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Non-standard applications of the Łojasiewicz-Simon theory: Stabilization to equilibria of solutions to phase-field models

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1 Introduction

1.1 Convergence to equilibria and the Łojasiewicz-Simon theory

We are interested in the long time behavior of trajectories of dynamical systems associated to various models of phase transitions. Such a complex problem can be viewed from several rather different perspectives. Typically, the systems in question possess a Lyapunov function - the free energy functional. In particular, the equilibrium states (stationary solutions) are the only candidates to belong to the ω-limit set of each individual trajectory.

In this paper, we focus on a simple question: Does any solution *converge* as $t \to \infty$ to a single stationary state? The answer is affirmative as soon as we know that the set of stationary solutions is discrete. In general, however, the structure of the set of equilibria may exhibit a rather complicated structure, in particular, it may contain curves or even more complicated objects having a positive Hausdorff dimension. If this is the case, the problem of convergence to a *single* equilibrium state becomes highly non-trivial. There are several examples of finite or infinite-dimensional dynamical systems, where solutions approach the set of equilibria but do not converge to one of them (see AULBACH [5], POLÁČIK and RYBAKOWSKI [33], POLÁČIK and SIMONDON [34], among others).

On the other hand, Łojasiewicz in his celebrated work on semi-analytic and subanalytic sets indicated a way how to prove convergence to equilibria of bounded solutions to general gradient-like systems (see [29]). In particular, he was able to show the following inequality:

$$|F(z) - F(a)|^{1-\theta} \leq c\|\nabla_x F(z)\| \text{ provided } |z-a| < \varepsilon, \text{ where } \theta \in (0, 1/2], \tag{1.1}$$

for any real analytic function $F : R^N \to R$. Very roughly indeed, an analytic function behaves like a polynomial (of sufficiently high degree) in a neighborhood of any point, at which its gradient vanishes.

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The idea that inequality (1.1) may force convergence in gradient systems was subsequently developed by SIMON [37] who generalized Łojasiewicz's original result to a certain class of analytic functionals on a Banach space. In particular, his result yields convergence to a single equilibrium of bounded solutions to a semilinear parabolic problem $u_t - \Delta u = f(x, u)$ provided f is analytic in u. His sophisticated and highly complicated approach has been considerably simplified and subsequently adapted by JENDOUBI [27] to obtain plausible convergence results for a much larger class of equations including semilinear hyperbolic systems with weak damping. Simon's technique, now better understood, has been used quite recently to obtain affirmative convergence results for a broad variety of equations of mathematical physics where the hypothesis of analyticity does not seem to be very restrictive ; in particular for the Cahn-Hilliard equation [26]. The same method has been also successfully modified to deal with degenerate parabolic equations of porous media type (see [17]). As pointed out by CHILL [11], it is the inequality (1.1) rather than analyticity of F that plays a decisive role in the analysis, however, analyticity of the function F proved to be the only efficient tool to verify (1.1).

The goal of the present paper is to review some non-standard applications of the Łojasiewicz-Simon method developed quite recently to deal with systems of evolutionary equations arising in the mathematical theory of phase transitions.

1.2 On the phase-field models

To begin with, let us point out that modeling the phase transition phenomena is a complex problem far from being fully understood. The phase-field approach involves the use of an order parameter χ in addition to the temperature ϑ as the basic state variables. This approach has been under extensive mathematical study since 1986 (see CAGINALP [8], [10], PENROSE and FIFE [31], [32] , BATES and CHMAJ [6], among many others).

In general, given a functional $E[\chi, \vartheta]$ characterizing the free energy, the time evolution of χ is a gradient flow with respect to χ of E while the heat balance is governed by the equation

$$\partial_t \Big(\vartheta + \lambda(\chi) \Big) + \mathrm{div}_x \mathbf{q} = 0, \tag{1.2}$$

where $\mathbf{q}$ is the heat flux and λ' represents the latent heat. Equation (1.2) is supplemented with suitable boundary conditions depending on the constitutive equation relating ϑ to $\mathbf{q}$. In particular, the resulting evolutionary system is *never* of purely gradient-type as required in the classical Łojasiewicz-Simon framework.

The most studied examples are listed below:

1.2.1 A Penrose-Fife model (see [31], [32])

The order parameter $\chi = \chi(t, x)$ satisfies

$$\partial_t \chi - \Delta \chi + W'(\chi) = \lambda'(\chi) \Big(\frac{1}{\vartheta_c} - \frac{1}{\vartheta} \Big), \ t > 0, \ x \in \Omega \subset R^3, \tag{1.3}$$

where W is typically a double-well potential $W(\chi) = (\chi^2 - 1)^2$, and ϑ_c stands for the critical phase change temperature.

The time evolution of the absolute temperature is governed by equation (1.2), with

$$\mathbf{q} = -\frac{1}{\vartheta^2} \nabla_x \vartheta. \tag{1.4}$$

Note that (1.4) may be viewed a generalized Fourier law with a singular heat conductivity coefficient proportional to ϑ^{-2}.

1.2.2 Caginalp models (see [8], [10])

In the Caginalp model proposed in [8], the time evolution of the order parameter is determined by equation

$$\partial_t \chi - \Delta \chi + W'(\chi) = \lambda'(\chi)\vartheta, \ t > 0, \ x \in \Omega \subset R^3, \tag{1.5}$$

while ϑ satisfies (1.2), where

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.6}$$

Here, in contrast with the Penrose-Fife model discussed above, the temperature is scaled so that $\vartheta = 0$ is the planar melting temperature.

The order parameter χ is typically a macroscopically observable property of the system, which can be traced to a statistical mechanical average serving to distinguish the two separate phases (see STANLEY [38]). Moreover, it is known that order parameters for some systems are conserved quantities, while in others are non-conserved (see [38]). Accordingly, CAGINALP [9] proposed a conserved analogue of (1.5) based on Cahn-Hilliard type ideas, specifically,

$$\partial_t \chi + \Delta \Big(\Delta \chi - W'(\chi) + \lambda'(\chi)\vartheta \Big) = 0, \ t > 0, \ x \in \Omega \subset R^3. \tag{1.7}$$

In certain applications, it is more appropriate to replace the classical Fourier law (1.6) by a general Coleman-Gurtin-type constitutive equation

$$\mathbf{q} = -\kappa \nabla_x \vartheta - \int_0^\infty k(s)\vartheta(t-s) \ \mathrm{d}s, \tag{1.8}$$

where $\kappa > 0$ stands for the "instantaneous" heat conductivity coefficient and k is a suitable dissipative kernel (cf. COLEMAN and GURTIN [12]).

1.2.3 Nonlocal phase-field models (see [6], [7])

One of the simplest models which takes into account long-range interactions, for example an Ising system with the so-called Kac potential is represented by the equation

$$\partial_t \chi + \chi - J * \chi + W'(\chi) = \lambda'(\chi)\vartheta, \ t > 0, \ x \in \Omega \subset R^3, \tag{1.9}$$

where

$$J * \chi(x,t) = \int_\Omega J(x-y)\chi(y,t) \ \mathrm{d}y, \tag{1.10}$$

for a suitable kernel J (see also WANG [39]).

As usual, equation (1.9) is supplemented with (1.2) describing the evolution of the temperature ϑ. The resulting system may be viewed as a simple model of phase transitions, where the convolution kernel accounts for interactions between states in both short and long scales (for related models see GAJEWSKI and ZACHARIAS [19], KREJČÍ et al. [28]).

Unlike the situation described in Sections 1.2.1, 1.2.2, equation (1.9) is of "neutral" type with no smoothing effect on χ. On one hand, such a setting allows for discontinuities - sharp phase transitions - propagating in time; on the other hand, the lack of regularity makes the analysis of the model rather delicate.

1.2.4 Parabolic-hyperbolic phase-field models (see [20])

There are rapid phase transformation processes in non-equilibrium dynamics, for which it is appropriate to take the inertial term proportional to $\partial_{tt}\chi$ into account. Accordingly, the time derivative $\partial_t\chi$ in (1.5), (1.7), and (1.9) has to be replaced by $(\partial_t + \varepsilon\partial_{tt})\chi$, where $\varepsilon > 0$ denotes a (possibly small) parameter. Consequently, the resulting system becomes hyperbolic-parabolic changing considerably the mathematical features of the problem.

2 Free energy

The free energy functional $E(\chi, \vartheta)$ plays the role of a Lyapunov function in the phase-field models. In order to see this, let us start with the Penrose-Fife system (1.2 - 1.4).

We shall assume that $\Omega \subset R^3$ is a bounded sufficiently regular domain and prescribe the homogeneous Neumann boundary conditions

$$\nabla_x\chi \cdot \mathbf{n}|_{\partial\Omega} = 0, \text{ with } \mathbf{n} \text{ the outer normal vector,} \tag{2.11}$$

for the order parameter χ, while the absolute temperature ϑ satisfies nonhomogeneous Dirichlet boundary conditions:

$$\vartheta|_{\partial\Omega} = \vartheta_\Gamma. \tag{2.12}$$

Assume, for the sake of simplicity, that

$$\vartheta_c = \vartheta_\Gamma = 1.$$

Multiplying equation (1.3) on $\partial_t\chi$ and using (1.2), (1.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{PF}(\chi,\vartheta) + \int_\Omega \left|\nabla_x\Big(1 - \frac{1}{\vartheta}\Big)\right|^2 \,\mathrm{d}x + \int_\Omega |\partial_t\chi|^2 \,\mathrm{d}x = 0, \tag{2.13}$$

where the free energy functional reads

$$E_{PF}(\chi,\vartheta) = \int_\Omega \Big[\vartheta - \log(\vartheta) + \frac{1}{2}|\nabla_x\chi|^2 + W(\chi)\Big] \,\mathrm{d}x \tag{2.14}$$

(see Section 4 in [16]).

Consequently, the integral

$$\int_0^\infty \int_\Omega \left|\nabla_x\Big(1 - \frac{1}{\vartheta}\Big)\right|^2 + |\partial_t\chi|^2 \,\mathrm{d}x\mathrm{d}t$$

is finite for any trajectory for which $E(\chi,\vartheta)$ remains bounded for $t \to \infty$. Thus we conjecture, at least intuitively, that

$$\partial_t\chi_\infty = 0, \ \vartheta_\infty = 1$$

for any $(\chi_\infty, \vartheta_\infty)$ belonging to the $\omega-$limit set of the trajectory $\cup_{t>0}(\chi(t), \vartheta(t))$, that means, $\chi_\infty = \chi_\infty(x)$ is a solution of the elliptic problem

$$-\Delta\chi_\infty + W'(\chi_\infty) = 0 \text{ in } \Omega,\ \nabla_x\chi_\infty \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.15}$$

Similarly, for the Caginalp model (1.5), (1.6), supplemented with the boundary condition 2.11, we recover the free energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} E_C(\chi, \vartheta) + \int_\Omega \mathrm{div}_x \mathbf{q}\vartheta \ \mathrm{d}x + \int_\Omega |\partial_t\chi|^2 \ \mathrm{d}x = 0, \tag{2.16}$$

with

$$E_C(\chi, \vartheta) = \int_\Omega \left[\frac{1}{2}|\vartheta|^2 + \frac{1}{2}|\nabla_x\chi|^2 + W(\chi)\right] \ \mathrm{d}x. \tag{2.17}$$

Assuming that $\mathbf{q}$ obeys the classical Fourier law (1.6), together with no-flux boundary condition

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.18}$$

we get

$$\int_\Omega \mathrm{div}_x \mathbf{q}\vartheta \ \mathrm{d}x = \int_\Omega \kappa|\nabla_x\vartheta|^2 \ \mathrm{d}x; \tag{2.19}$$

whence the $\omega-$limit set of any bounded trajectory consists of pairs of functions $\chi_\infty = \chi_\infty(x), \vartheta_\infty = \text{const}$, where

$$-\Delta\chi_\infty + W'(\chi_\infty) = \lambda'(\chi_\infty)\vartheta_\infty \text{ in } \Omega,\ \nabla_x\chi_\infty \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.20}$$

If we replace (2.18) by the homogeneous Dirichlet boundary conditions

$$\vartheta|_{\partial\Omega} = 0, \tag{2.21}$$

we get $\vartheta_\infty = 0$, and equation (2.20) reduces to (2.15).

In the case when the heat flux is given through the non-local (in time) constitutive equation (1.8), the term

$$\int_\Omega \mathbf{q} \cdot \nabla_x\vartheta \ \mathrm{d}x \text{ may not be negative (!), however, } -\int_0^\tau \int_\Omega \mathbf{q} \cdot \nabla_x\vartheta \ \mathrm{d}x \ \mathrm{d}t \geq 0$$

in accordance with the second law of thermodynamics provided the convolution kernel k is of positive type (see Chapter 2.6.6 in PRUESS [35]). Accordingly, the $\omega-$limit sets are still characterized through (2.20) (cf. Proposition 2.4 in [2]).

The situation becomes more delicate for the conserved phase-field system, where the order parameter χ satisfies the fourth order equation (1.7). Accordingly, the boundary condition (2.11) has to be replaced by

$$\nabla_x\chi \cdot \mathbf{n}|_{\partial\Omega} = \nabla_x(\Delta\chi) \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{2.22}$$

Applying the inverse operator Δ_N^{-1}, where Δ_N denotes the Laplace operator supplemented with the homogeneous Neumann boundary conditions, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} E_C(\chi, \vartheta) + + \int_\Omega |[-\Delta_N]^{-1/2}\partial_t\chi|^2 \ \mathrm{d}x - \int_\Omega \mathbf{q} \cdot \nabla_x\vartheta \ \mathrm{d}x = 0. \tag{2.23}$$

Here, necessarily, we have to impose the no-flux boundary condition for the heat flux in the form

$$\nabla_x\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{2.24}$$

which, together with (2.22), implies that

$$\int_\Omega \chi(t)\ \mathrm{d}x \text{ is a constant of motion.}$$

In particular, we can assume $\int_\Omega \chi\ \mathrm{d}x = 0$.

Consequently, the $\omega-$limit sets of bounded trajectories consist of pairs $\chi_\infty, \vartheta_\infty$ solving

$$\Delta\Big(\Delta\chi_\infty - W'(\chi_\infty) + \lambda'(\chi_\infty)\vartheta_\infty\Big) = 0,\ \vartheta_\infty = -\frac{1}{|\Omega|}\int_\Omega \lambda(\chi_\infty)\ \mathrm{d}x, \tag{2.25}$$

where χ_∞ satisfies the boundary conditions (2.22) (see Section 1 in [4]).

The free energy balance for the "non-local" phase-field model introduced in (1.9) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{NL}(\chi,\vartheta) + \int_\Omega |\partial_t\chi|^2\ \mathrm{d}x - \int_\Omega \mathbf{q}\cdot\nabla_x\vartheta\ \mathrm{d}x = 0, \tag{2.26}$$

where

$$E_{NL}(\chi,\vartheta) = \int_\Omega \Big[\frac{1}{2}|\vartheta|^2 + \frac{1}{2}|\chi|^2 - \frac{1}{2}\chi(J*\chi) + W(\chi)\Big]\ \mathrm{d}x. \tag{2.27}$$

Thus the candidates χ_∞, ϑ_∞ for the $\omega-$limit sets must satisfy

$$\chi_\infty - J*\chi_\infty + W'(\chi_\infty) = \lambda'(\chi_\infty)\vartheta_\infty,\ \vartheta_\infty = \text{const in } \Omega. \tag{2.28}$$

In particular, taking $\vartheta_\infty = 0$ and J a Green kernel associated to some elliptic operator, say,

$$-\Delta(J*\chi) = \chi \text{ in } \Omega,\ J*\chi|_{\partial\Omega} = 0,$$

equation (2.28) can be transformed to

$$\Delta w + \Gamma(w) = 0 \text{ in } \Omega,\ w|_{\partial\Omega} = 0, \tag{2.29}$$

where

$$w = W'(\chi_\infty) + \chi_\infty,\ \Gamma = (W' + \mathrm{Id})^{-1}$$

(see Section 1 in [14]).

On the point of conclusion, we infer that structure of the $\omega-$limit sets is intimately related to the solution set of the semi-linear elliptic equation

$$-\Delta w + F(w) = g(x) \text{ in } \Omega \tag{2.30}$$

supplemented with either Dirichlet or Neumann boundary conditions. It is known that the topology of the set of solutions to problem (2.30) can be non-trivial, in particular, there may be a continuum of solutions even if Ω is a ball (see HARADA et al. [24], MALCHIODI et al. [30], SENBA and SUZUKI [36], for the Neumann case, and AFTALION and PACELLA [1], for the Dirichlet case, to name only a few). In the light of these results, the problem of convergence to a single equilibrium becomes highly non-trivial even in very simple geometries of the underlying spatial domain.

3 The long-time behavior of bounded trajectories

3.1 Basic ideas

To begin with, let us show, on the intuitive level, how the Łojasiewicz-Simon theory applies to the convergence problem. We consider the situation described in (2.16 - 2.19), where, in addition, we suppose that the temperature ϑ satisfies the homogeneous Dirichlet boundary conditions (2.21).

Thus $\vartheta_\infty = 0$, and, integrating (2.16) we deduce

$$I(\chi_\infty) - I(\chi(\tau)) - \frac{1}{2}\int_\Omega \vartheta^2(\tau)\ \mathrm{d}x + \int_\tau^\infty \Big[\int_\Omega |\partial_t \chi|^2\ \mathrm{d}x + \int_\Omega \kappa |\nabla_x \vartheta|^2\ \mathrm{d}x \Big]\ \mathrm{d}t = 0, \tag{3.31}$$

where we have set

$$I(\chi) = \int_\Omega \frac{1}{2}|\nabla_x \chi|^2 + W(\chi)\ \mathrm{d}x. \tag{3.32}$$

Note that the value of $I(\chi_\infty)$ is the same for all χ_∞ belonging to a fixed $\omega-$limit set.

Supposing that the functional I satisfies the Łojasiewicz-Simon inequality (1.1) we get, by means of the Poincarè inequality,

$$\int_\tau^\infty \int_\Omega \Big[|\partial_t \chi|^2 + \kappa \vartheta^2 \Big]\ \mathrm{d}x\ \mathrm{d}t \le c\Big(|\partial I(\chi(\tau))|^{\frac{1}{1-\theta}} + \kappa \int_\Omega \vartheta^2(\tau)\ \mathrm{d}x \Big). \tag{3.33}$$

On the other hand, it follows from (1.5) that

$$\partial I \approx -\partial_t \chi + \lambda'(\chi)\vartheta;$$

whence, taking into account that $\int_\Omega \vartheta^2(\tau)\ \mathrm{d}x \to 0$ for $\tau \to \infty$, we conclude that

$$\int_\tau^\infty \int_\Omega \Big[|\partial_t \chi|^2 + \kappa \vartheta^2 \Big]\ \mathrm{d}x\ \mathrm{d}t \le c\Big(\int_\Omega \Big[|\partial_t \chi(\tau)|^2 + \kappa \vartheta^2(\tau) \Big]\ \mathrm{d}x \Big)^\beta, \text{ with } 1 < \beta < 2. \tag{3.34}$$

It is an easy exercise to show that (3.34) implies

$$\int_0^\infty \Big[\int_\Omega |\partial_t \chi|^2 + \kappa \vartheta^2\ \mathrm{d}x \Big]^{\frac{1}{2}}\ \mathrm{d}t < \infty,$$

in particular,

$$\chi(t) \to \chi_\infty \text{ in } L^2(\Omega) \text{ for } t \to \infty$$

(see Lemma 7.1 in [17]).

The reader will have noticed that many "details" have been omitted in the above discussion, in particular, we did not specify the associated function spaces in which ∂I should be understood. Another issue left open was compactness of trajectories in $C(\overline{\Omega})$, which is in fact necessary for the $\omega-$limit sets to be non-empty. We refer to [3, Theorem 2.1] for a complete proof of the following result.

Theorem 3.1 *Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\mu}$. Assume that $W \in C^2(R)$ such that*

$$W'(z)z > 0 \textit{ for all } |z| > 1,\ W|_{(-1,1)} \textit{ is real analytic.} \tag{3.35}$$

Furthermore, we suppose that $\lambda \in C^2(R)$ is globally Lipschitz on R, and $\lambda(0) = 0$. Let χ, ϑ be a classical solution of problem (1.2), (1.5), (1.6) for $t > 0$ satisfying the boundary conditions (2.11), (2.21) and such that

$$|\chi(t,x)| + |\vartheta(t,x)| \leq M \text{ for all } t > 0,\ x \in \Omega. \tag{3.36}$$

Then

$$\chi(t) \to \chi_\infty,\ \vartheta(t) \to 0 \text{ in } C(\overline{\Omega}) \text{ for } t \to \infty, \tag{3.37}$$

where χ_∞ solve (2.15).

The *existence* of global-in-time solutions for the phase-field model is not an issue in the present paper. The relevant results for the model considered in Theorem 3.1 are discussed in [3]. In particular, it can be shown that the problem in question possesses a global-in-time solution for any initial data $\chi(0,\cdot)$, $\vartheta(0,\cdot) \in L^2(\Omega)$ that becomes smooth for $t > 0$ because of the regularizing effect of the diffusion semigroup. As a matter of fact, the convergence claimed in (3.37) takes place in a much stronger topology, say, $C^2(\overline{\Omega})$ depending on the regularity of $\partial\Omega$.

A similar result can be proved for the Penrose-Fife system (1.2 - 1.4), supplemented with the homogeneous Neumann boundary condition (2.11) for χ, and a general (non-homogeneous) Dirichlet boundary condition (2.12) to be satisfied by ϑ. Here the main stumbling block is to keep the (absolute) temperature ϑ bounded below away from zero (see [16, Theorem 2.5]).

3.2 Conserved phase-field models of Caginalp's type

We focus on the situation described by system (1.2), (1.6), (1.7), supplemented with the boundary conditions (2.22), (2.24). Similarly to the above, the analysis is based on the free energy balance (2.23). However, new difficulties arise due to the fact that (i) the order parameter belongs to the factor space of functions of zero mean, (ii) the presence of the "negative" norm in (2.23) requires the Łojasiewcz-Simon inequality to be proved in the same space. Fortunately, the trajectories are precompact in relatively strong topologies as equation (1.7) is of fourth order. We report the following result (see Theorem 1.1 in [15]).

Theorem 3.2 *Let $\Omega \subset R^N$, $N = 1,2,3$ be a bounded domain of class $C^{2+\mu}$. Let the function W be real analytic on R, $\lambda(\chi) = \lambda\chi$ for a positive λ. Let χ, ϑ be a classical solution of problem (1.2), (1.6), (1.7) on $(0,\infty) \times \Omega$ satisfying the boundary conditions (2.22), (2.24), and such that*

$$|\chi(t,x)| + |\vartheta(t,x)| \leq M \text{ for all } t > 0,\ x \in \Omega.$$

Then

$$\chi(t) \to \chi_\infty,\ \vartheta(t) \to \vartheta_\infty \text{ in } C(\overline{\Omega}) \text{ for } t \to \infty,$$

where $\vartheta_\infty = \mathrm{const}$ and χ_∞ solve (2.25), with the boundary condition (2.22).

3.3 Problems with terms of memory type

We will be concerned with the situation, where the heat flux $\mathbf{q}$ in equation (1.2) is given through the constitutive equation (1.8). As already pointed out, the main difficulty lies in the fact that the "instantaneous" energy dissipation term

$$-\int_\Omega \mathbf{q} \cdot \nabla_x \vartheta \ \mathrm{d}x = \kappa \int_\Omega |\nabla_x \vartheta|^2 \ \mathrm{d}x + \int_\Omega \Big(\int_0^\infty k(s) \nabla_x \vartheta(t-s) \ \mathrm{d}s\Big) \nabla_x \vartheta \ \mathrm{d}x$$

appearing in (2.16) is not necessarily positive.

On the other hand, however, the second law of thermodynamics requires

$$\int_0^\tau \int_\Omega \Big(\int_0^\infty k(s) \nabla_x \vartheta(t-s) \, \mathrm{d}s \Big) \nabla_x \vartheta \, \mathrm{d}x \, \mathrm{d}t \geq 0 \text{ for } \tau \geq 0, \tag{3.38}$$

that means, the kernel k must be of positive type in the sense of PRUESS [35, Chapter 2.6.6]. A typical example of such a kernel reads

$$k(s) = s^{-\alpha} \exp(-\beta s), \text{ with } 0 \leq \alpha < 1, \ \beta > 0. \tag{3.39}$$

Because of the afore-mentioned difficulties, the free energy balance (2.16) (or its analogues) cannot be used in a direct fashion. A way to attack the problem is to consider the so-called summed past history of ϑ

$$\eta(t, s, x) = \int_{t-s}^t \vartheta(z, x) \, \mathrm{d}z, \ s \geq 0 \tag{3.40}$$

introduced by DAFERMOS [13]. Accordingly, the convolution term can be written as

$$\int_\Omega \Big(\int_0^\infty k(s) \nabla_x \vartheta(t-s) \, \mathrm{d}s \Big) \cdot \nabla_x \vartheta(t) \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \frac{1}{2} (-k')(s) \| \nabla_x \eta(t,s) \|^2_{L^2(\Omega; R^3)} \, \mathrm{d}s +$$

$$\frac{1}{2} \int_0^\infty \| \nabla_x \eta(t,s) \|^2_{L^2(\Omega; R^3)} \, \mathrm{d}k'(s)$$

(see Section 2 in [2]).

Thus we can introduce a new "free energy" functional

$$\tilde{E}_C(\chi, \vartheta) = E_C(\chi, \vartheta) + \int_0^\infty \frac{1}{2} (-k')(s) \| \nabla_x \eta(t,s) \|^2_{L^2(\Omega; R^3)} \, \mathrm{d}s,$$

for which the corresponding balance equation can be handled in a similar way as in Section 3.1 provided the kernel k meets certain decay properties specified in the following theorem (see Theorem 1.2 in [2]).

Theorem 3.3 *Let $\Omega \subset R^3$ be a domain of class $C^{2+\mu}$. Suppose, in addition, the λ, W belong to $C^2(R)$ and satisfy:*

$$\lambda(0) = 0, \ |\lambda'(z)| \leq \text{const } \textit{for all } z \in R,$$

$$W'(z) z > 0 \textit{ for } |z| > 1, \ W'(z) \mathrm{sgn} z > a|z| - b, \ a > 0, \ b \geq 0 \textit{ for } z \in R,$$

and W is real analytic on the interval $(-1, 1)$. Furthermore, assume that $\kappa > 0$ and k satisfies:

$$k \in L^1(0, \infty), \ k \textit{ convex}, \ \mathrm{d}k'(s) + \delta k'(s) \, \mathrm{d}s \geq 0 \textit{ for a certain } \delta > 0.$$

Then for any global-in-time classical solution χ, ϑ of problem (1.2), (1.5), (1.8), with the boundary conditions (2.11), (2.21), there exists χ_∞ - a solution of problem (2.15) such that

$$\chi(t) \to \chi_\infty, \ \vartheta(t) \to 0 \textit{ in } C(\overline{\Omega}) \textit{ for } t \to \infty.$$

Similar results for the conserved phase-field models were obtained in [4].

In [21], the authors consider the so-called Maxwell-Cattaneo heat conduction law:

$$\sigma \partial \mathbf{q}_t + \mathbf{q} = -\nabla_x \vartheta$$

that can be interpreted as a time convolution of $\nabla_x \vartheta$ with an exponential kernel and vanishing instantaneous heat conductivity. They obtain convergence results similar to Theorem 3.3.

3.4 Non-local phase-field models

There is a substantial difficulty when dealing with the non-local phase-field models and the associated free energy balance described through (2.26), (2.27). In all the previously known applications of the Łojasiewicz-Simon theory, the energy functional takes a form

$$E[v] \approx \int_\Omega \Big(\frac{1}{2} |\nabla_x v|^2 + F(v) \Big) \, \mathrm{d}x,$$

that means, a *quadratic form* perturbed by a non-linear *compact* functional. The underlying function spaces are the Sobolev spaces $W^{1,p}$ on which E is analytic provided p is large enough and F is an analytic function. Similarly to Theorems 3.1 - 3.3, the convergence takes place place in an arbitrarily smooth norm (cf. SIMON [37]).

On the other hand, the energy functional E_{NL} related to our problem is given by (2.27). As we do not expect, or at least it seems very hard to prove compactness of the trajectories $\cup_{t \geq 0} \chi(t)$ in $L^\infty(\Omega)$ (where E_{NL} is analytic together with W); the natural domain of definition of E_{NL} is the Hilbert space $L^2(\Omega)$. It is easy to see that $E_{NL} \in C^1(L^2(\Omega); R)$, however, it is well-known that, in general, $E_{NL} \notin C^2(L^2(\Omega); R)$ no matter how smooth W is. Consequently, Lojasiewicz-Simon's approach based on approximate linearizations must be considerably modified.

In [14], a "non-smooth" version of the Lojasiewicz-Simon theorem is proved. It applies basically to functionals of the form "maximal monotone operator + linear compact perturbation". It is of independent interest and we believe more applications can be found. Recently, the theory was further developed and used in a different context by GAJEWSKI and GRIEPENTROG [18].

Based on the "non-smooth" variant of the Łojasiewicz-Simon theory, the following convergence result was proved in Section [14, Theorem 1.2].

Theorem 3.4 *Assume that $\Omega \subset R^N$ is a bounded domain of class $C^{2+\mu}$. Let*

$$\chi \mapsto J * \chi \text{ be a compact self-adjoint operator on } L^2(\Omega)$$

taking bounded sets in $L^\infty(\Omega)$ in compact sets in $C(\overline{\Omega})$. In addition suppose that W is a real analytic function on R such that

$$W(0) = W'(0) = 0, \; W''(z) \geq -\beta \text{ for all } z \in R \text{ and a certain } \beta < 1,$$

$$\lim_{|z| \to \infty} W''(z) = \infty.$$

Then any solution χ, ϑ of problem (1.2), (1.6), (1.9), supplemented with the homogeneous Dirichlet boundary conditions (2.21), belonging to the class

$$\chi \in L^\infty(0, T; L^\infty(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)),$$

$$\vartheta \in L^\infty(0, T; W^{1,2}_0(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega))$$

for all $T > 0$ satisfies

$$\chi(t) \to \chi_\infty \text{ in } L^2(\Omega), \; \vartheta(t) \to 0 \text{ in } W^{1,2}_0(\Omega) \text{ for } t \to \infty,$$

where χ_∞ is a solution of the stationary problem (2.28).

3.5 Parabolic-hyperbolic problems

All the convergence results stated in Theorems 3.2 - 3.4 can be adapted to the case when the $\partial_t \chi$ in the equations for the order parameter is replaced by the "hyperbolic" operator $\varepsilon \partial_{tt} \chi + \partial_t \chi$ (see [21], [22], [23]). Here the basic tool is a variant of the Łojasiewicz-Simon theory for hyperpolic problems with damping developed by HARAUX and JENDOUBI [25].

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