_2 \tag{6.14}$$

holds for $\boldsymbol{g}\in H_2(\mathbb{R}^3)$, $\lambda\in\mathbb{C}\smallsetminus\{0\}$ with $\operatorname{Re}\lambda\ge 0$, $|\lambda|\ge C_1$, and for $\lambda\in S_{\vartheta,a}$ with $\operatorname{Re}\lambda<0$ and $|\lambda|\ge C_1$ and

2) the estimate

$$|\lambda|\;\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}\|_2 \;\le\; C\,\|\nabla\boldsymbol{g}\|_2 \tag{6.15}$$

holds for $\boldsymbol{g}\in H_2(\mathbb{R}^3)\cap W^{1,2}(\mathbb{R}^3)^3$, and for λ as in (6.14).

Proof. Let $\lambda\in\mathbb{C}\smallsetminus\{0\}$ with $\operatorname{Re}\lambda\ge 0$ or $\lambda\in S_{\vartheta,a}$. This means by (6.2) that $\lambda\in\varrho(\mathcal{L})$. Let $\boldsymbol{g}\in H_2(\mathbb{R}^3)\cap W^{1,2}(\mathbb{R}^3)^3$, and put $\boldsymbol{u} := (\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{g}$. Then

$$-\Delta\boldsymbol{u}+\lambda\,\boldsymbol{u} \;=\; \boldsymbol{g}-\tau\,\partial_1\boldsymbol{u}+\mathcal{P}\mathfrak{B}\boldsymbol{u}. \tag{6.16}$$

Multiplying this equation by $-\Delta\overline{\boldsymbol{u}}$, integrating over $\mathbb{R}^3$, separating real and imaginary parts, and then applying Hölder's inequality and Lemma 8, we get

$$\|\Delta\boldsymbol{u}\|_2^2+\operatorname{Re}\lambda\,\|\nabla\boldsymbol{u}\|_2^2 \;\le\; C\,\big(\|\nabla\boldsymbol{g}\|_2\,\|\nabla\boldsymbol{u}\|_2+\|\nabla\boldsymbol{u}\|_2\,\|\Delta\boldsymbol{u}\|_2\big), \tag{6.17}$$

$$|\mathrm{Im}\,\lambda|\;\|\nabla \boldsymbol{u}\|_2^2 \;\le\; C\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2\;\|\Delta \boldsymbol{u}\|_2\big). \tag{6.18}$$

Now we distinguish between the cases $\mathrm{Re}\,\lambda \ge 0$ and $\mathrm{Re}\,\lambda < 0$. First consider the (more difficult) case $\mathrm{Re}\,\lambda < 0$. Then $\lambda \in S_{\vartheta,a}$, hence

$$\mathrm{Re}\,(\lambda - a) \;>\; -\cos(\pi-\vartheta)\cdot|\lambda - a|, \qquad |\mathrm{Im}\,\lambda| \;\ge\; |\lambda - a|\cdot\sin(\pi-\vartheta).$$

We may thus deduce from (6.17) and (6.18), respectively, that

$$\begin{aligned}
&\|\Delta \boldsymbol{u}\|_2^2 + \big(-\cos(\pi-\vartheta)\cdot|\lambda-a| + a\big)\;\|\nabla \boldsymbol{u}\|_2^2 \;\le\; \|\Delta \boldsymbol{u}\|_2^2 + \mathrm{Re}\,\lambda\;\|\nabla \boldsymbol{u}\|_2^2\\
&\quad\le\;\; C\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2\;\|\Delta \boldsymbol{u}\|_2\big),\\
&|\lambda - a|\cdot\sin(\pi-\vartheta)\cdot\|\nabla \boldsymbol{u}\|_2^2 \;\le\; C\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2\;\|\Delta \boldsymbol{u}\|_2\big).
\end{aligned}$$

The second inequality is multiplied by $2\,\cot(\pi-\vartheta)$ and then added to the first one. It follows

$$\begin{aligned}
&\|\Delta \boldsymbol{u}\|_2^2 + \big(\cos(\pi-\vartheta)\cdot|\lambda-a| + a\big)\;\|\nabla \boldsymbol{u}\|_2^2\\
&\quad\le\;\; C\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2\;\|\Delta \boldsymbol{u}\|_2\big)\\
&\quad\le\;\; \alpha_1\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2^2\big) + \tfrac{1}{2}\,\|\Delta \boldsymbol{u}\|_2^2,
\end{aligned} \tag{6.19}$$

with a constant α_1 depending on τ, $\boldsymbol{U}$ and ϑ. Now suppose in addition that $|\lambda| \ge 2\alpha_1/\cos(\pi-\vartheta)$. Then $\cos(\pi-\vartheta)\cdot|\lambda-a| + a \ge \cos(\pi-\vartheta)\cdot|\lambda| \ge 2\alpha_1$, hence from (6.19)

$$\begin{aligned}
&\|\Delta \boldsymbol{u}\|_2^2 + \big(\,\cos(\pi-\vartheta)\cdot|\lambda-a|/2 + a/2 + \alpha_1\,\big)\;\|\nabla \boldsymbol{u}\|_2^2\\
&\quad\le\;\; \alpha_1\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2^2\big) + \tfrac{1}{2}\,\|\Delta \boldsymbol{u}\|_2^2,
\end{aligned}$$

so that

$$\tfrac{1}{2}\,\|\Delta \boldsymbol{u}\|_2^2 + \big(\,\cos(\pi-\vartheta)\cdot|\lambda-a|/2 + a/2\,\big)\;\|\nabla \boldsymbol{u}\|_2^2 \;\le\; \alpha_1\;\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2.$$

Since $\cos(\pi-\vartheta)\cdot|\lambda-a| + a \;\ge\; |\lambda|\cdot\cos(\pi-\vartheta)$, we now get $|\lambda|\cdot\|\nabla \boldsymbol{u}\|_2^2 \;\le\; C\,\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2$, hence $|\lambda|\,\|\nabla \boldsymbol{u}\|_2 \;\le\; C\,\|\nabla \boldsymbol{g}\|_2$. Recall that this inequality was proved under the assumptions $\lambda \in S_{\vartheta,a}$, $\mathrm{Re}\,\lambda < 0$, $|\lambda| \ge 2\alpha_1/\cos(\pi-\vartheta)$.

Now we consider the case $\mathrm{Re}\,\lambda \ge 0$. Adding (6.17) and (6.18), we obtain

$$\|\Delta \boldsymbol{u}\|_2^2 + \big(\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda|\big)\;\|\nabla \boldsymbol{u}\|_2^2 \;\le\; C\,\big(\|\nabla \boldsymbol{g}\|_2\,\|\nabla \boldsymbol{u}\|_2 + \|\Delta \boldsymbol{u}\|_2\;\|\nabla \boldsymbol{u}\|_2\big).$$

But $\mathrm{Re}\,\lambda + |\mathrm{Im}\,\lambda| \ge |\lambda|$, so we may conclude that

$$\begin{aligned}
&\|\Delta \boldsymbol{u}\|_2^2 + |\lambda|\;\|\nabla \boldsymbol{u}\|_2^2 \;\le\; C\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2\;\|\Delta \boldsymbol{u}\|_2\big)\\
&\quad\le\;\; \alpha_2\,\big(\|\nabla \boldsymbol{g}\|_2\;\|\nabla \boldsymbol{u}\|_2 + \|\nabla \boldsymbol{u}\|_2^2\big) + \tfrac{1}{2}\,\|\Delta \boldsymbol{u}\|_2^2,
\end{aligned}$$

with a constant $\alpha_2 > 0$ depending on τ and $\boldsymbol{U}$. Thus, if $|\lambda| \geq 2\alpha_2$, we arrive at the inequality

$$\tfrac{1}{2}\,\|\Delta \boldsymbol{u}\|_2^2 + |\lambda|\,\|\nabla \boldsymbol{u}\|_2^2 \,\leq\, \alpha_2\,\|\nabla \boldsymbol{g}\|_2\,\|\nabla \boldsymbol{u}\|_2.$$

It follows $|\lambda|\,\|\nabla \boldsymbol{u}\|_2 \leq C\,\|\nabla g\|_2$. This completes the proof of (6.15). In order to show (6.14), we multiply (6.16) by $\overline{\boldsymbol{u}}$ instead of $-\Delta\overline{\boldsymbol{u}}$. Then we obtain (6.14) by repeating the previous arguments with obvious modifications. □

Note that Theorem 21 presents the resolvent estimates related to the operator $\mathcal{L}$ for the case that $|\lambda|$ is small, whereas Theorem 22 deals with the case of large $|\lambda|$. Lemma 14 might be considered as an (obvious) result for intermediate values.

7. Estimates of the semigroup $e^{\mathcal{L}t}$.

We recall that our convention at the beginning of Section 6 with respect to generic constants remains valid in this section. Furthermore, we recall that we assume that the operator $\mathcal{L}$ satisfies conditions (A1) (Section 5) and (A2) (Section 6).

By Theorem 20, (6.14) and [17, Theorem 1.3.4], the operator $\mathcal{L}$ defined in (6.1) generates an analytic semigroup in $H_2(\mathbb{R}^3)$ ([17, Definition 1.3.3]), which we denote by $e^{\mathcal{L}t}$. In what follows, we will exploit the resolvent estimates from Section 6 in order to evaluate this semigroup. We begin by introducing the constant

$$C_2 \,:=\, \max\big\{\,C_1;\ \epsilon_4(\tfrac{1}{16});\ 1/\sqrt{2};\ 2a\,\tan(\pi-\vartheta)\,\big\},$$

where C_1 was chosen in Theorem 22, and $\epsilon_4(\frac{1}{16})$ in Corollary 5. For the quantities a and ϑ, we refer to Theorem 20. Since $C_2 \geq 2a\,\tan(\pi-\vartheta)$, we may choose $\vartheta_0 \in (\pi/2,\,\vartheta)$ so close to $\pi/2$ that for any $s \in [C_2,\infty)$, the inclusion

$$\big\{\,s\,e^{i\,\varphi};\ \varphi \in [-\vartheta_0,\,-\pi/2] \cup [\pi/2,\,\vartheta_0]\,\big\} \cup \big\{\,r\,e^{i\,\vartheta_0};\ r \in [s,\infty)\,\big\} \,\subset\, S_{\vartheta,a} \tag{7.1}$$

holds. Let $\alpha, \beta \in (0,\infty)$ with $\alpha < \beta$, $\beta \geq C_2$. Then we define the curves $\Gamma_i^{(\alpha,\beta)} \subset \mathbb{C}$, with $i \in \{1, ..., 5\}$, by setting

$$\Gamma_1^{(\alpha,\beta)} := \big\{\,\alpha \cdot e^{i\,\varphi};\ \varphi \in [-\pi/2,\,\pi/2]\,\big\}, \qquad \Gamma_2^{(\alpha,\beta)} \,:=\, \big\{\,i\,r;\ r \in [\alpha,\beta]\,\big\},$$
$$\Gamma_3^{(\alpha,\beta)} := \big\{\,i\,\beta + r\,e^{i\,\vartheta};\ r \in [0,\infty)\,\big\}, \qquad \Gamma_i^{(\alpha,\beta)} \,:=\, \big\{\,\overline{y};\ y \in \Gamma_{i-2}^{(\alpha,\beta)}\,\big\}$$

for $i \in \{4; 5\}$. Let $s \in [C_2,\infty)$ and define

$$\Lambda_1^{(s)} \,:=\, \big\{\,s\,e^{i\,\varphi};\ \varphi \in [-\vartheta_0,\vartheta_0]\,\big\}, \qquad \Lambda_2^{(s)} \,:=\, \big\{\,r\,e^{i\,\vartheta_0};\ r \in [s,\infty)\,\big\},$$
$$\Lambda_3^{(s)} \,:=\, \big\{\,\overline{y};\ y \in \Lambda_2^{(s)}\,\big\}.$$

Then, in view of (6.2), (7.1) and Theorem 20, and since $\beta \geq C_2 > a\,\tan(\pi-\vartheta)$, we have $\Gamma_\nu^{(\alpha,\beta)},\ \Lambda_\mu^{(s)} \subset \varrho(\mathcal{L})$ $(1 \leq \nu \leq 5,\ 1 \leq \mu \leq 3)$. As a consequence of these relations and [17, Theorem 1.3.4], we obtain

$$\begin{aligned} e^{\mathcal{L}t}(\boldsymbol{w}) &= (2\pi i)^{-1} \sum_{\nu=1}^{5} \int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda t}\,(\lambda\, I - \mathcal{L})^{-1} \boldsymbol{w}\, d\lambda \\ &= (2\pi i)^{-1} \sum_{\mu=1}^{3} \int_{\Lambda_\mu^{(s)}} e^{\lambda t}\,(\lambda\, I - \mathcal{L})^{-1} \boldsymbol{w}\, d\lambda \end{aligned} \tag{7.2}$$

for $t \in (0,\infty)$, $\boldsymbol{w} \in H_2(\mathbb{R}^3)$. As to the arguments we present in this section, we can note: The main difficulty consists in showing that for large t and for ξ, σ, $\boldsymbol{f}$ as in Theorem 15, the term $\|\nabla e^{\mathcal{L}t}(\boldsymbol{f})\|_{2;\,B_R}$ is bounded by $C(R,\xi,\sigma)\, t^{-1-\epsilon}\,\|\boldsymbol{f}\|_2$, for some $\epsilon > 0$. (Incidentally we will choose $\epsilon = \frac{1}{8}$, but this will only be for definiteness.) We will obtain such an estimate by considering the first sum on the right–hand side of (7.2). This means in particular that we have to show that

$$\Big\| \int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda t}\, \nabla(\lambda\, I - \mathcal{L})^{-1} \boldsymbol{f}\, d\lambda \Big\|_{2;\,B_R} \le C(\xi,\sigma,R)\, \|\boldsymbol{f}\|_2\, t^{-1-\epsilon}$$

for large t. In view of (6.10), this should require $\alpha \le t^{-1-\epsilon}$. On the other hand, in order to produce a factor $t^{-\gamma}$ for some $\gamma > 0$ in the estimate of the integral $\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda t}\, \nabla(\lambda\, I - \mathcal{L})^{-1} \boldsymbol{f}\,|_{B_R}\, d\lambda$ for $\nu = 2$ and $\nu = 4$, we introduce the local parameter $\varphi(r) := i\, r$ $(r \in [\alpha,\beta])$, and then integrate by parts with respect to r, so that the factor e^{irt} is transformed into $e^{irt}\,(it)^{-1}$. But this means that a single partial integration does not suffice to generate a factor $t^{-1-\epsilon}$. On the other hand, after two such integrations, we obtain a term $\nabla(ir\, I - \mathcal{L})^{-3} \boldsymbol{f}\,|_{B_R}$, which gives rise to a factor $r^{-2-\delta}$ for some $\delta > 0$ (see (6.12)). The integration of this term on the interval $[\alpha,\beta]$ leads to the factor $\alpha^{-1-\delta} = t^{(1+\epsilon)(1+\delta)}$ which cancels the effect of the second partial integration. Therefore, recalling that the term $\nabla(ir\, I - \mathcal{L})^{-2} \boldsymbol{f}\,|_{B_R}$ only produces a factor $r^{-\delta}$ (see (6.11)), we perform a kind of interpolation between one and two partial integrations. To this end, we use fractional derivatives, as introduced in the next lemma.

Lemma 15. *Let κ, $b \in \mathbb{R}$ with $\kappa < b$, $\mu \in (0,1)$, $h \in C^1([\kappa,b])$ with $h(b) = 0$. Define $\overline{h} : [\kappa,b] \mapsto \mathbb{C}$ by*

$$\overline{h}(r) := \Gamma(1-\mu)^{-1} \int_r^b (s-r)^{-1+\mu}\, h(s)\, ds \quad \text{for } r \in [\kappa,b].$$

Then $\overline{h} \in C^1([\kappa,b])$ with

$$\overline{h}'(r) = \Gamma(1-\mu)^{-1} \int_r^b (\alpha-r)^{-1+\mu}\, h'(\alpha)\, d\alpha \quad \text{for } r \in [\kappa,b]. \tag{7.3}$$

Define $\gamma :\ [\kappa,b] \ni r \longmapsto \Gamma(\mu)^{-1} \int_r^b (s-r)^{-\mu}\, \overline{h}'(s)\, ds \in \mathbb{C}.$ *Then $h = -\gamma$.*

(Note that Γ without any subscript or superscript denotes the Gamma function.) We leave the proof of this lemma to the reader, and only note that the equation $\gamma = -h$ may be reduced to the relation $B(\mu, 1-\mu) = \Gamma(\mu) \cdot \Gamma(1-\mu)$ for $\mu \in (0,1)$, where B denotes the usual beta function.

Now we can prove an inequality which will be the key element in our estimate of the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$.

Lemma 16. *Let $\delta \in (0, \frac{1}{4})$ and abbreviate $b := \min\{\, \epsilon_4(\delta);\, 1/\sqrt{2}\,\}$, with $\epsilon_4(\delta)$ from Corollary 5. Then, for ξ, σ, $\boldsymbol{f}$ as in Theorem 15, $R \in (0,\infty)$, $\kappa \in (0,b)$, $t \in (0,\infty)$,*

$$\Big\| \int_\kappa^b \mathrm{e}^{\mathrm{i}rt}\, \nabla(\mathrm{i}r\mathfrak{I} - \mathcal{L})^{-2} \boldsymbol{f}\, \mathrm{d}r \Big\|_{2;\, B_R} \;\le\; C(\xi,\sigma,\delta,R)\, t^{-1/4}\, \kappa^{-\delta}\, \|\boldsymbol{f}\|_2.$$

Proof. Take ξ, σ, $\boldsymbol{f}$, R, κ, t as in the lemma. Note that by (6.2), we have $\{\mathrm{i}r;\ r \in [\kappa, b]\} \subset \varrho(\mathcal{L})$. Therefore the mapping $\boldsymbol{g} : [\kappa, b] \ni r \mapsto \nabla(\mathrm{i}r\,\mathfrak{I} - \mathcal{L})^{-1}\boldsymbol{f}\,|_{B_R} \in L^2(B_R)^9$ is in particular twice continuously differentiable, with

$$\boldsymbol{g}^{(\nu)}(r) = (-\mathrm{i})^\nu\, \nu\, \nabla(\mathrm{i}r\,\mathfrak{I} - \mathcal{L})^{-(\nu+1)}(\boldsymbol{f})\,|_{B_R}$$

for $\nu \in \{1;\, 2\}$, $r \in [\kappa, b]$. Thus, due to the assumption $b \le \epsilon_4(\delta)$, inequalities (6.11) and (6.12) yield

$$r^\delta\, \|\boldsymbol{g}'\|_2 + r^{2+\delta}\, \|\boldsymbol{g}''(r)\|_2 \;\le\; C(\xi,\sigma,\delta,R)\, \|\boldsymbol{f}\|_2 \tag{7.4}$$

for $r \in [\kappa, b]$. Put $h(r) := (\mathrm{i}t)^{-1}\,(\mathrm{e}^{\mathrm{i}rt} - \mathrm{e}^{\mathrm{i}bt})$ for $r \in [\kappa, b]$. Define $\overline{h}$ and γ as in Lemma 15, with $\mu = \frac{1}{4}$. Then we get a partial integration, using the equation $\gamma' = -h'$ (Lemma 15), and changing the order of integration,

$$\begin{aligned}
&\int_\kappa^b \mathrm{e}^{\mathrm{i}rt}\, \nabla(\mathrm{i}r\mathfrak{I} - \mathcal{L})^{-2}(\boldsymbol{f})\,|_{B_R}\, \mathrm{d}r \;=\; (-\mathrm{i}) \int_\kappa^b \gamma'(r)\, \boldsymbol{g}'(r)\, \mathrm{d}r \\
&\quad= \;\; \mathrm{i}\,\Gamma(\tfrac{1}{4})^{-1} \int_\kappa^b \overline{h}'(s) \left(\int_\kappa^s (s-r)^{-1/4}\, \boldsymbol{g}''(r)\, \mathrm{d}r \right) \mathrm{d}s \\
&\qquad + \mathrm{i}\,\Gamma(\tfrac{1}{4})^{-1} \int_\kappa^b (s-\kappa)^{-1/4}\, \overline{h}'(s)\, \mathrm{d}s\, \boldsymbol{g}'(\kappa).
\end{aligned} \tag{7.5}$$

Note that $\overline{h}'$ is a fractional derivative of h (of order $\frac{3}{4}$). Thus we have transformed an integral of the form $\int_\kappa^b h' \cdot \boldsymbol{g}'\, \mathrm{d}r$ involving the derivative h' of h, into an integral of the form $\int_\kappa^b \overline{h}' \cdot \psi\, dr$ (modulo boundary terms) involving a fractional derivative of h, in contrast to the function h itself, which would arise by the standard partial integration.

The lemma follows from (7.4), (7.5) and the inequalities $|h(r)| \le 2/t$, $|h'(r)| \le 2$ for $r \in [\kappa, b]$. We omit the details because they were already elaborated in [7, proof of Lemma 6.2]. $\square$

In the following theorem, we estimate $\nabla e^{\mathcal{L}t}(\boldsymbol{f})\,|_{B_R}$ for large values of t, with $\boldsymbol{f}$ given as in Theorem 15.

Theorem 23. *Put* $b := \min\{\,\epsilon_4(\frac{1}{16});\ 1/\sqrt{2}\,\}$, *with* $\epsilon_4(\frac{1}{16})$ *from Corollary 5. Let* $R \in (0,\infty)$, $t \in [b^{-1},\infty)$, *and take* ξ, σ, $\boldsymbol{f}$ *as in Theorem 15. Then*

$$\|\nabla e^{\mathcal{L}t}(\boldsymbol{f})\|_{2;\,B_R} \;\le\; C(\xi,\sigma,R)\;\|\boldsymbol{f}\|_2\cdot t^{-9/8}.$$

Proof. We start from the first equation in (7.2), with $\alpha = t^{-2}$. The latter assumption means in particular that $\alpha = t^{-2} \le t^{-1} \le b$. We further take $\beta \in [C_2,\infty)$, where C_2 was introduced at the beginning of this section. We find that

$$\begin{aligned}
&\Big\|\nabla \int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda t}\,(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\,d\lambda\Big\|_{2;\,B_R}\\
&\le\;\alpha\int_{-\pi/2}^{\pi/2} |e^{t\alpha\, e^{i\varphi}}|\;\big\|\nabla(\alpha\, e^{i\varphi}\,\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\big\|_{2;\,B_R}\,d\varphi\\
&\le\;C(\xi,\sigma)\;\|\boldsymbol{f}\|_2\;\alpha\, e^{\alpha t}\;\le\;C(\xi,\sigma)\;\|\boldsymbol{f}\|_2\;t^{-2},
\end{aligned}\tag{7.6}$$

where the last but one inequality holds because of (6.10). The last one is a consequence of the choice $\alpha = t^{-2}$. If $\lambda \in \Gamma_3^{(\alpha,\beta)} \cup \Gamma_5^{(\alpha,\beta)}$, we have $|\lambda| \ge |\beta| \ge C_2 \ge C_1$ and $\lambda \in S_{\vartheta,a}$, $\operatorname{Re}\lambda \ge 0$, so inequality (6.15) is valid for such λ. This allows us to conclude that

$$\begin{aligned}
&\Big\|\nabla\Big(\sum_{\nu\in\{3;\,5\}}\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda t}\,(\lambda\,\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\,d\lambda\Big)\Big\|_{2;\,B_R}\\
&\le\;C\,\|\nabla\boldsymbol{f}\|_2\int_0^\infty \big|e^{i\beta+re^{i\vartheta}}\big|\;|i\beta+r\,e^{i\vartheta}|^{-1}\,dr\\
&\le\;C\,\|\nabla\boldsymbol{f}\|_2\;\beta^{-1}\int_0^\infty e^{rt\,\cos\vartheta}\,dr\;\le\;C(\xi)\;\|\boldsymbol{f}\|_2\;(\beta t)^{-1},
\end{aligned}\tag{7.7}$$

where the last estimate follows from the first inequality in (4.26). This leaves us to deal with the main difficulty of this proof, that is, the estimate of the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$. To this end, we perform a partial integration. Noting that $b \le C_2 \le \beta$, we obtain

$$\begin{aligned}
&\nabla\Big(\int_{\Gamma_2^{(\alpha,\beta)}} e^{\lambda t}\,(\lambda\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\,d\lambda\Big)\Big|_{B_R}\\
&=\;i\int_\alpha^\beta e^{irt}\,\nabla(ir\,\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\Big|_{B_R}\,dr\;=\;\sum_{j=1}^4\mathfrak{N}_j,
\end{aligned}\tag{7.8}$$

where

$$\mathfrak{N}_1 \;:=\; t^{-1}\,e^{it\beta}\,\nabla(i\beta\,\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\Big|_{B_R},$$

$$\begin{aligned}
\mathfrak{N}_2 &:= -t^{-1}\,\mathrm{e}^{\mathrm{i}t\alpha}\,\nabla(\mathrm{i}\alpha\,\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\big|_{B_R},\\
\mathfrak{N}_3 &:= \frac{\mathrm{i}}{t}\int_\alpha^b \mathrm{e}^{\mathrm{i}tr}\,\nabla(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-2}(\boldsymbol{f})\,\big|_{B_R}\,\mathrm{d}r,\\
\mathfrak{N}_4 &:= \frac{\mathrm{i}}{t}\int_b^\beta \mathrm{e}^{\mathrm{i}tr}\,\nabla(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-2}(\boldsymbol{f})\,\big|_{B_R}\,\mathrm{d}r.
\end{aligned}$$

The integral over $\Gamma_4^{(\alpha,\beta)}$ is split into the sum $\sum_{j=1}^4 \overline{\mathfrak{N}_j}$, where $\overline{\mathfrak{N}_j}$ is defined in an analogous way as $\mathfrak{N}_j$, for $j\in\{1;\,...;\,4\}$. Recalling that $C_1\le C_2\le\beta$, we get from (6.15) and (4.26),

$$\|\mathfrak{N}_1\|_1+\|\overline{\mathfrak{N}}_1\|_2 \;\le\; C(\xi)\,(t\beta)^{-1}\,\|\boldsymbol{f}\|_2. \tag{7.9}$$

We further find, using the standard resolvent equation, (6.10) and (6.11) with $\delta=\frac{1}{16}$, that

$$\begin{aligned}
\|\mathfrak{N}_2+\overline{\mathfrak{N}}_2\|_2 \;&\le\; t^{-1}\,|\mathrm{e}^{-\mathrm{i}\alpha t}-\mathrm{e}^{\mathrm{i}\alpha t}|\;\|\nabla(-\mathrm{i}\alpha\,\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\|_{2;\,B_R}\\
&\quad+t^{-1}\,|\mathrm{e}^{\mathrm{i}\alpha t}|\;\big\|2\mathrm{i}\alpha\,\nabla\big[\,(-\mathrm{i}\alpha\,\mathfrak{I}-\mathcal{L})^{-1}\circ(\mathrm{i}\alpha\,\mathfrak{I}-\mathcal{L})^{-1}(\boldsymbol{f})\,\big]\big\|_{2;\,B_R}\\
&\le\; C(\xi,\sigma,R)\,\|\boldsymbol{f}\|_2\,t^{-1}\,\big(\,|\sin(\alpha t)|+\alpha^{15/16}\,\big)\\
&\le\; C(\xi,\sigma,R)\,\|\boldsymbol{f}\|_2\,(\alpha+\alpha^{15/16}\,t^{-1}).
\end{aligned}\tag{7.10}$$

Lemma 16 with $\delta=\frac{1}{16}$ yields

$$\|\mathfrak{N}_3\|_2+\|\overline{\mathfrak{N}}_3\|_2 \;\le\; C(\xi,\sigma,R)\,\|\boldsymbol{f}\|_2\,t^{-5/4}\,\alpha^{-1/16}. \tag{7.11}$$

As to $\mathfrak{N}_4$, we perform an additional partial integration, to obtain

$$\begin{aligned}
\mathfrak{N}_4 &= 2\mathrm{i}t^{-2}\int_b^\beta \mathrm{e}^{\mathrm{i}rt}\,\nabla(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-3}(\boldsymbol{f})\,\big|_{B_R}\,\mathrm{d}r+t^{-2}\,\mathrm{e}^{\mathrm{i}\beta t}\,\nabla(\mathrm{i}\beta\,\mathfrak{I}-\mathcal{L})^{-2}(\boldsymbol{f})\,\big|_{B_R}\\
&\quad-t^{-2}\,\mathrm{e}^{\mathrm{i}bt}\,\nabla(\mathrm{i}b\,\mathfrak{I}-\mathcal{L})^{-2}(\boldsymbol{f})\,\big|_{B_R}.
\end{aligned}$$

Now we apply (6.15) with $\lambda=\mathrm{i}\beta$, (6.11) with $\lambda=\mathrm{i}b$, the inequality $b\le\epsilon_4(\frac{1}{16})$ and (4.26), to obtain

$$\|\mathfrak{N}_4\|_2 \;\le\; C\,t^{-2}\int_b^\beta \|\nabla(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_2\,\mathrm{d}r+C(\xi,\sigma,R)\,t^{-2}\,\|\boldsymbol{f}\|_2. \tag{7.12}$$

The remaining integral in (7.12) is split into an integral from b to C_2 and into another one from C_2 to β. (Recall that $b\le C_2\le\beta$.) But $\int_b^{C_2}\|(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_2\,\mathrm{d}r\le C\,\|\boldsymbol{f}\|_2$ by Lemma 14 with $\gamma_1=b$, $\gamma_2=C_2$, whereas

$$\int_{C_2}^b \|\nabla(\mathrm{i}r\,\mathfrak{I}-\mathcal{L})^{-3}\boldsymbol{f}\|_2\,\mathrm{d}r \;\le\; C\,\|\nabla\boldsymbol{f}\|_2\int_{C_2}^\beta r^{-3}\,\mathrm{d}r \;\le\; C\,\|\nabla\boldsymbol{f}\|_2$$

by (6.15). Thus, referring to (4.26), we obtain from (7.12),

$$\|\mathfrak{N}_4\|_2 \;\leq\; C(\xi,\sigma,R)\;\|\boldsymbol{f}\|_2\,t^{-2}. \tag{7.13}$$

An analogous estimate for $\|\overline{\mathfrak{N}}_4\|_2$ may be derived by the same arguments. Now, combining (7.8), the analogue of (7.8) for the integral over $\Gamma_4^{(\alpha,\beta)}$, (7.9)–(7.11), (7.13) and the analogue of (7.13) for $\overline{\mathfrak{N}}_4$, we obtain

$$\begin{aligned}
&\Big\|\nabla\Big(\sum_{\nu\in\{2;\,4\}}\int_{\Gamma_\nu^{(\alpha,\beta)}} \mathrm{e}^{\lambda t}\,(\lambda\,\mathfrak{I}-\mathcal{L})^{-1}\boldsymbol{f}\;\mathrm{d}\lambda\Big)\Big\|_{2;\,B_R}\\
&\quad\leq\; C(\xi,\sigma,R)\;\|\boldsymbol{f}\|_2\;\big((t\beta)^{-1}+\alpha+t^{-1}\,\alpha^{15/16}+t^{-5/4}\,\alpha^{-1/16}+t^{-2}\big)\\
&\quad\leq\; C(\xi,\sigma,R)\;\|\boldsymbol{f}\|_2\;\big((t\beta)^{-1}+t^{-9/8}\big),
\end{aligned} \tag{7.14}$$

where the last inequality holds because we chose $\alpha = t^{-2}$. By referring to (7.2), (7.6), (7.7), (7.14), we may conclude that $\|\nabla\mathrm{e}^{\mathcal{L}t}(\boldsymbol{f})\|_{2;\,B_R} \leq C(\xi,\sigma,R)\;\big((t\beta)^{-1} + t^{-9/8}\big)$. Letting β tend to infinity, we obtain the statement of the theorem. □

Theorem 24. *Choose b as in Theorem 23. Then, for $t\in(0,b^{-1}]$, $\Phi\in H_2(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)^3$, the inequality $\|\nabla\mathrm{e}^{\mathcal{L}t}(\Phi)\|_2 \leq C(\vartheta_0)\;\|\nabla\Phi\|_2$ holds.*

Proof. Take t,Φ as in the theorem. Put $s_0 := 1/t$ if $t\leq 1/C_2$, and $s_0 := C_2$ if $t > 1/C_2$. Then we have $s_0 \geq C_2$ in any case, so we may represent $\mathrm{e}^{\mathcal{L}t}(\Phi)$ by the second sum in (7.2). Moreover, for $\lambda\in\Lambda_i^{(s_0)}$, $1\leq i\leq 3$, we have $|\lambda|\geq s_0\geq C_2\geq C_1$, and in the case $\operatorname{Re}\lambda\leq 0$ in addition $\lambda\in S_{\vartheta,a}$ (see (7.1)), hence

$$\|\nabla(\lambda\mathfrak{I}-\mathcal{L})^{-1}\Phi\|_2 \;\leq\; C\;|\lambda|^{-1}\;\|\nabla\Phi\|_2 \tag{7.15}$$

by (6.15). In addition, we observe that $s_0t = 1$ if $t\leq 1/C_2$, and $s_0t\leq C_2\,b^{-1}$ else. As a consequence, $s_0t\leq C$ in any case. Choosing $\psi_1(\varphi) := s_0\,\mathrm{e}^{\mathrm{i}\varphi}$ $(\varphi\in[-\vartheta_0,\vartheta_0])$ as a representation of $\Lambda_1^{(s_0)}$, we get with (7.15):

$$\begin{aligned}
&\|\mathrm{e}^{\psi_1(\varphi)\,t}\;\psi_1'(\varphi)\;\nabla\big(\psi_1(\varphi)\,\mathfrak{I}-\mathcal{L}\big)^{-1}\Phi\|_2\\
&\quad\leq\;\; \mathrm{e}^{s_0\,\cos\varphi t}\;s_0\;C\,|s_0\,\mathrm{e}^{\mathrm{i}\varphi}|^{-1}\;\|\nabla\Phi\|_2 \;\leq\; C\;\|\nabla\Phi\|_2,
\end{aligned}$$

where we have used that $s_0t\leq C$, as noted above. Moreover, introducing the local representation $\psi_2(r) := r\,\mathrm{e}^{\mathrm{i}\vartheta_0}$ $(r\in[s_0,\infty))$ of $\Lambda_2^{(s_0)}$, we find with (7.15) that

$$\|\mathrm{e}^{\psi_2(r)\,t}\;\psi_2'(r)\;\nabla\big(\psi_2(r)\,\mathfrak{I}-\mathcal{L}\,\big)^{-1}\Phi\|_2 \;\leq\; C(\vartheta_0)\;\mathrm{e}^{rt\,\cos\vartheta_0}\;r^{-1}\;\|\nabla\Phi\|_2.$$

Furthermore, observing that $s_0t\geq 1$,

$$\int_{s_0}^{\infty}\mathrm{e}^{rt\,\cos\vartheta_0}\;r^{-1}\;\mathrm{d}r \;=\; \int_{s_0t}^{\infty}\mathrm{e}^{\alpha\,\cos\vartheta_0}\;\alpha^{-1}\;\mathrm{d}\alpha \;\leq\; \int_{1}^{\infty}\mathrm{e}^{\alpha\,\cos\vartheta_0}\;\alpha^{-1}\;\mathrm{d}\alpha \;\leq\; C(\vartheta_0).$$

The same argument also works for an analogous representation of $\Lambda_3^{(s_0)}$. Combining the preceding results, we get

$$\left\| \nabla \left(\int_{\Lambda_i^{(s_0)}} \mathrm{e}^{\lambda t} \, (\lambda \mathfrak{I} - \mathcal{L})^{-1}(\Phi) \, d\lambda \right) \right\|_2 \;\le\; C(\vartheta_0) \, \|\nabla \Phi\|_2$$

for $i \in \{1; 2; 3\}$. This proves the theorem. □

Theorem 25. *Let $\xi \in \mathbb{R}$ and $R \in (0,\infty)$. There exists a non–increasing function φ belonging to $L^1\big((0,\infty)\big) \cap L^2\big((0,\infty)\big)$, depending on τ, $\boldsymbol{U}$, ϑ, a, ϑ_0, ξ and R, such that*

$$\|\nabla \mathrm{e}^{\mathcal{L}t}(\boldsymbol{f})\|_{2;\,B_R} \;\le\; \varphi(t) \, \|\boldsymbol{f}\|_2$$

for $t \in (0,\infty)$ and for $\boldsymbol{f} \in H_2'$ (i.e. for $\boldsymbol{f}$ being an eigenfunction of the operator $\Delta + \xi\,\mathcal{P}\mathfrak{B}_{\mathrm{sym}}$, associated with a positive eigenvalue).

Proof. Again we abbreviate $b := \min\{\epsilon_4(\frac{1}{16});\, 1/\sqrt{2}\}$. By Theorem 23, there is $\gamma_1 > 0$ depending on τ, $\boldsymbol{U}$, R, ϑ, a, ξ, σ such that $\|\nabla \mathrm{e}^{\mathcal{L}t}(\boldsymbol{f})\|_{2;\,B_R} \le \gamma_1 \, t^{-9/8} \, \|\boldsymbol{f}\|_2$ for $t \in [b^{-1},\infty)$ and for $\boldsymbol{f} \in D(\mathcal{L})$ verifying the differential equation stated at the end of Theorem 25. Moreover Theorems 24 and 15 yield the existence of a constant $\gamma_2 > 0$ depending on the same quantities and also on ϑ_0 such that $\|\nabla \mathrm{e}^{\mathcal{L}t}(\boldsymbol{f})\|_2 \le \gamma_2 \, \|\boldsymbol{f}\|_2$ for $t \in (0, b^{-1}]$ and for $\boldsymbol{f}$ as before. Thus, the function φ defined by $\varphi(t) := \gamma_1 \, t^{-9/8}$ for $t \in [b^{-1},\infty)$, $\varphi(t) := \gamma_2$ for $t \in (0, b^{-1})$, has all the desired properties. □

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