Some structural properties of low rank matrices related to computational complexity

Bruno Codenotti* Pavel Pudlák † Giovanni Resta*

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Abstract

We consider the problem of the presence of short cycles in the graphs of nonzero elements of matrices which have sublinear rank and nonzero entries on the main diagonal, and analyze the connection of these properties to the rigidity of matrices. In particular, we exhibit a family of matrices which shows that sublinear rank does not imply the existence of triangles. This family can also be used to give a constructive bound of the order of $k^{3/2}$ on the Ramsey number $R(3,k)$, which matches the best known bound. On the other hand we prove that sublinear rank implies the existence of 4-cycles. We prove some partial results towards establishing lower bounds on matrix rigidity and consequently on the size of logarithmic depth circuits for computing certain explicit linear transformations.

1 Introduction

The problem of relating the rank of a matrix to its structural properties given by the pattern of its nonzero entries is a classical problem in mathematics. In complexity theory the most famous instance of this problem is the relation between the communication complexity of a $\{0,1\}$ matrix and its rank over the field of reals [15, 11]. In this paper we consider general matrices over arbitrary fields and we study cycles in the graphs of nonzero elements. Our goal is to prove lower bounds on the rigidity of matrices which would imply nonlinear lower bounds on some algebraic circuits. This research goes in the direction proposed by Valiant [18], who defined the concept of rigidity of matrices and proved that lower bounds on the size of logarithmic depth circuits can be proved by constructing matrices with high rigidity. The *rigidity of a matrix* M is defined as the function $R_M(r)$, which for a given r gives the minimum number of entries of M which one has to change in order to reduce its rank to r or less. Valiant proved the following result.

Theorem 1 ([18]) *If for some $\varepsilon > 0$, the $n \times n$ matrix M_n has rigidity $R_{M_n}(\varepsilon n) \geq n^{1+\varepsilon}$, then the transformation $x \rightarrow M_n x$ cannot be computed by linear size and logarithmic depth circuits with gates computing linear functions over a given field.*

Another relation of this type was found by Razborov [13]. He proved that some weaker bounds on rigidity would imply that a $\{0,1\}$ matrix defines a function which is not in the communication complexity version of the polynomial hierarchy. The existing lower bounds on the rigidity are, however, insufficient even for that.

*Istituto di Matematica Computazionale del CNR, Via S. Maria 46, 56126-Pisa (Italy). e-mail: {codenotti,resta}@imc.pi.cnr.it.

†Mathematical Institute, AVČR, Žitná 25, 115 67 Praha 1, Czech Republic. e-mail: pudlak@math.cas.cz. Partially supported by grant A1019602 of the Academy of Sciences of the Czech Republic and by a cooperative research grant INT-9600919/ME-103 from the NSF (USA) and the MŠMT (Czech Republic), part of this research was done while visiting Istituto di Matematica Computazionale del CNR, Pisa.

Although both a random matrix and a matrix whose entries are different indeterminates have rigidity even larger than required by Theorem 1 (close to n^2), very little is known about *explicit* matrices. The best known lower bounds on the rigidity of explicit matrices are of the form $\Omega(\frac{n^2}{r}\log\frac{n}{r})$ [5], which gives only linear lower bounds on $R_M(\varepsilon n)$. It seems that Hadamard matrices have large rigidity over the real field, but the best bound is so far only $\Omega(n^2/r)$, (Kashin and Razborov [8]).

Let us call an *alternating cycle* an oriented graph which is a cycle, when the orientation is forgotten, and such that the orientation of the arcs on the cycle alternates with one exception (put otherwise, there is a vertex v on the cycle such that if we go around the cycle from v to v, the orientation of the edges alternates).

Let $A = (a_{ij})$ be an $n \times n$ matrix. The graph of nonzero entries of A is the directed graph with vertex set $\{1, \dots, n\}$, where (i, j) is an arc iff $a_{ij} \neq 0$.

Given a matrix, we call $[2, 2]$ *configuration* a 2×2 submatrix consisting of nonzero elements. In graph theoretical terms a $[2, 2]$ configuration corresponds to either an alternating 3-cycle, which is usually called a transitive triangle, (when the 2×2 submatrix has one entry on the main diagonal), or an alternating 4-cycle (when none of the entries of the 2×2 submatrix is on the main diagonal), or a 2-cycle (when the 2×2 submatrix has two entries on the main diagonal).

We are interested in odd alternating cycles as subgraphs of the graph of nonzero entries of low rank matrices, because of a connection to matrix rigidity. The reason for considering *odd* lengths is that an odd alternating cycle corresponds to a configuration in the matrix where one element is on the main diagonal, while an even alternating cycle is not connected to it. In particular, the following conjecture implies large rigidity, hence nonlinear lower bounds on the size of some circuits, for certain explicitly defined matrices. Note that we have to use some nontriviality assumption, such as having nonzero elements on the main diagonal or having a support, in order to get any interesting implication from the low rank assumption.

Conjecture 1 (The Odd Alternating Cycle Conjecture) *For every field F, there exist an odd k and $\varepsilon > 0$ such that every $n \times n$ matrix M with nonzero entries on the main diagonal, and such that rank$(M) \leq \varepsilon n$, contains an alternating cycle of length k.*

Furthermore, the Odd Alternating Cycle Conjecture implies a nonlinear lower bound on the computation of cyclic shifts on semilinear circuits studied in [12].

In Section 2 we shall show that Conjecture 1 does not hold for $k = 3$ for any field. We shall show that this example can be modified to give a very simple constructive lower bound on the Ramsey number $R(3, k) \geq k^{3/2}$, which matches the best known one, due to Alon [1]. If the field has characteristic different from 2, the counterexample can be provided by a symmetric matrix. For GF_2 we describe a family of symmetric matrices of rank $\frac{n}{4} + 2$ with 1's on the main diagonal and without a triangle, which is so far the best bound in this case. We also analyze some special cases, obtained by making further assumptions, for which matrices with sublinear rank must contain transitive triangles.

Our results should be contrasted with those by Rosenfeld [16], and Alon and Szegedy [2]. (Their results are stated in terms of vectors in E^r, but can be easily translated to statements on real valued matrices.) Rosenfeld proved that a symmetric positive definite $n \times n$ matrix of rank $\leq n/2$ with ones on the main diagonal contains a triangle. Hence the assumption of being positive semidefinite is essential. On the other hand, Alon and Szegedy proved that, for every $\delta > 0$ there exists k such that there are symmetric positive semidefinite matrices with ones on the main diagonal and rank $\leq n^\delta$ with no $k \times k$ principal submatrices of nonzero elements. The minimal k for which they get sublinear rank can be computed from their proof, but for small values, in particular for $k = 4$, it is open whether such matrices must contain K_4.

In Section 3 we prove that matrices with sublinear rank over real or complex numbers must contain a $[2, 2]$ configuration, in fact they must contain an alternating 4-cycle. For fields of nonzero characteristic the corresponding statement is an open problem.

In section 4 we give an explicit construction of a family of rigid matrices, assuming the Odd Alternating Cycle Conjecture is true. (We call a family of matrices rigid, if for some fixed $\varepsilon > 0$ the rigidity $R_M(\varepsilon n)$

of any $n \times n$ matrix M from the family is superlinear.) Even if the conjecture fails, these matrices may be good candidates for large rigidity.

Although our results do not improve on any current lower bound on circuit complexity, we nevertheless think that we made a visible progress in that area. Fundamental problems in circuit complexity cannot be solved by gradually increasing lower bounds. We need to make progress in associated combinatorial and algebraic problems, and this paper is a step in this direction.

2 Triangles in low rank matrices

2.1 Non symmetric case

We show here the construction of an $n \times n$ matrix with ones on the main diagonal, rank of the order of $n^{2/3}$, and such that the graph obtained by associating edges to nonzero entries of the matrix does not have transitive triangles. As a byproduct of our construction, we find another constructive bound on the Ramsey number $R(3, n)$ which is simpler than the construction obtained by Alon in [1].

Theorem 2 *For every m, there is an explicitly definable square matrix M of dimension $n \approx m^{3/2}$ such that M has 1's on the main diagonal, has rank $\leq m$ and the associated graph of nonzero elements does not contain a transitive triangle.*

Proof. We shall use an auxiliary symmetric graph G with no cycles of length less than 6, and with a nonlinear number of edges, e.g., the bipartite graph given by a projective plane. Note that it is easy to construct such graphs with m vertices and $O(m^{3/2})$ edges. Starting from G, we first describe an oriented graph H, and then a matrix which have nonzero entries corresponding to edges of H. For the sake of a simpler description, in the following we take G as the bipartite graph of a projective plane.

The vertices of our graph H will be pairs (P, L), where P corresponds to a point on a line L. The edges of H are given by pairs $((P, L), (P', L'))$, where P, P', L, L' are all different and $P \in L'$, i.e., the point P is incident to the line L'. It is straightforward to verify that there is no transitive triangle in H (using only the fact that there are no cycles of length less than 6 in G).

Now we associate to H a matrix as follows. We index both rows and columns by pairs (P, L). To a row (P, L), we assign the vector whose coordinates are the vertices of G (that is, points and lines) and which has an entry equal to -1 on P and an entry equal to 1 on L, all the other entries being zero. To a column (P, L), we assign the characteristic vector of the set $(L \setminus P) \cup \{L\}$, (i.e., the set consisting of the vertex L and all neighbours of L except P). Thus the matrix obtained as the product of the vectors associated to its rows and those associated with its columns has 1's on the main diagonal, -1's on the entries corresponding to the edges of H, and 0 elsewhere. The rank of this matrix is at most m, the number of vertices of G, while its size is equal to the number of edges of G, which is of the order of $m^{3/2}$. □

Notice that the above construction does not produce $\{0, 1\}$ matrices, except for the field $GF[2]$.

It is an easy observation that the graph H has oriented 3-cycles if G contains 6-cycles. This prevents from applying the construction to the symmetric case, where oriented 3-cycles become triangles. However, to get a graph without oriented 3-cycles, we can start from a dense bipartite graph without cycles of length less than 8 (the density becomes smaller in this case, but still nonlinear, i.e., of the order of $n^{4/3}$). Then it is possible to proceed as above, and then symmetrize the construction by adding the matrix and its transpose, thus getting an $n \times n$ symmetric matrix still of rank $O(n^{3/4})$, and without triangles. However the above approach fails over $GF[2]$, because symmetrization produces zeros on the main diagonal.

Using the fact that the rank (over any field) of a matrix is an upper bound on the size of the maximal independent set of a graph associated with the zero-nonzero pattern of the matrix, the above construction - after symmetrization - provides an explicit Ramsey graph. More precisely, it gives an n-vertex graph without triangles and with independent sets of size $O(n^{3/4})$. As it is, the bound is worse than Alon's bound. However, it is possible to work on the original construction, do another kind of symmetrization, and get the same asymptotic bound as Alon's, while significantly gaining in simplicity. The idea is to consider only the upper triangle of the matrix obtained from the graph H, and to copy it in the lower part of the matrix. In this way, we obtain a symmetric matrix without triangles, without having to start with a sparser graph G. On the other hand, we loose the "low rank" property, whereas the rank of the original matrix still bounds the size of the maximal independent set. We summarize these considerations in the following theorem.

Theorem 3 *Let G be the graph of a projective geometry with $2m$ vertices. Take a linear ordering of pairs (P, L), $P \in L$ and define a* symmetric *graph J on this ordering by connecting (P, L) with (P', L'), where (P, L) is less than (P', L') in the ordering, iff P, P', L, L' are all different and $P \in L'$. Then the graph J neither contains a triangle nor an independent set of size $2m + 1$, while the number of vertices is of the order of $m^{3/2}$.*

On the other hand, one can prove an $\Omega(\sqrt{n \log n})$ lower bound on the rank matrices with ones on the main diagonal and without triangles using the well-known bound on the Ramsey number $R(3, k) = \Theta(k^2/\log k)$. Namely, let an $n \times n$ matrix M be given, and $n \geq R(3, k)$. Color the edges of the complete graph on n vertices blue, if the corresponding entry in the right upper half of M is nonzero, and red otherwise. If M does not contain a transitive triangle in the right upper part, the complete graph does not contain a blue triangle. Hence there must be a red complete subgraph on k elements. This corresponds to a $k \times k$ principal submatrix with zeros above the main diagonal, which has rank k. Thus the least rank which implies the existence of triangles is between $c_1\sqrt{n \log n})$ and $c_2 n^{2/3}$, for some constants c_1, c_2.

2.2 Symmetric case over $GF[2]$

We now turn our attention to symmetric $\{0, 1\}$ matrices with low rank over $GF[2]$, a case for which we cannot apply the result of the previous section. In the following, we describe a family of matrices of rank $\frac{n}{4} + 2$ and without triangles.

Let I_k, J_k, and P_k denote the identity matrix, the matrix with all the entries equal to 1, and the matrix with the $(i, k-i)$-th entries equal to 1, respectively, all of size k. Let us consider the following $n \times n$ matrix, for $n = 4k$, written in block form.

$$A_n = I_n + B_n = \begin{pmatrix} I_k & I_k & I_k & J_k - P_k \\ I_k & I_k & J_k - I_k & P_k \\ I_k & J_k - I_k & I_k & P_k \\ J_k - P_k & P_k & P_k & I_k \end{pmatrix}$$

This is a family of symmetric matrices with the following properties:

- A_n is triangle-free. This property can be easily verified computing the trace of B_n^3. We have

$$B_n^2 = \begin{pmatrix} (k-2)J_k + 3I_k & 2(J_k - I_k) & 2(J_k - I_k) & 2P_k \\ 2(J_k - I_k) & (k-2)J_k + 3I_k & J_k - I_k & 2(J_k - P_k) \\ 2(J_k - I_k) & J_k - I_k & (k-2)J_k + 3I_k & 2(J_k - P_k) \\ 2P_k & 2(J_k - P_k) & 2(J_k - P_k) & (k-2)J_k + 3I_k \end{pmatrix},$$

  from which we readily see that $\mathrm{Tr}(B_n^3) = 4\mathrm{Tr}(6J_k - 6I_k) = 0$.

- $\mathrm{Rank}_2(A) = r = n/4 + 2$. Indeed the matrices A_n can be obtained as $A_n = UU^\top$, where $U^\top$ is the following $r \times n$ matrix.

$$\left(\begin{array}{c|ccc|c|ccc|c|ccc|ccc|c}
1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\hline
0 & & & & 0 & & & & 0 & & & & & & & 1 \\
\vdots & & I_{r-3} & & \vdots & & I_{r-3} & & \vdots & & I_{r-3} & & & J_{r-3} - P_{r-3} & & \vdots \\
0 & & & & 0 & & & & 0 & & & & & & & 1
\end{array}\right). \tag{1}$$

- A_n is regular of degree r.
- B_n has independent sets of size $k = r - 2$.
- $\mathrm{Rank}_{\mathbf{R}}(A_n) = n$, and A_n has exactly 5 distinct integer eigenvalues. More precisely A_n has the following eigensystem.
  - $\lambda_1 = r$ with multiplicity 1 and eigenvector $(1, \cdots, 1)^\top$.
  - $\lambda_2 = r - 4$ with multiplicity 1 and eigenvector
  $$(\overbrace{-1, \cdots, -1}^{k}, \overbrace{1, \cdots, 1}^{k}, \overbrace{1, \cdots, 1}^{k}, \overbrace{-1, \cdots, -1}^{k})^\top .$$
  - $\lambda_3 = 4 - r$ with multiplicity 2 and eigenvectors
  $$(\overbrace{-1, \cdots, -1}^{k}, \overbrace{0, \cdots, 0}^{k}, \overbrace{0, \cdots, 0}^{k}, \overbrace{1, \cdots, 1}^{k})^\top$$
  and
  $$(\overbrace{-1, \cdots, -1}^{k}, \overbrace{1, \cdots, 1}^{k}, \overbrace{-1, \cdots, -1}^{k}, \overbrace{1, \cdots, 1}^{k})^\top .$$
  - $\lambda_4 = -2$ with multiplicity $k - 1$ and eigenvectors
  $$\left(\begin{array}{c|c|c|c|c|c|c|c}
1 & & -1 & & -1 & & & 1 \\
\vdots & -P_{k-1} & \vdots & P_{k-1} & \vdots & P_{k-1} & -I_{k-1} & \vdots \\
1 & & -1 & & -1 & & & 1
\end{array}\right)^\top .$$
  - $\lambda_5 = 2$ with multiplicity $n - r - 1 = 3(k-1)$ and eigenvectors
  $$\left(\begin{array}{c}
\begin{array}{c|c|c|c|c|c}
 & 1 & & & -1 & \\
-I_{k-1} & \vdots & 0 & 0 & \vdots & P_{k-1} \\
 & 1 & & & -1 &
\end{array} \\
\hline
\begin{array}{c|c|c|c|c|c}
-1 & & & -1 & & \\
\vdots & P_{k-1} & 0 & \vdots & P_{k-1} & 0 \\
-1 & & & -1 & &
\end{array} \\
\hline
\begin{array}{c|c|c|c|c|c}
-1 & & -1 & & & \\
\vdots & P_{k-1} & \vdots & P_{k-1} & 0 & 0 \\
-1 & & -1 & & &
\end{array}
\end{array}\right)^\top .$$

The verification of the above properties can now be done by direct inspection.

It is an open problem whether or not the above family is extremal, i.e., whether all symmetric $\{0, 1\}$ matrices with rank over $GF[2]$ at most $\frac{n}{4} + 1$ must contain triangles.

2.3 Special cases

We now consider sublinear rank matrices satisfying additional constraints which guarantee that they must contain triangles.

Low rank and sparse factors.

It is known that a symmetric $\{0,1\}$ matrix M can be factored over $GF[2]$ as $M = AA^\top$, where the number of columns of A is equal to either r or $r+1$, and r is the rank of M over $GF[2]$, see [10]. This representation allows us to investigate the presence of triangles in M in a purely combinatorial way, by treating the rows of the matrix U as a set system. Since we assume that all the diagonal entries are nonzero, we have that all sets have odd cardinalities. A triangle corresponds to three sets, every two of which intersecting in odd sets. More precisely, the above decomposition of a symmetric matrix M over $GF[2]$ can be interpreted as representing M as an intersection matrix as follows. The rows and columns are indexed by sets of some family of subsets of $\{1,\ldots,r\}$, where r is the rank of M. The (i,j)-th entry of M is 1 iff the intersection of the index sets corresponding to row i and column j is odd. The rows of U are the characteristic vectors of the sets. We shall call set systems also *hypergraphs.* If all sets have size k, then we speak of *k-hypergraphs.*

Definition 1 *A* sunflower *(also called a* delta system*, or a* star*) with l petals and core Y is a family of sets $X_1,\ldots,X_l$ such that $X_i \cap X_j = Y$ for every $i \neq j$.*

Note that the assumption that $X_1,\ldots,X_l$ form a sunflower with an odd core is stronger than the assumption that every two sets intersect in an odd set. A classical result of Erdös and Radó [4] states that for a given k and l, every sufficiently large k-hypergraph contain a sunflower with l petals. It has been observed in [7] that a k-hypergraph, k odd, on an n element set of vertices with at least $n^{(k-1)/2}l^{(k+1)/2}\frac{1\cdot 3\cdots k}{2\cdot 4\cdots(k-1)}$ edges contains a sunflower with l petals and with an *even* core. A k-hypergraph can be easily constructed which shows that this bound cannot be essentially improved. On the other hand the next theorem shows that the case of odd cores is essentially different. The theorem is an easy consequence of a theorem of Füredi [6]. We are indebted to V. Rödl for suggesting this proof, which replaces our original, selfcontained, but more complicated one.

Theorem 4 *For every positive integers k,l, with k odd and $l \geq 3$, there exists an integer K such that for every $n \geq 1$ and every k-hypergraph H on n vertices with at least Kn edges, there exists a sunflower in H with l petals and an odd core.*

Proof. Let a k-hypergraph on n vertices be given. By Füredi's theorem there exists a subhypergraph H' with $|H'| \geq \varepsilon|H|$ such that for every two different sets $X,Y \in H'$, the intersection $X \cap Y$ is a core of a sunflower with l petals contained in H'. The constant $\varepsilon > 0$ depends only on k and l. Thus we only need to show that there is at least one odd intersection. Suppose $K\varepsilon > 1$, then $|H'| > 1$. Hence, by the odd town theorem (an easy algebraic argument) there is at least one odd intersection and we are done. $\square$

This theorem implies that if $M = AA^\top$, where the $n \times \frac{n}{K}$ matrix A is "sparse" and M has 1's on the main diagonal, then the associated matrix must contain a complete graph on l vertices. We shall show that a similar statement holds also with $M = AB^\top$, where A and B may be different sparse matrices.

Theorem 5 $\forall k,l$, $\exists\varepsilon > 0$ *such that* $\forall n$, *if $M = AB$ (over $GF[2]$), where M is an $n \times n$ matrix with ones on the main diagonal, A is an $n \times r$ matrix, B is an $r \times n$ matrix, $r \leq \varepsilon n$, A (resp. B) has at most k ones in each column (resp. row), then there exists an $l \times l$ principal submatrix of ones in M. (Note that M does not need to be symmetric.)*

Theorem 5 can be reformulated in an equivalent way, in set intersection terms:

Theorem 6 *$\forall k, l$, $\exists \varepsilon > 0$, $\forall n$, and for all sets $(A_1, B_1), \ldots, (A_n, B_n)$, $|A_i|, |B_i| \leq k$, $A_i, B_i \subseteq X$, $|X| = r \leq \varepsilon n$, where for all i, if $|A_i \cap B_i|$ is odd, then there exist $i_1, \ldots, i_l$, such that for all $1 \leq \alpha, \beta \leq l$, with $\alpha \neq \beta$, we have that $|A_{i_\alpha} \cap B_{i_\beta}|$ is odd.*

Theorem 6 follows from a stronger result, which we prove below.

Theorem 7 *$\forall k, l$, $\exists \varepsilon > 0$, $\forall n$, and for all sets $(A_1, B_1), \ldots, (A_n, B_n)$, $|A_i|, |B_i| \leq k$, $A_i, B_i \subseteq X$, $|X| = r \leq \varepsilon n$, if for all i, $|A_i \cap B_i|$ is odd, then there exists a set D of odd cardinality and $i_1, \ldots, i_l$ such that for all $1 \leq \alpha, \beta \leq l$, with $\alpha \neq \beta$, we have that $A_{i_\alpha} \cap B_{i_\beta} = D$.*

Proof. Let $C_i = A_i \cap B_i$, i.e., $|C_i|$ is odd. By Theorem 4, we have that there exist $j_1, \ldots, j_m$ and D such that $\forall \alpha \neq \beta$, $C_{j_\alpha} \cap C_{j_\beta} = D$.

Now, for every i, choose the mappings

$f_i : P(A_i) \to \{0, \ldots, 2^k - 1\}$,

$g_i : P(B_i) \to \{0, \ldots, 2^k - 1\}$,

and assign the colour $(f_i(A_i \cap B_j), g_j(A_i \cap B_j))$ to the pair (i, j), for $i < j$. By Ramsey Theorem, there exists $\{i_1, \ldots, i_l\} \subseteq \{j_1, \ldots, j_m\}$ such that all pairs have the same colour.

Claim: $\forall i, i', j, j' \in \{i_1, \ldots, i_l\}, i < j, i' < j'$, we have $A_i \cap B_j = A_{i'} \cap B_{j'}$.

Proof: $A_i \cap B_j = A_i \cap B_{j'} = A_{i'} \cap B_{j'}$.

Thus there exists a set D' of odd cardinality such that $A_i \cap B_j = D'$, for all $i, j \in \{i_1, \ldots, i_l\}$, $i < j$. Symmetrically, there exists D'' such that $A_i \cap B_j = D''$, for all $i, j \in \{i_1, \ldots, i_l\}$, $i > j$. W.l.o.g. we can assume that $l \geq 4$. Then $D' \subseteq A_1 \cap B_2 \cap A_2 \cap B_3 \cap A_3 \cap A_4 \subseteq A_2 \cap B_2 \cap A_3 \cap B_3 = D$. But also $D \subseteq A_1 \cap B_1 \cap A_2 \cap B_2 \subseteq A_1 \cap B_2 = D'$. Therefore $D = D'$. By symmetry $D = D''$.

□

Note that sunflower theorems had been used at least twice to prove lower bounds on the size of circuits [14, 7].

Rank equal to the maximal independent set.

We show that an $2n \times 2n$ 0-1 matrix M, with rank n, maximal independent set of size n, and without 'triangles' must be bipartite. Let I denote the identity matrix. We represent M (or $P^\top M P$, where P is a suitable permutation matrix) as a 2×2 block matrix, with $n \times n$ blocks:

$$M = \begin{pmatrix} I & B \\ C & E \end{pmatrix}$$

such that $E \geq I$, (element-wise). Note that the case $E = I$ would imply that the graph is bipartite.

We will show that M (in order not to have triangles) must be bipartite.

We can write

$$M = \begin{pmatrix} I & B \\ C & E \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} I & B \\ 0 & E - CB \end{pmatrix},$$

where M' is obtained from M by Gaussian elimination, and thus has the same rank as M. It is easy to see that if $E - CB \neq 0$ then the rank of M' would exceed n, hence we can assume that $E = CB$. If $E = I$ then M is clearly bipartite. Assuming by contradiction that $E \neq I$ we obtain that M must have triangles. In fact, since $E \geq I$, $E \neq I$ implies that E must have at least an off-diagonal nonzero entry, say e_{pq}, with $p \neq q$. Since $E = CB$ we have

$$1 = E_{pq} = \sum_{k=1}^{n} c_{pk} b_{kq},$$

which implies that there is at least one index k such that both c_{pk} and b_{kq} are equal to 1. The thesis follows by observing that e_{pq}, c_{pk} and b_{kq} form a triangle in M.

3 4-cycles in low rank matrices

Triangles are special cases of a $[2,2]$ configuration. We have seen so far that only in special cases sublinear rank matrices must contain them. In this section we consider general $[2,2]$ configurations and show that they must appear. We first analyze the special case of $\{0,1\}$ matrices with constant row sums, and then generalize the result obtained in this case to arbitrary matrices.

Lemma 8 *Let $B = I + A$ be a $\{0,1\}$ $n \times n$ symmetric matrix with 1's on the main diagonal and row sums all equal to $d \geq 2$.*

1. *If B does not contain 2×2 full submatrices, except for those with two entries on the main diagonal, then the following equalities hold*

$$\begin{array}{ll} \mathrm{tr}(A) \ = 0 & \mathrm{tr}(B) \ = n \\ \mathrm{tr}(A^2) = n(d-1) & \mathrm{tr}(B^2) = nd \\ \mathrm{tr}(A^3) = 0 & \mathrm{tr}(B^3) = n(3d-2) \\ \mathrm{tr}(A^4) = n(2d^2-5d+3) & \mathrm{tr}(B^4) = n(2d^2+d-2) \end{array} \tag{2}$$

2. *If B does not contain 2×2 full submatrices, except for those with one or two entries on the main diagonal, then the following equalities hold*

$$\begin{array}{ll} \mathrm{tr}(A) \ = 0 & \mathrm{tr}(B) \ = n \\ \mathrm{tr}(A^2) = n(d-1) & \mathrm{tr}(B^2) = nd \\ \mathrm{tr}(A^3) = n(d^2-3d+2) & \mathrm{tr}(B^3) = nd^2 \\ \mathrm{tr}(A^4) = n(2d^2-5d+3) & \mathrm{tr}(B^4) = n(6d^2-11d+6) \end{array} \tag{3}$$

Proof.

1. Let us consider the undirected graph G associated with A. Since A does not contain 2×2 full submatrices, G cannot have cycles of length 1, 3 or 4. It is well known that the entry $a_{ii}^{(k)}$ on the main diagonal of A^k is equal to the number of closed walks of length k in G, which originate and terminate at node i. Since the nodes of G have degree $d-1$, then $a_{ii}^{(2)} = d-1$. The lack of self-loops and 3-cycles implies that $a_{ii}^{(3)} = 0$. The closed walks of length 4 from i to i consist of the $(d-1)^2$ walks $i \to j \to i \to h \to i$, with $j, h \neq i$, and of the $(d-1)(d-2)$ walks $i \to j \to h \to j \to i$, for $j \neq i$ and $h \neq i, j$.
2. We evaluate $\mathrm{tr}(A^3)$ and $\mathrm{tr}(A^4)$ under the assumption that A can contain 3-cycles. The number of closed walks of length 3 from i to i is equal to the number of 3-cycles, since A does not contain self-loops, and thus we obtain $\mathrm{tr}(A^3) = n(d-1)(d-2) = n(d^2-3d+2)$. It is easy to see that the number of closed walks of length 4 from i to i does not increase w.r.t. the previous case, hence we still have $\mathrm{tr}(A^4) = n(2d^2-5d+3)$.

In both cases, the values of $\mathrm{tr}(B^k)$ are easily obtained by expanding $B^k = (I+A)^k$. □

Theorem 9 *Let B be a $\{0,1\}$ $n \times n$ symmetric matrix with 1's on the main diagonal constant row sums.*

1. *If $\mathrm{rk}(B) < \frac{8}{17}n$, then B contains a 2×2 full submatrix without two entries on the main diagonal;*
2. *if $\mathrm{rk}(B) < \frac{1}{6}n$, then B contains a 2×2 full submatrix with no entries on the main diagonal.*

Proof Assuming by contradiction that B does not contain a 2×2 full submatrix, then equalities (2) hold. Since B is symmetric we have $\mathrm{rk}(B)=\mathrm{rk}(B^2)$ and we can apply the know inequality

$$\mathrm{rk}(C)\geq \frac{(\mathrm{tr}(C))^2}{\mathrm{tr}(C^2)}\,,$$

to B^2, obtaining

$$\mathrm{rk}(B)\geq \frac{(nd)^2}{n(2d^2+d-2)}=\frac{nd^2}{2d^2+d-2}\,, \tag{4}$$

for $d\geq 2$.

Since

$$\min_{d>2}\frac{d^2}{d^2+d-2}=\frac{8}{17}\,,$$

(where the minimum is reached for $d=4$), we obtain a contradiction with the assumption $\mathrm{rk}(B)<\frac{8}{17}n$, and the thesis follows.

The proof of the second case, $\mathrm{rk}(B)<\frac{1}{6}n$, is obtained similarly, applying equalities (3). $\square$

We now proceed to generalizing the above results. The intermediate step is provided by Lemma 10 below, which gives a lower bound on the rank of doubly stochastic matrices without $[2,2]$ configurations. This Lemma is then instrumental to obtain a more general result.

In the following we denote by $A\cdot B$ the element-wise product of two matrices.

Lemma 10 *Let B be an $n\times n$ matrix such that the matrix $B\cdot B$ is doubly stochastic, i.e., $\sum_j b_{ij}^2=\sum_i b_{ij}^2=1$ for $i,j=1,\ldots,n$.*

- *If B does not contain a full 2×2 submatrix, then $\mathrm{rk}(B)>n/2$;*
- *if B does not contain full 2×2 submatrices, except for those with two entries on the main diagonal, then $\mathrm{rk}(B)>n/4$;*
- *if B does not contain full 2×2 submatrices, except for those with one or two entries on the main diagonal, then $\mathrm{rk}(B)>n/6$.*

Proof. We have $\mathrm{rk}(B)\geq\mathrm{rk}(BB^\top)$, and we can apply the inequality

$$\mathrm{rk}(C)\geq \frac{(\mathrm{tr}(C))^2}{\mathrm{tr}(C^2)}$$

to $C=BB^\top$.

The value of $\mathrm{tr}(BB^\top)=\mathrm{tr}(C)$ can be easily calculated, since $\sum_j b_{ij}^2=1$ for $i=1,\ldots,n$. Indeed we have $c_{ij}=\sum_h b_{ih}b_{jh}$, hence $c_{ii}=\sum_h b_{ih}^2=1$, and $\mathrm{tr}(C)=\sum_i c_{ii}=n$. The trace of C^2 can be computed similarly:

$$\mathrm{tr}(C^2)=\sum_{i,j}c_{ij}c_{ji}=\sum_{i,j}\left(\sum_h b_{ih}b_{jh}\right)\left(\sum_k b_{ik}b_{jk}\right)=\sum_{i,j,h,k}b_{ih}b_{jh}b_{ik}b_{jk}\,.$$

Let δ_{ij} be equal to 1 if $i=j$, and 0 otherwise. Then we have

$$\mathrm{tr}(C^2)=[\delta_{ij}+\delta_{hk}-\delta_{ij}\delta_{hk}+(1-\delta_{ij})(1-\delta_{hk})]\sum_{i,j,h,k}b_{ih}b_{jh}b_{ik}b_{jk},$$

and thus

$$\mathrm{tr}(C^2) = \sum_{i,h,k} b_{ih}^2 b_{ik}^2 + \sum_{i,j,h} b_{ih}^2 b_{jh}^2 - \sum_{i,h} b_{ih}^4 + \sum_{i\neq j,h\neq k} b_{ih}b_{jh}b_{ik}b_{jk}.$$

Exploiting the fact that $B \cdot B$ is doubly stochastic, we get

$$\sum_{i,h,k} b_{ih}^2 b_{ik}^2 = \sum_i \left[\sum_j b_{ij}^2\right]^2 = n \quad \text{and} \quad \sum_{i,j,h} b_{ih}^2 b_{jh}^2 = \sum_h \left[\sum_k b_{kh}^2\right]^2 = n\,,$$

and, since $\sum_{ij} b_{ij}^4 > 0$, we have $\mathrm{tr}(C^2) < 2n + T$, with

$$T = \sum_{i\neq j,h\neq k} b_{ih}b_{jh}b_{ik}b_{jk}\,.$$

The products in T correspond exactly to all 2×2 nonzero submatrices of B (each counted 4 times due to symmetries). Hence, if B does not contain such submatrices, we have $T = 0$ and $\mathrm{tr}(C^2) < 2n$, from which

$$\mathrm{rk}(B) = \mathrm{rk}(C) \geq \frac{\mathrm{tr}(C)^2}{\mathrm{tr}(C^2)} > \frac{n^2}{2n} = \frac{n}{2}\,, \tag{5}$$

as claimed.

If there are 2×2 nonzero submatrices with two entries on the main diagonal of B, then the nonzero terms in T are those obtained setting either $i = h$, $j = k$ or $i = k$, $j = h$. The two cases are disjoint (since $i \neq j$ and $h \neq k$) and thus we get

$$\sum_{i\neq j,h\neq k} b_{ih}b_{jh}b_{ik}b_{jk} = 2\sum_{i\neq j} b_{ii}b_{ij}b_{ji}b_{jj} \leq 2\sum_{i,j} b_{ij}b_{ji} \leq 2\sum_{i,j} \frac{b_{ij}^2 + b_{ji}^2}{2} = 2n\,,$$

by stochasticity and using the inequality $xy \leq (x^2+y^2)/2$. Summarizing we obtain $\mathrm{tr}(C^2) < 4n$, from which, proceeding as in (5), we have $\mathrm{rk}(B) > n/4$.

If there are also the 2×2 submatrices with one entry on the main diagonal, then at least one of $i = h$, $i = k$, $j = h$, $j = k$ holds, so that we can write

$$\begin{aligned}\sum_{i\neq j,h\neq k} b_{ih}b_{ik}b_{jh}b_{jk} \quad &\leq \quad \sum_{ijh} b_{ih}b_{ii}b_{jh}b_{ji} + \sum_{ijk} b_{ij}b_{ik}b_{jj}b_{jk} + \\ &+ \quad \sum_{ijh} b_{ih}b_{ij}b_{jh}b_{jj} + \sum_{ijk} b_{ii}b_{ik}b_{jh}b_{jk}\,.\end{aligned}$$

We have

$$\sum_{ijh} b_{ih}b_{ii}b_{jh}b_{ji} = \sum_{ij} b_{ii}b_{ji}\sum_h b_{ih}b_{jh} \leq \sum_{ij} b_{ii}b_{ji}\sum_h \frac{b_{ih}^2 + b_{jh}^2}{2} = \sum_{ij} b_{ii}b_{ji},$$

and

$$\sum_{ij} b_{ii}b_{ji} \leq \sum_i\sum_j \frac{b_{ii}^2 + b_{ji}^2}{2} \leq n\,.$$

The other 3 terms can be bounded analogously, thus giving $\mathrm{tr}(C^2) < 2n + 4n = 6n$. Proceeding as in (5), we have $\mathrm{rk}(B) > n/6$. □

Definition 2 *If A is an $n \times n$ matrix and σ is a permutation of $\{1, \ldots, n\}$, then the sequence of elements $a_{1,\sigma(1)}, \ldots, a_{n,\sigma(n)}$ is called the* diagonal *corresponding to σ.*

Definition 3 *Let A be a square matrix. A is said to have* support *if it contains a diagonal of non-zero elements. A is said to have* total support *if every nonzero element of A lies on a diagonal of nonzero elements.*

Lemma 11 *Let A be a square matrix with total support, then there exists a matrix B with the same rank and nonzero pattern of A such that $B \cdot B$ is doubly stochastic.*

Proof. The matrix $A' = A \cdot A$, whose nonzero pattern is the same of A, is nonnegative and has total support. By [17] there exist two positive diagonal matrices D_1 and D_2 such that $D_1 A' D_2$ is doubly stochastic. It is easy to verify that the matrix $B = D_1^{1/2} A D_2^{1/2}$ meets our conditions, since it has the same pattern and rank of A and

$$B \cdot B = (D_1^{\frac{1}{2}} A D_2^{\frac{1}{2}}) \cdot (D_1^{\frac{1}{2}} A D_1^{\frac{1}{2}}) = (D_1^{\frac{1}{2}} \cdot D_1^{\frac{1}{2}})(A \cdot A)(D_2^{\frac{1}{2}} \cdot D_2^{\frac{1}{2}}) = D_1 A' D_2 .$$

□

Theorem 12 *Let A be an $n \times n$ matrix with total support.*

- *If $\mathrm{rk}(A) \leq n/2$, then A contains at least one full 2×2 submatrix;*
- *if $\mathrm{rk}(A) \leq n/4$, then A contains at least one full 2×2 submatrix with at most one entry on the main diagonal;*
- *if $\mathrm{rk}(A) \leq n/6$, then A contains at least one full 2×2 submatrix with no entry on the main diagonal.*

Proof. Let us suppose, by contradiction, that A does not contain a full 2×2 submatrix. Then by Lemmas 10 and 11 there exists a matrix B such that $\mathrm{rk}(A) = \mathrm{rk}(B) > n/2$. This contradicts the assumption $\mathrm{rk}(A) \leq n/2$, and hence proves the thesis. The other two cases, $\mathrm{rk}(A) \leq n/4$ and $\mathrm{rk}(A) \leq n/6$, are similar. □

Corollary 13 *Let A be an $n \times n$ matrix with support.*

- *If $\mathrm{rk}(A) \leq n/2$, then A contains at least one full 2×2 submatrix;*
- *if $\mathrm{rk}(A) \leq n/4$, then A contains at least one full 2×2 submatrix with at most one entry on the main diagonal;*
- *if $\mathrm{rk}(A) \leq n/6$, then A contains at least one full 2×2 submatrix with no entry on the main diagonal.*

Proof. If A does not have total support, then there exist two permutation matrices P and Q such that $B = PAQ$ is a block triangular matrix, whose diagonal blocks $B_1, B_2, \ldots, B_t$ have total support [3]. The matrices B_i are uniquely determined within arbitrary permutation of their rows, but their ordering in B is not necessarily unique. As a consequence we cannot guarantee that the multiplication by P and Q maps all the diagonal elements of A inside the union of the blocks B_i. However, we can choose P and Q in such a way that every diagonal element of A is either outside the union of the diagonal matrices B_i or on the diagonal of one of them.

The sum of the ranks of the diagonal blocks of B does not exceed the rank of A, and thus, if $\mathrm{rk}(A) \leq n/k$, there is at least a block, say B_j, of size m and rank less than m/k. Depending on k we consider 3 cases.

If $\mathrm{rk}(B_j) \leq m/2$, by Theorem 12, we have that B_j, and thus $A = P^{-1} B Q^{-1}$ contains a 2×2 submatrix of nonzeros, as claimed. In fact it is easy to see that the multiplication by permutation matrices, like P and Q and their inverses, preserves such a structure.

If $\mathrm{rk}(B_j) \leq m/4$, by Theorem 12, we have that B_j contains a 2×2 submatrix of nonzeros, with at most one entry on the main diagonal of B_j. Since all the diagonal elements of A which appear inside B_j are located on the main diagonal of B_j, we have that the 2×2 submatrix of B_j, once mapped again in A, can contain at most one diagonal entry, as required.

The case $\mathrm{rk}(B_j) \leq m/6$ is similar to the case $\mathrm{rk}(B_j) \leq m/4$. □

4 New candidates for high rigidity

We now describe an explicit construction of circulant matrices which have rigidity of the order of $n(\log n)^{1/(k-1)}$, provided that the the Odd Alternating Cycle Conjecture is true. Recall that a circulant matrix is a matrix fully determined by its first row, each other row being a cyclic shift of the previous one. For technical reasons, we number rows and columns of the matrices starting from 0, rather than 1. We construct a circulant matrix $C_n^{'}$ whose first row has nonzero entries in columns $1, b, b^2, \ldots, b^{m-1}$, where the choices of b and m are described below.

Lemma 14 *Let $n = 2^{2m} - 1$, and define $a = 2^{2m-1} + 2^{m-1}$, and $b = a + 1$. The following relations hold over $\mathbf{Z}_n$ for $1 \leq h \leq m$:*

$$\begin{aligned} a^2 &\equiv_n a & (6) \\ b^h &\equiv_n 2^{2m-1} - 2^{m-1} + 2^{h-1} + 2^{h+m-1}. & (7) \end{aligned}$$

Proof. From $2^{2m} \equiv_n 1$, we easily obtain (6), since

$$a^2 = 2^{4m-2} + 2 \cdot 2^{2m-1} 2^{m-1} + 2^{2m-2} \equiv_n 2 \cdot 2^{2m-2} + 2^{m-1} \equiv_n a\,.$$

Hence we also have that $a^h \equiv_n a$, for $h > 0$. Relation (7) is obtained as follows

$$\begin{aligned} b^h &= (a+1)^h = 1 + \sum_{i=1}^{h} \binom{h}{i} a^i = (2^h - 1)a + 1 \\ &\equiv_n 2^{h-1} + 2^{h+m-1} - 2^{2m-1} - 2^{m-1} + 1 \equiv_n 2^{2m-1} - 2^{m-1} + 2^{h-1} + 2^{h+m-1}\,. \end{aligned}$$

where we used (6) and $2^{2m} \equiv_n 1$ to simplify the expressions. □

Corollary 15 *The set $\{1, b, b^2, \ldots, b^{m-1}\}$, with the elements taken modulo n, has size m and it is a subgroup of the multiplicative group $\mathbf{Z}_n^*$.*

Proof. The size is immediate from (7). To see that it is a subgroup, just check that $b^m \equiv_n 1$. □

Let us consider, for an integer α invertible over $\mathbf{Z}_n$, a matrix C_n'' defined by

$$c''_{i,j} = c'_{\alpha i, \alpha j}\,, \tag{8}$$

where indices run from zero and are computed over $\mathbf{Z}_n$.

It is easy to see that the effect of (8) is to permute the diagonals in such a way that C_n'' is still circulant. In particular, if $\alpha = b^{-j}$, with $1 \leq j \leq m-1$, the elements of the diagonal corresponding to b^j are moved to diagonal 1, and, since $\{1, b, b^2, \ldots, b^{m-1}\}$ and $\{b^{-j}, b^{-j+1}, \ldots, b^{m-1-j}\}$ coincide (by Corollary 15), we have $C_n'' = C_n'$.

We summarize relevant properties of C_n' in the following observations.

Observation 1 *Let $n = 2^{2m-1}$. There are $m-1$ permutation matrices Q_k such that the automorphism $Q_k C_n^{'} Q_k^T = C_n^{''} = C_n^{'}$ corresponds to the transformation (8). In particular the permutation matrix Q_h, defined as $q_{ij} = 1$ iff $j = b^{-h} i$ and 0 elsewhere, takes the elements of diagonal b^h onto diagonal 1.*

Observation 2 *Let M be the matrix obtained from $C_n^{'}$ by deleting its first column and last row. M has a principal submatrix of order $\frac{n}{4}$ which is an identity matrix, since it is easy to verify, from (7), that $n/2 < b^j \pmod{n} < 3n/4$, for $1 \leq j \leq m-1$.*

The above two observations can be used to prove the following Theorem.

Theorem 16 *Assuming the Odd Alternating Cycle Conjecture for a field F and an odd k there exists an $\varepsilon > 0$ such that*
$$R_{C'_n}(\varepsilon n) = \Omega(n(\log n)^{1/(k-1)}).$$

Proof. By Observation 2, we have that the submatrix M (associated to the first diagonal of C'_n) contains an $\frac{n}{4} \times \frac{n}{4}$ identity matrix.

Let us assume that the Odd Alternating Cycle Conjecture is true, for an odd k and a constant $\varepsilon > 0$. Then it is easy to see that matrices of rank at most $\frac{1}{2}\varepsilon n$ must contain a linear number of alternating k-cycles. In order to decrease the rank of M below $\frac{1}{2}\varepsilon n$, we must thus either introduce a linear number of alternating k-cycles or change a linear number of the diagonal entries to 0. By Observation 2, we actually have a linear number of alternating k-cycles which do not contain entries from other diagonals of C'_n.

By Observation 1, we can rearrange C'_n by means of permutations so that the elements of each diagonal can in turn be moved to the first diagonal. This implies that we can repeat the previous argument for all the m *diagonals* of C'_n. Thus either more than half of the elements on more than half of the diagonals are changed to 0, in which case we are done, as this gives $\Omega(nm) = \Omega(n \log n)$ changes, or there are $\Omega(nm)$ alternating k-cycles. To get a lower bound on the number of changes in the latter case, we let d be the average number of changes in a row. We may assume that each row and each column contains at most $4d$ changes. The number of alternating k-cycles can be easily upper bounded by a function of the order of nd^{k-1}. Thus we get $nd^{k-1} = \Omega(nm)$, whence the number of changes must be $\Omega(n(\log n)^{1/(k-1)})$. □

5 Conclusions

The Odd Alternating Cycle Conjecture fails for $k = 3$, and a similar statement is false for complete graphs on k vertices, for k sufficiently large (even for positive semidefinite matrices). However, in spite of some effort, we were not able to disprove the conjecture even for $k = 5$. Furthermore, it is true for some special cases (products of sparse matrices) and the statement is true for $k = 4$ over reals. These results do not give better bounds on the size of linear circuits, but we are thinking of some generalizations which may have such consequences. Thus we believe that this is a promising research area, which will eventually lead to results on the complexity of linear circuits and solve problems posed more than 20 years ago. As the construction of a Ramsey graph shows, this research is also part of mainstream research in combinatorics.

To conclude, we shall insist on the three most important open problems, among those mentioned in this paper:

1. Does the Odd Alternating Cycle Conjecture hold for $k = 5$?
2. Does the Odd Alternating Cycle Conjecture hold for $k = 3$ for symmetric matrices over GF_2?
3. Does sublinear rank and nonzero elements on the main diagonal imply the existence of a 2×2 submatrix of nonzero elements also for fields of nonzero characteristic?

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