A nice choice function in a countable model

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Our aim is to show that in a countable structure we can find choice functions which are in some sense consistent with definable sets. In general, unless a choice function is definable, by adding such a function we introduce new definable relations. We shall show that that it is always possible to add a choice function such the preimages of definable sets by the function are still definable in the original structure, ie., the choice function is "measurable" with respect to definable sets.

Theorem 1 *Let M be a structure with countable support, with constants for every element of the support and at most countably many relations. Suppose R is a binary relation definable in M such that $\forall x \exists y\ R(x,y)$. Then there exists a function f such that $\forall x\ R(x, f(x))$ and for every set A definable in M, the set $\{a;\ f(a) \in A\}$ is also definable in M.*

Proof. Let $V = \{v_1, v_2, \ldots\}$ be the support of M, let $A_1, A_2, \ldots$ be all subsets of V definable in M. Define f as follows: for $n = 1, 2, \ldots$ put $f(v_n) = v_m$ where m is the least number such that v_m belongs to $R\text{"}v_n \cap \bigcap_{i \in I} A_i$, where I is the lexicographically first subset of $\{1, \ldots, n\}$ such that $R\text{"}v_n \cap \bigcap_{i \in I} A_i \neq \emptyset$. Here $R\text{"}v_n$ stands for $\{a;\ R(a, v_n)\}$; the intersection $\bigcap_{i \in I} A_i$ is defined to be V if all sets A_i, $i = 1, \ldots, n$, are disjoint with $R\text{"}v_n$; the lexicographic ordering is determined by the ordering of the elements as $v_1 < v_2 < \ldots$.

First observe that for $n < m$ and $B \subseteq V$, if I is the lexicographically first subset of $\{1, \ldots, n\}$ such that $A \cap \bigcap_{i \in I} A_i \neq \emptyset$ and J is the lexicographically first subset of $\{1, \ldots, m\}$ such that $A \cap \bigcap_{i \in J} A_i \neq \emptyset$, then $I \subseteq J$ (in fact $I = J \cap \{1, \ldots, n\}$). The observation implies immediately the following claim.

Claim. For $j \geq n$, we have $f(v_m) \in A_n$ iff

$$n \in I \text{ where } I \text{ is the lexicographically first subset of } \{1, \ldots, n\} \text{ such that } R\text{"}v_m \cap \bigcap_{i \in I} A_i \neq \emptyset. \tag{1}$$

Since we have constants for elements of M, every finite subset is definable. Hence we only need to show that the inverse image of A_n by the function f is

*I proved this lemma some 15 years ago, but only now I wote it down.

definable in M up to a finite error. The latter fact follows from the claim, since the condition (1) is a condition on certain intersections of sets $R"v_m, A_1, \ldots, A_n$, which are definable in M. Namely, the condition (1) can be expressed as follows

$$\bigvee_{I;n\in I} \bigwedge_{J;J<I} R"v_m \cap \bigcap_{i\in J} A_i = \emptyset \ \wedge \ R"v_m \cap \bigcap_{i\in I} A_i \neq \emptyset.$$

Thus the preimage of every definable set in M by f is also definable in M. $\square$

The intended application was in constructions of ultrapowers using ultrafilters on definable subsets. To prove Łoś' theorem, one needs choice functions. The above theorem shows that these functions can be chosen so that we still remain in the boolean algebra of definable sets.