*Pattern Recognition*, Vol. 26, No. 1, pp. 167 174, 1993
Printed in Great Britain

0031 3203/93 $6.00 + .00
Pergamon Press Ltd


# PATTERN RECOGNITION BY AFFINE MOMENT INVARIANTS

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(*Received* 21 *November* 1991; *in revised form* 24 *April* 1992; *received for publication* 3 *June* 1992)

**Abstract**—The paper deals with moment invariants, which are invariant under general affine transformation and may be used for recognition of affine-deformed objects. Our approach is based on the theory of algebraic invariants. The invariants from second- and third-order moments are derived and shown to be complete. The paper is a significant extension and generalization of recent works. Several numerical experiments dealing with pattern recognition by means of the affine moment invariants as the features are described.

Feature extraction Affine transform Algebraic invariants Moment invariants
Pattern recognition Image matching

## 1. INTRODUCTION

A feature-based recognition of objects or patterns independent of their position, size, orientation and other variations has been the goal of much recent research. Finding efficient invariant features is the key to solving this problem. There have been several kinds of features used for recognition. These may be divided into four groups as follows:

(1) visual features (edges, textures and contours);
(2) transform coefficient features (Fourier descriptors,[1,2] Hadamard coefficients;[3]
(3) algebraic features (based on matrix decomposition of image, see reference (4) for details); and
(4) statistical features (moment invariants).

In this paper, attention is paid to statistical features. Moment invariants are very useful tools for pattern recognition. They were derived by Hu[5] and they were successfully used in aircraft identification,[6] remotely sensed data matching[7] and character recognition.[8] Further studies were made by Maitra[9] and Hsia[10] in order to reach higher reliability. Several effective algorithms for fast computation of moment invariants were recently described in references (11–13).

All the above-mentioned features are invariant only under translation, rotation and scaling of the object. In this paper, our aim is to find features which are invariant under general affine transformations and which may be used for recognition of affine-deformed objects. Our approach is based on the theory of algebraic invariants.[14] The first attempt to find affine invariants in this way was made by Hu,[5] but his affine moment invariants were derived incorrectly.

Several correct affine moment invariants are derived in Section 2, and their use for object recognition and scene matching is experimentally proved in Section 3.

## 2. AFFINE MOMENT INVARIANTS

The affine moment invariants are derived by means of the theory of algebraic invariants. They are invariant under general affine transformation

$$u = a_0 + a_1 x + a_2 y$$
$$v = b_0 + b_1 x + b_2 y. \quad (1)$$

The general two-dimensional $(p+q)$th order moments of a density distribution function $\rho(x, y)$ are defined as:

$$m_{pq} = \iint_{-\infty}^{\infty} x^p y^q \rho(x, y)\,dx\,dy \quad p, q = 0, 1, 2, \ldots \quad (2)$$

For simplicity we deal only with binary objects in this paper, then $\rho$ is a characteristic function of object $G$, and

$$m_{pq} = \iint_G x^p y^q\,dx\,dy \quad p, q = 0, 1, 2, \ldots \quad (3)$$

It is possible to generalize all the following relations and results for grey-level objects.

The affine transformation (1) can be decomposed into six one-parameter transformations:

1. $u = x + \alpha$, $v = y$
2. $u = x$, $v = y + \beta$
3. $u = \omega \cdot x$, $v = \omega \cdot y$
4. $u = \delta \cdot x$, $v = y$
5. $u = x + t \cdot y$, $v = y$
6. $u = x$, $v = t' \cdot x + y$.

Any function $F$ of moments which is invariant under these six transformations will be invariant under the general affine transformation (1).

From the requirement of invariantness under these transformations we can derive the type and parameters of the function $F$.

If we use central moments instead of general moments (2) or (3), any function of them will be invariant under the translations 1 and 2. The central moments $\mu_{pq}$ are defined as:

$$\mu_{pq} = \iint_{-\infty}^{\infty} (x-\bar{x})^p (y-\bar{y})^q \rho(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad p,q = 0,1,2,\ldots \tag{4}$$

where $\bar{x} = m_{10}/m_{00}$ and $\bar{y} = m_{01}/m_{00}$ are the co-ordinates of the centre of gravity of a given object.

We will assume the function $F$ has the form of a polynomial of the central moments

$$F = \sum_i k_i \mu_{p_1(i)q_1(i)} \cdots \mu_{p_{c(i)}(i)q_{c(i)}(i)}. \tag{5}$$

The demand of invariantness under scaling 3 implies the condition of correct normalization of members of the polynomial $F$. The simplest way is to divide the members of $F$ by the correct power of $\mu_{00}$, i.e. the area of object $G$. For moments we have

$$\mu'_{pq} = \omega^{p+q+2}\mu_{pq} \tag{6}$$

especially

$$\mu'_{00} = \omega^2 \mu_{00}$$

where $\mu'_{pq}$ are the central moments after a transformation. The function $F$ of quotients $\mu_{pq}/\mu_{00}^{((p+q)/2)+1}$ is invariant under scaling. Then the function $F$ does not have the form (5), but

$$F = \sum_i k_i \mu_{p_1(i)q_1(i)} \cdots \mu_{p_{c(i)}(i)q_{c(i)}(i)} / \mu_{00}^{z(i)} \tag{7}$$

where

$$z(i) = \left( \sum_{j=1}^{c(i)} (p_j(i) + q_j(i)) \right) \Big/ 2 + c(i).$$

For $F$ to be invariant under one-axis scaling 4, the members of the polynomial $F$ must be isobars, i.e. the sum of $p$th indexes of each member must be equal to the sum of $q$th indexes. Substituting 4 into integral (4), we obtain the following relation between $\mu_{pq}$ and $\mu'_{pq}$:

$$\mu'_{pq} = \delta^p |\delta| \mu_{pq}. \tag{8}$$

Introducing it into (7) we obtain the condition

$$\sum_{j=1}^{c(i)} p_j(i) = \sum_{j=1}^{c(i)} q_j(i). \tag{9}$$

We will denote this sum as $w$. If a member consists of $r$ $k$th-order moments, $r'$ $k'$th-order moments, etc., for this sum $w$ holds

$$2w = kr + k'r' + \cdots. \tag{10}$$

The number $w$ is called the weight of the invariant. For instance, the member $\mu_{30}\mu_{12}\mu_{02}$ can be a member of an invariant of weight $w = 4$ ($2.4 = 3.2 + 2.1$).

If $\delta = -1$, the transformation is a mirror reflection. Then

$$F' = (-1)^w F. \tag{11}$$

If $w$ is odd, the invariant changes sign under transformations that include mirror reflection, i.e. where the determinant of the transformation $J = a_1 b_2 - a_2 b_1$ is negative. These are called skew invariants.

If $F$ is invariant under skew transformation 5 with parameter $t$, the derivative of $F$ with respect to $t$ is equal to zero

$$\mathrm{D}F' = \frac{\mathrm{d}F'}{\mathrm{d}t} = \sum_p \sum_q \frac{\partial F'}{\partial \mu'_{pq}} \frac{\mathrm{d}\mu'_{pq}}{\mathrm{d}t} = 0. \tag{12}$$

Because it holds that

$$\begin{aligned}\frac{\mathrm{d}m'_{pq}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\iint_G (x+ty)^p y^q\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_G p(x+ty)^{p-1} y^{q+1}\,\mathrm{d}x\,\mathrm{d}y \\ &= pm'_{p-1,q+1}\end{aligned} \tag{13}$$

and because a similar relation is satisfied also for the central moments, then it holds that

$$\mathrm{D}F' = \sum_p \sum_q p\mu'_{p-1,q+1} \frac{\partial F'}{\partial \mu'_{pq}} = 0$$

and also

$$\mathrm{D}F = \sum_p \sum_q p\mu_{p-1,q+1} \frac{\partial F}{\partial \mu_{pq}} = 0. \tag{14}$$

Equation (14) is called the Cayley–Aronholdschen differential equation. It determines coefficients of members of the polynomial $F$. Similarly, from transformation 6 we can derive the equation

$$\sum_p \sum_q q\mu_{p+1,q-1} \frac{\partial F}{\partial \mu_{pq}} = 0. \tag{15}$$

From the mirror reflection

$$\begin{aligned} u &= y \\ v &= x \end{aligned} \tag{16}$$

we can derive the condition of symmetry

$$F(\mu'_{pq}, \mu'_{p'q'}, \ldots) = (-1)^w F(\mu_{qp}, \mu_{q'p'}, \ldots). \tag{17}$$

If this condition and equation (14) are satisfied, then equation (15) is satisfied too.

There are some special theorems about invariants, e.g.

(I) Apolar:

$$\tfrac{1}{2} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \mu_{i,k-i}\mu_{k-i,i} \tag{18}$$

after normalization is invariant if $k$ is even.

(II) Hankel determinant:

$$\begin{vmatrix} \mu_{0k} & \mu_{1,k-1} & \cdots & \mu_{q-1,k-q+1} \\ \mu_{1,k-1} & \mu_{2,k-2} & \cdots & \mu_{q,k-q} \\ \vdots & & & \\ \mu_{u-1,k-u+1} & \mu_{u,k-u} & \cdots & \mu_{u+q-2,k-u-q+2} \\ \mu_{0k'} & \mu_{1,k'-1} & \cdots & \mu_{q-1,k'-q+1} \\ \mu_{1,k'-1} & \mu_{2,k'-2} & \cdots & \mu_{q,k'-q} \\ \vdots & & & \\ \mu_{v-1,k'-v+1} & \mu_{v,k'-v} & \cdots & \mu_{v+q-2,k'-v-q+2} \\ \vdots & & & \end{vmatrix} \tag{19}$$

after normalization is invariant, $k = q + u - 2$, $k' = q + v - 2, \ldots$ are orders of moments.

(III) Discriminant of polynomial:

$$\sum_{i=0}^{k} \binom{k}{i} \mu_{i,k-i} x^{k-i} \tag{20}$$

after dividing by $\mu_{0k}$ and normalization is invariant.

*Note*: Discriminant of a general polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \tag{21}$$

is equal to

$$(-1)^{n(n-1)/2} \begin{vmatrix} a_0 & a_1 & \cdots & a_n & 0 & 0 \cdots 0 \\ 0 & a_0 & \cdots & a_{n-1} & a_n & 0 \cdots 0 \\ & & & \vdots & & \\ 0 & \cdots & 0 \; a_0 \, a_1 & a_2 & a_3 & \cdots a_n \\ b_0 & b_1 \cdots b_{n-1} & & 0 & 0 & \cdots 0 \\ 0 & b_0 \cdots b_{n-2} & & b_{n-1} & 0 & \cdots 0 \\ & & \vdots & & & \\ 0 & \cdots & 0 \; b_0 & b_1 & b_2 & \cdots b_{n-1} \end{vmatrix} \tag{22}$$

where $b_0, b_1, \ldots, b_{n-1}$ are coefficients of $f'(x)$, i.e. $b_j = a_j(n-j)$.

Proofs of assertions (I)–(III) can be found in reference (14).

### 2.1. *Invariants from second- and third-order moments*

The simplest invariant consists of second-order moments. It is the apolar

$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4. \tag{23}$$

(We normalized by fourth power of $\mu_{00}$, because $2((p+q)/2+1) = 2(2/2+1) = 4$.)

For the third-order moments, the discriminant of the polynomial

$$\mu_{03}x^3 + 3\mu_{12}x^2 + 3\mu_{21}x + \mu_{30} \tag{24}$$

is, according to (22), equal to

$$-\begin{vmatrix} \mu_{03} & 3\mu_{12} & 3\mu_{21} & \mu_{30} & 0 \\ 0 & \mu_{03} & 3\mu_{12} & 3\mu_{21} & \mu_{30} \\ 3\mu_{03} & 6\mu_{12} & 3\mu_{21} & 0 & 0 \\ 0 & 3\mu_{03} & 6\mu_{12} & 3\mu_{21} & 0 \\ 0 & 0 & 3\mu_{03} & 6\mu_{12} & 3\mu_{21} \end{vmatrix}. \tag{25}$$

The corresponding invariant is

$$I_2 = (\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3 + 4\mu_{21}^3\mu_{03} - 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}. \tag{26}$$

From second- and third-order moments we can make up the Hankel determinant

$$\begin{vmatrix} \mu_{02} & \mu_{11} & \mu_{20} \\ \mu_{03} & \mu_{12} & \mu_{21} \\ \mu_{12} & \mu_{21} & \mu_{30} \end{vmatrix}. \tag{27}$$

The corresponding invariant is

$$I_3 = (\mu_{20}(\mu_{21}\mu_{03} - \mu_{12}^2) - \mu_{11}(\mu_{30}\mu_{03} - \mu_{21}\mu_{12}) + \mu_{02}(\mu_{30}\mu_{12} - \mu_{21}^2))/\mu_{00}^7. \tag{28}$$

There is another invariant, whose members consist of three second-order moments and two third-order moments. Its weight is $(2.3 + 3.2)/2 = 6$; see Relation (10). According to the condition of isobars we look for all possible partitions of the number 6 to a sum of three integers from 0 to 2 $(p_1, p_2, p_3)$ and two integers from 0 to 3 $(p_4, p_5)$:

| $p_1$ $p_2$ $p_3$ $p_4$ $p_5$ | Members of the invariant |
|---|---|
| $2+2+2+0+0$ | $k_1\mu_{20}^3\mu_{03}^2$ |
| $2+2+1+1+0$ | $k_2\mu_{20}^2\mu_{11}\mu_{12}\mu_{03}$ |
| $2+2+0+2+0$ | $k_3\mu_{20}^2\mu_{02}\mu_{21}\mu_{03}$ |
| $2+2+0+1+1$ | $k_4\mu_{20}^2\mu_{02}\mu_{12}^2$ |
| $2+1+1+2+0$ | $k_5\mu_{20}\mu_{11}^2\mu_{21}\mu_{03}$ |
| $2+1+1+1+1$ | $k_6\mu_{20}\mu_{11}^2\mu_{12}^2$ |
| $2+1+0+3+0$ | $k_7\mu_{20}\mu_{11}\mu_{02}\mu_{30}\mu_{03}$ |
| $2+1+0+2+1$ | $k_8\mu_{20}\mu_{11}\mu_{02}\mu_{21}\mu_{12}$ |
| $2+0+0+3+1$ | $k_3\mu_{20}\mu_{02}^2\mu_{30}\mu_{12}$ |
| $2+0+0+2+2$ | $k_4\mu_{20}\mu_{02}^2\mu_{21}^2$ |
| $1+1+1+3+0$ | $k_9\mu_{11}^3\mu_{30}\mu_{03}$ |
| $1+1+1+2+1$ | $k_{10}\mu_{11}^3\mu_{21}\mu_{12}$ |
| $1+1+0+3+1$ | $k_5\mu_{11}^2\mu_{02}\mu_{30}\mu_{12}$ |
| $1+1+0+2+2$ | $k_6\mu_{11}^2\mu_{02}\mu_{21}^2$ |
| $1+0+0+3+2$ | $k_2\mu_{11}\mu_{02}^2\mu_{30}\mu_{21}$ |
| $0+0+0+3+3$ | $k_1\mu_{02}^3\mu_{30}^2$ |

(29)

$$q_1 = 2 - p_1, \quad q_2 = 2 - p_2, \quad q_3 = 2 - p_3, \quad q_4 = 3 - p_4, \quad q_5 = 3 - p_5.$$

The form of the invariant is given in the right column. According to the condition of symmetry (17), the symmetric members have the same coefficients.

From equation (14) we can derive a system of linear equations with 10 unknown coefficients, whose rank is 8. If we choose coefficients $k_1$ and $k_{10}$, the solution is

$$\begin{aligned} & k_1 \\ & k_2 = -6k_1 \\ & k_3 = -6k_1 - k_{10} \\ & k_4 = 9k_1 + k_{10} \\ & k_5 = 12k_1 + k_{10} \\ & k_6 = -k_{10} \\ & k_7 = 6k_1 + k_{10} \\ & k_8 = -18k_1 - k_{10} \\ & k_9 = -8k_1 - k_{10} \\ & k_{10}. \end{aligned} \tag{30}$$

If we choose $k_1 = 0$, $k_{10} = 1$, we obtain invariant $-I_1 \cdot I_3$. If we choose $k_1 = 1$, $k_{10} = 0$, we obtain a new independent invariant

$$I_4 = (\mu_{20}^3\mu_{03}^2 - 6\mu_{20}^2\mu_{11}\mu_{12}\mu_{03} - 6\mu_{20}^2\mu_{02}\mu_{21}\mu_{03} + 9\mu_{20}^2\mu_{02}\mu_{12}^2 + 12\mu_{20}\mu_{11}^2\mu_{21}\mu_{03} + 6\mu_{20}\mu_{11}\mu_{02}\mu_{30}\mu_{03} - 18\mu_{20}\mu_{11}\mu_{02}\mu_{21}\mu_{12} - 8\mu_{11}^3\mu_{30}\mu_{03} - 6\mu_{20}\mu_{02}^2\mu_{30}\mu_{12} + 9\mu_{20}\mu_{02}^2\mu_{21}^2 + 12\mu_{11}^2\mu_{02}\mu_{30}\mu_{12} - 6\mu_{11}\mu_{02}^2\mu_{30}\mu_{21} + \mu_{02}^3\mu_{30}^2)/\mu_{00}^{11}. \quad (31)$$

### 2.2. *The number of invariants*

The number of invariants is dealt with in the Cayley–Sylvester theorem, whose statement and proof are in reference (14). It implies that if we denote as $A(k,r,w)$ the number of partitions of the number $w$ to the sum of $r$ integers from 0 to $k$, the number $N(k,r,w)$ of independent non-constant invariants consisting of members with $r$ $k$th-order moments is

$$N(k,r,w) = A(k,r,w) - A(k,r,w-1) \quad (32)$$

where $w = kr/2$ is the weight of the invariant. A similar relation is satisfied for invariants with more complex structure:

$$N(k,r;k',r';k'',r'';\ldots;w) = A(k,r;k',r';k'',r'';\ldots;w) - A(k,r;k',r';k'',r'';\ldots;w-1). \quad (33)$$

For instance, in the case of $I_4$ we have the number of partitions of the number 6 to the sum of three integers from 0 to 2 and two integers from 0 to 3

$$A(2,3;3,2;6) = 16 \qquad \text{see the left column of (29)}$$
$$A(2,3;3,2;5) = 14$$
$$N(2,3;3,2;6) = 16 - 14 = 2. \quad (34)$$

The number of invariants can be also found by means of Gauss polynomials. They are defined as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-m+1})}{(1-x)(1-x^2)\cdots(1-x^m)}. \quad (35)$$

For $A(k,r,w)$ holds

$$A(k,r,w) = \begin{bmatrix} k+r \\ r \end{bmatrix}_w = \begin{bmatrix} k+r \\ k \end{bmatrix}_w \quad (36)$$

where $[\ ]_w$ denotes the coefficient at $w$th power of the polynomial. Then the number of invariants is

$$N(k,r,w) = (1-x)\begin{bmatrix} k+r \\ k \end{bmatrix}_w = \begin{bmatrix} k+r \\ k \end{bmatrix}^*_w. \quad (37)$$

Generally

$$A(k,r;k',r';k'',r'';\ldots;w) = \left.\begin{bmatrix} k+r \\ k \end{bmatrix}\begin{bmatrix} k'+r' \\ k' \end{bmatrix}\begin{bmatrix} k''+r'' \\ k'' \end{bmatrix}\cdots\right|_w$$
$$N(k,r;k',r';k'',r'';\ldots;w) = \left.\begin{bmatrix} k+r \\ k \end{bmatrix}^*\begin{bmatrix} k'+r' \\ k' \end{bmatrix}\begin{bmatrix} k''+r'' \\ k'' \end{bmatrix}\cdots\right|_w. \quad (38)$$

The entire number of independent non-constant invariants with second- and third-order moments of weight $w$ is equal to

$$\sum_{\substack{r,s \\ r+3s=w}} N(2,r;3,2s;w) = \sum_{\substack{r,s \\ r+3s=w}} \left.\begin{bmatrix} 2+r \\ 2 \end{bmatrix}\begin{bmatrix} 3+2s \\ 3 \end{bmatrix}^*\right|_w$$
$$= \sum_{\substack{r,s \\ r+3s=w}} \left.\frac{(1-x^{2+r})(1-x^{1+r})(1-x^{3+2s})(1-x^{2+2s})(1-x^{1+2s})}{(1-x)(1-x^2)^2(1-x^3)}\right|_w.$$

After relatively complex modification this becomes

$$\left.\frac{1+x^9}{(1-x^2)(1-x^4)(1-x^6)^2}\right|_w. \quad (39)$$

This means that there is one invariant of weight 2 ($I_1$), one invariant of weight 4 ($I_3$) and two invariants of weight 6 ($I_2$ and $I_4$). There is also one skew invariant of weight 9, but its second and higher powers are dependent. Therefore its sign is independent, but its absolute value is algebraically dependent on invariants $I_1, I_2, I_3$ and $I_4$.

For complete proofs of theorems and detailed discussion of the properties of invariants see reference (15).

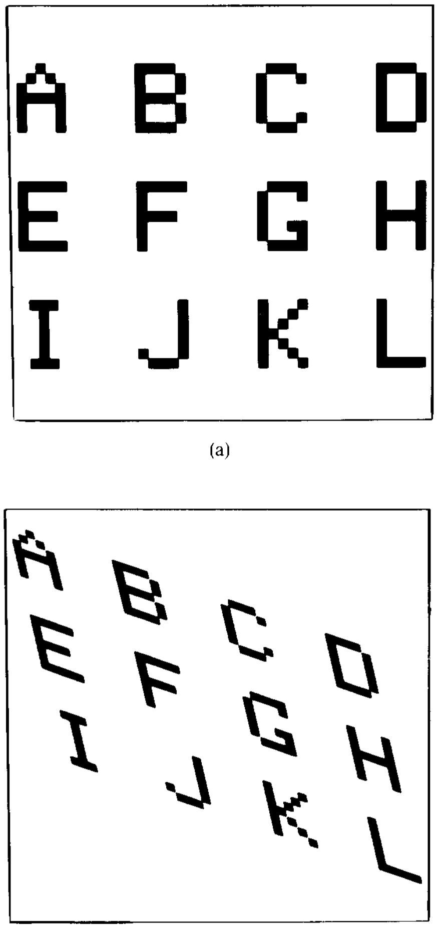
## 3. NUMERICAL EXPERIMENTS

In order to show the performance of the affine moment invariants as features for pattern recognition, several experiments on test and real images were carried out. It was proved that in the case of affine-deformed patterns we can reach very accurate results. Moreover, it is shown that affine moment invariants are successfully applicable for recognition of projective-deformed patterns.

In the following experiments, three features $I_1$, $I_2$ and $I_3$ were used for object recognition. Classification was performed by minimum distance in 3D Euclidean feature space. Of course, more precise classification algorithms which consider contextual information could be used in the case of scene matching (see references (15, 16) for details).

### 3.1. *Experiment 1—letters*

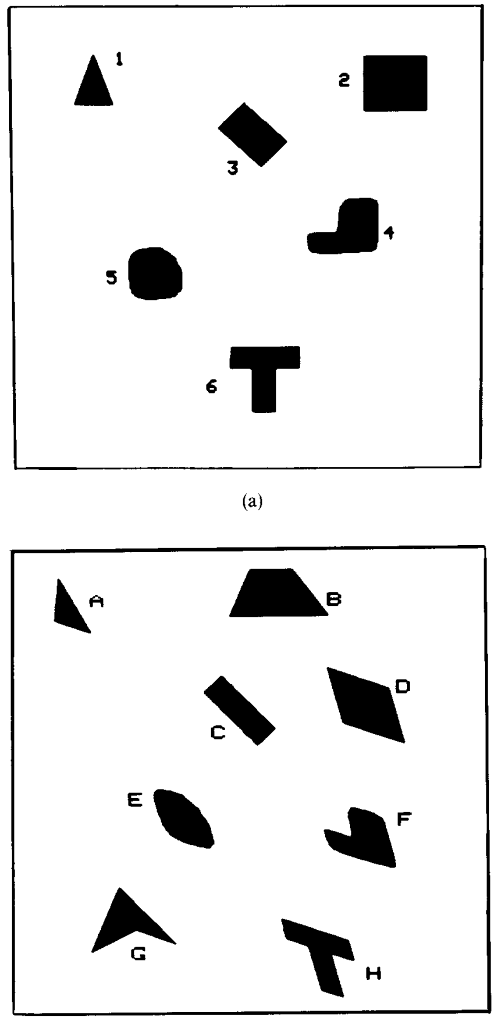
The first experiment deals with recognition of capital letters. Figure 1(a) shows original letters—templates.




(a)

(b)

Fig. 1. (a) Templates; (b) the letters to be recognized.




(a)

(b)

Fig. 2. (a) Templates; (b) the shapes to be recognized.

Table 1. Affine moment invariants of letters in Experiment 1

| Letter | Fig. 1(a) $I_1[10^{-4}]$ | Fig. 1(a) $I_2[10^{-8}]$ | Fig. 1(a) $I_3[10^{-6}]$ | Fig. 1(b) $I_1[10^{-4}]$ | Fig. 1(b) $I_2[10^{-8}]$ | Fig. 1(b) $I_3[10^{-6}]$ |
|---|---|---|---|---|---|---|
| A | 330 | 6 | −133 | 336 | 10 | −137 |
| B | 292 | 0 | −15 | 294 | 0 | −16 |
| C | 775 | 1286 | 192 | 779 | 1290 | 186 |
| D | 468 | −8 | −64 | 470 | −8 | −65 |
| E | 367 | 120 | 62 | 376 | 123 | 62 |
| F | 315 | −245 | −318 | 315 | −234 | −311 |
| G | 516 | −30 | −139 | 519 | −33 | −146 |
| H | 408 | 0 | 0 | 411 | 0 | 0 |
| I | 218 | 0 | 0 | 219 | 0 | 0 |
| J | 601 | 3423 | −1240 | 599 | 3405 | −1227 |
| K | 511 | 550 | 187 | 518 | 586 | 195 |
| L | 523 | −2295 | −1950 | 526 | −2110 | −1952 |

Figure 1(b) shows the same set of letters deformed by affine transform. The task was to recognize the deformed letters. The values of invariants $I_1$, $I_2$ and $I_3$ are given in Table 1. It can be seen clearly that the moment invariants really are invariant under affine transform and that the classification is performed without any errors.

### 3.2. *Experiment 2—geometric shapes*

The test image of the size $512 \times 512$ pixels (see Fig. 2(a)) consists of six simple geometric shapes. The image was deformed by affine transform and two new objects were added (see Fig. 2(b)). Then the classification of objects in Fig. 2(b) into six shape classes given in Fig. 2(a) was performed by means of moment invariants $I_1$, $I_2$ and $I_3$.

The values of moment invariants are shown in Table 2. Table 3 contains the distances between each two objects in the feature space. The results of classification by minimum distance are given in Table 4.

There are two misclassifications there: $2 \sim$ C and $3 \sim$ D. They are caused by low separability of classes 2 and 3 in the feature space.

### 3.3. *Experiment 3—satellite images*

The third experiment deals with the recognition of closed-boundary regions which were detected in satellite images. Moreover, we try to use the invariants $I_1$, $I_2$ and $I_3$ for recognition of projective-deformed regions and the regions extracted from a digitized map.

Four input digital images are given (see Fig. 3): original image of the size $512 \times 512$ pixels taken by satellite Landsat TM, its affine and projective transforms and digitized map of the same area on a scale 1:25,000.

Several closed-boundary regions were detected via adaptive thresholding and region growing in the original image; on the Earth's surface they represent fields and lakes. The same regions were found in both transformed images. Ten regions were manually selected in the map. All extracted regions are shown in Fig. 4. The regions from the original Landsat image served as templates.

Table 3. Distance matrix

| Fig. 2(b) | Fig. 2(a) 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 1.0 | 68.0 | 67.7 | 39.3 | 69.1 | 176.1 |
| B | 53.6 | 15.4 | 15.0 | 18.5 | 17.2 | 228.2 |
| C | 66.9 | 1.0 | 1.4 | 32.0 | 5.0 | 239.6 |
| D | 66.5 | 1.4 | 1.0 | 31.4 | 6.0 | 239.3 |
| E | 68.7 | 5.1 | 6.1 | 35.4 | 0.0 | 241.2 |
| F | 39.1 | 33.4 | 32.8 | 1.0 | 36.1 | 213.6 |
| G | 261.0 | 323.5 | 323.3 | 298.7 | 324.1 | 86.8 |
| H | 179.0 | 243.0 | 242.7 | 216.6 | 243.9 | 2.8 |

Table 4. Shape classification by minimum distance method

| Fig. 2(a) | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Fig. 2(b) | A | C | D | F | E | H |

Firstly, the recognition of the regions from the affine-deformed image was carried out. According to the theoretical presumptions, the classification was correct everywhere.

Secondly, we classified the regions from the projective-deformed image. Note that $I_1$, $I_2$ and $I_3$ theoretically are not invariant under projective transformation. Nevertheless, the result was also perfect—all regions were recognized correctly. Successful recognition resulted from the slight effect of non-linear terms of projective transform with respect to the size of the regions.

Finally, the recognition of the map regions was performed. The situation is much more complicated than in the previous cases, because the original image and map differ from each other by unknown shape-non-preserving distortion.

The distances between map regions and templates are shown in Table 5. The classification by minimum distance (with rejection threshold $t = 50$) yields six corresponding pairs (see Table 6). There is only one misclassification among them ($8 \sim$ D); all other regions are classified correctly.

Table 2. Affine moment invariants of shapes in Experiment 2

| Fig. 2(a) | | | | Fig. 2(b) | | | |
|---|---|---|---|---|---|---|---|
| Object | $I_1[10^{-4}]$ | $I_2[10^{-8}]$ | $I_3[10^{-6}]$ | Object | $I_1[10^{-4}]$ | $I_2[10^{-8}]$ | $I_3[10^{-6}]$ |
| 1 | 92 | −32 | −55 | A | 93 | −32 | −55 |
| 2 | 69 | 0 | 0 | B | 75 | −2 | −14 |
| 3 | 70 | 0 | 0 | C | 69 | 0 | −1 |
| 4 | 89 | −7 | −25 | D | 70 | 0 | −1 |
| 5 | 64 | 0 | −1 | E | 64 | 0 | −1 |
| 6 | 143 | −168 | −155 | F | 90 | −7 | −25 |
| | | | | G | 145 | −241 | −202 |
| | | | | H | 143 | −170 | −157 |

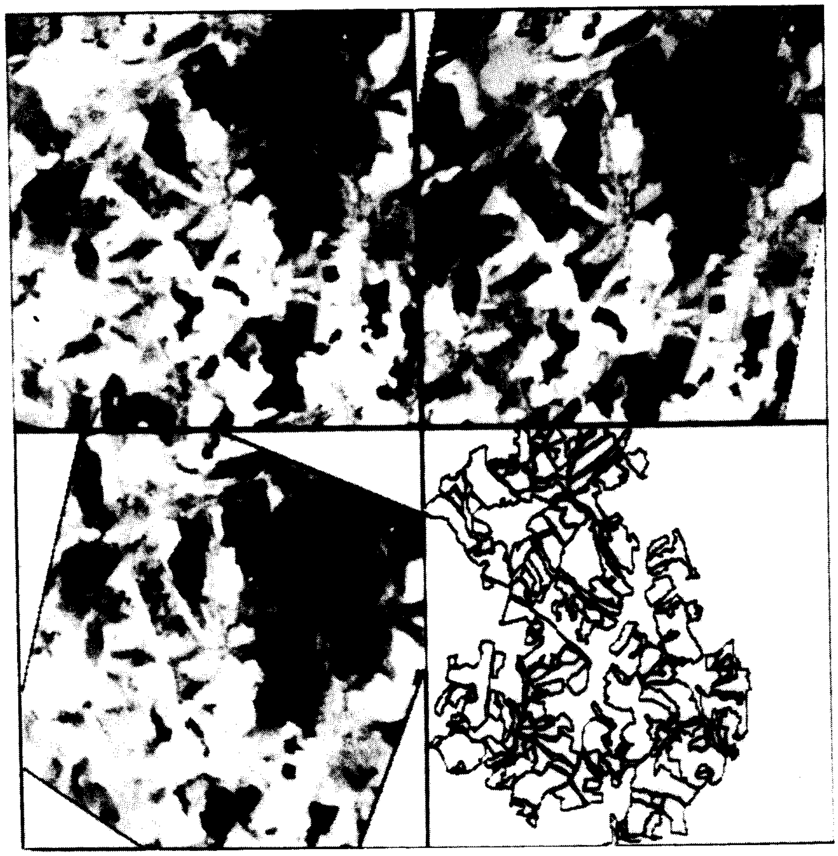


Fig. 3. From left to right and from top to bottom: original Landsat TM image, its affine transform, its projective transform, digitized map on a scale 1 : 25,000.

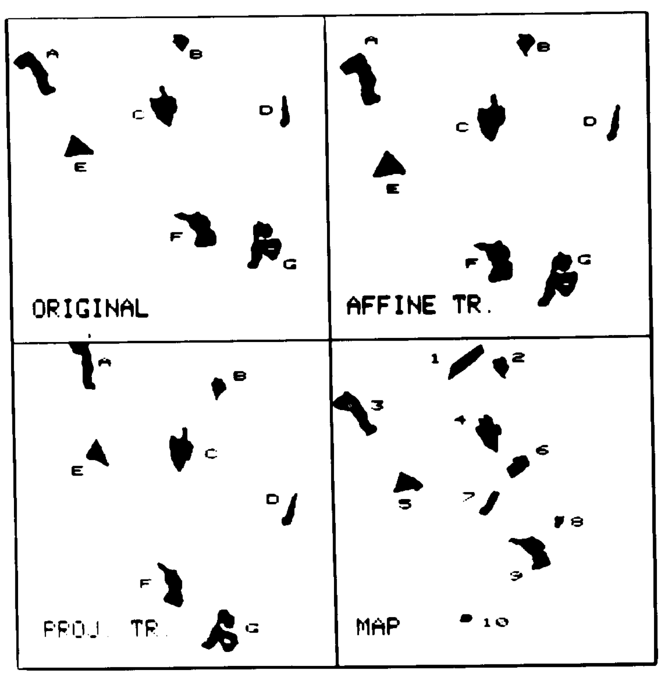



Fig. 4. Closed-boundary regions extracted from Fig. 3.

Table 5. Distance matrix

| Map | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| | | | | Image | | | |
| 1 | 55 | 17 | 13 | 37 | 108 | 34 | 152 |
| 2 | 39 | 1.4 | 13 | 20 | 94 | 23 | 136 |
| 3 | 18 | 41 | 50 | 24 | 67 | 36 | 99 |
| 4 | 46 | 11 | 3.2 | 32 | 104 | 24 | 146 |
| 5 | 43 | 53 | 64 | 33 | 43 | 56 | 85 |
| 6 | 54 | 16 | 12 | 36 | 107 | 33 | 152 |
| 7 | 46 | 11 | 4.5 | 31 | 104 | 25 | 146 |
| 8 | 37 | 4.1 | 16 | 17 | 91 | 23 | 133 |
| 9 | 20 | 31 | 30 | 35 | 98 | 9 | 129 |
| 10 | 54 | 16 | 13 | 36 | 107 | 34 | 151 |

Table 6. Correspondence of regions in the original image and in the map

| Map regions | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Templates | — | B | A | C | E | — | — | D | F | — |

The results of the above-described experiments prove that pattern recognition by affine moment invariants is sufficiently accurate and that it can be used even in the case of more general transformation of patterns.

#### 4. SUMMARY

The paper deals with feature-based recognition of affine-deformed objects and patterns. The set of new invariant features is presented. Affine moment invariants are used as features. The affine moment invariants were derived by means of decomposition of affine transformation into six one-parameter transformations. Results of the theory of algebraic invariants were used for this derivation. In this sense, our paper is a significant extension and generalization of recent works.[5–10]

Affine moment invariants were successfully used in several experiments for recognition of affine-deformed objects and patterns. It was proved that they can be applied even in the case of more general deformations.

Pattern recognition by affine moment invariants can be used in many practical tasks, for example in image matching, multitemporal image sequence analysis, shape classification, character recognition and so on. The approach has numerous applications in remote sensing and medical imaging.

*Acknowledgements*—We would like to thank Dr Stanislav Saic for the creation of appropriate conditions for our work and Mrs Jana Švarcová for excellent preparation of the typescript.

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