# The Independence of the Affine Moment Invariants

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**Abstract.** The affine moment invariants are important tool for recognition of geometrically deformed images for many years. Nevertheless, the proof of independence of a chosen set of them is still problem. This contribution presents a new approach to these proofs, the affine moment invariants are compared with the corresponding set of the normalized moments. The normalized moments are complete and independent and if the values of the normalized moments can be computed unambiguously from the values of the affine moment invariants and vice versa, then the set of the affine moment invariants is complete and independent, too. The proof for invariants up to the fourth order is presented directly. The proof for higher orders can be more complicated, but it can be simplified, if we compute the solution not in the whole space of the feature values, but in some suitably chosen specific values.

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## INTRODUCTION

Moment invariants have become a powerful tool for recognizing objects regardless of their particular position, orientation, viewing angle, and other variations. We have various sets of moment invariants, each of them is invariant to different type of transformation. We will deal with the geometric transformations only and there are two major groups among them: similarity transform and affine transform. The similarity transform includes translation, scaling and rotation and there is a well elaborated theory on rotation moment invariants [1, 2, 3, 4] including creation of complete and independent sets [5, 6], which have been successfully used in numerous applications. The affine transform includes the similarity transform and in addition to that stretching (anisotropic scaling) and second rotation. A projective transform, modeling photographing a planar scene by a pin-hole camera, can be approximated by an affine transform for small objects and large camera-to-scene distance. Thus, having powerful affine moment invariants for object description and recognition is in great demand. The theory of affine moment invariants has its pioneer era after it, too [7, 4], but problems with creation of complete and independent set of affine moment invariants remain until now.

The using incomplete set of features means we cannot recognize two objects that could be recognized by the same number of complete features. If we use dependent features, then they do not contribute to the result at all and if they are noisy, then they can even worsen the result. Therefore the using complete and independent sets of features in pattern recognition is very important.

The affine moment invariants can be derived by means of the classical theory of algebraic invariants, e.g. [8], by some better automated method (method of tensors [9], method of graphs [10, 11]) or by newly arisen method of normalization [12, 13, 14]. The invariants derived by the normalization are called *normalized moments* to distinguish them from the traditional *affine moment invariants*. The normalized moments have some advantages in comparison with the affine moment invariants, namely easy creation of complete and independent sets, and one drawback, they are discontinuous on symmetric objects. Therefore affine moment invariants remain important mean for recognition of the affinely distorted objects. The theme of this paper is using the normalized moments for verification of the independence of the affine moment invariants.

## AFFINE MOMENT INVARIANTS

The affine transformation can be expressed as

$$\begin{aligned} u &= a_0 + a_1 x + a_2 y \\ v &= b_0 + b_1 x + b_2 y. \end{aligned} \tag{1}$$

The geometric two-dimensional $(p+q)$-th order moments of an image $f(x,y)$ are defined as

$$m_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q f(x,y) dx dy \quad p,q = 0,1,2,\dots \tag{2}$$

Invariance to translation can be provided by using central moments. They are defined as

$$\mu_{pq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-x_t)^p (y-y_t)^q f(x,y) dx dy, \tag{3}$$

where $p,q = 0,1,2,\dots,$ and $x_t = m_{10}/m_{00}$, $y_t = m_{01}/m_{00}$ are the coordinates of the centroid. If we use quotients

$$\nu_{pq} = \mu_{pq}/\mu_{00}^{(p+q+2)/2}, \tag{4}$$

then their function is invariant under scaling, too.

Now, the affine moment invariants can be derived e.g. by the method of graphs [10]. Here is example of the invariants up to the fourth order:

$$I_1 = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$$

$$I_2 = (-\mu_{30}^2\mu_{03}^2 + 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} - 4\mu_{30}\mu_{12}^3 - 4\mu_{21}^3\mu_{03} + 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}$$

$$I_3 = (\mu_{20}\mu_{21}\mu_{03} - \mu_{20}\mu_{12}^2 - \mu_{11}\mu_{30}\mu_{03} + \mu_{11}\mu_{21}\mu_{12} + \mu_{02}\mu_{30}\mu_{12} - \mu_{02}\mu_{21}^2)/\mu_{00}^7$$

$$\begin{aligned} I_4 = \; & (-\mu_{20}^3\mu_{03}^2 + 6\mu_{20}^2\mu_{11}\mu_{12}\mu_{03} - 3\mu_{20}^2\mu_{02}\mu_{12}^2 - 6\mu_{20}\mu_{11}^2\mu_{21}\mu_{03} - 6\mu_{20}\mu_{11}^2\mu_{12}^2 \\ & +12\mu_{20}\mu_{11}\mu_{02}\mu_{21}\mu_{12} - 3\mu_{20}\mu_{02}^2\mu_{21}^2 + 2\mu_{11}^3\mu_{30}\mu_{03} + 6\mu_{11}^3\mu_{21}\mu_{12} \\ & -6\mu_{11}^2\mu_{02}\mu_{30}\mu_{12} - 6\mu_{11}^2\mu_{02}\mu_{21}^2 + 6\mu_{11}\mu_{02}^2\mu_{30}\mu_{21} - \mu_{02}^3\mu_{30}^2)/\mu_{00}^{11} \end{aligned}$$

$$\begin{aligned}
I_5 = \ & (\mu_{20}^3\mu_{30}\mu_{03}^3 - 3\mu_{20}^3\mu_{21}\mu_{12}\mu_{03}^2 + 2\mu_{20}^3\mu_{12}^3\mu_{03} - 6\mu_{20}^2\mu_{11}\mu_{30}\mu_{12}\mu_{03}^2 \\
& + 6\mu_{20}^2\mu_{11}\mu_{21}^2\mu_{03}^2 + 6\mu_{20}^2\mu_{11}\mu_{21}\mu_{12}^2\mu_{03} - 6\mu_{20}^2\mu_{11}\mu_{12}^4 + 3\mu_{20}^2\mu_{02}\mu_{30}\mu_{12}^2\mu_{03} \\
& - 6\mu_{20}^2\mu_{02}\mu_{21}^2\mu_{12}\mu_{03} + 3\mu_{20}^2\mu_{02}\mu_{21}\mu_{12}^3 + 12\mu_{20}\mu_{11}^2\mu_{30}\mu_{12}^2\mu_{03} \\
& - 24\mu_{20}\mu_{11}^2\mu_{21}^2\mu_{12}\mu_{03} + 12\mu_{20}\mu_{11}^2\mu_{21}\mu_{12}^3 - 12\mu_{20}\mu_{11}\mu_{02}\mu_{30}\mu_{12}^3 \\
& + 12\mu_{20}\mu_{11}\mu_{02}\mu_{21}^3\mu_{03} - 3\mu_{20}\mu_{02}^2\mu_{30}\mu_{21}^2\mu_{03} + 6\mu_{20}\mu_{02}^2\mu_{30}\mu_{21}\mu_{12}^2 \\
& - 3\mu_{20}\mu_{02}^2\mu_{21}^3\mu_{12} - 8\mu_{11}^3\mu_{30}\mu_{12}^3 + 8\mu_{11}^3\mu_{21}^3\mu_{03} - 12\mu_{11}^2\mu_{02}\mu_{30}\mu_{21}^2\mu_{03} \\
& + 24\mu_{11}^2\mu_{02}\mu_{30}\mu_{21}\mu_{12}^2 - 12\mu_{11}^2\mu_{02}\mu_{21}^3\mu_{12} + 6\mu_{11}\mu_{02}^2\mu_{30}^2\mu_{21}\mu_{03} \\
& - 6\mu_{11}\mu_{02}^2\mu_{30}^2\mu_{12}^2 - 6\mu_{11}\mu_{02}^2\mu_{30}\mu_{21}^2\mu_{12} + 6\mu_{11}\mu_{02}^2\mu_{21}^4 - \mu_{02}^3\mu_{30}^3\mu_{03} \\
& + 3\mu_{02}^3\mu_{30}^2\mu_{21}\mu_{12} - 2\mu_{02}^3\mu_{30}\mu_{21}^3)/\mu_{00}^{16}
\end{aligned}$$

$$I_6 = \ (\mu_{40}\mu_{04} - 4\mu_{31}\mu_{13} + 3\mu_{22}^2)/\mu_{00}^6$$

$$I_7 = \ (\mu_{40}\mu_{22}\mu_{04} - \mu_{40}\mu_{13}^2 - \mu_{31}^2\mu_{04} + 2\mu_{31}\mu_{22}\mu_{13} - \mu_{22}^3)/\mu_{00}^9$$

$$I_8 = \ (\mu_{20}^2\mu_{04} - 4\mu_{20}\mu_{11}\mu_{13} + 2\mu_{20}\mu_{02}\mu_{22} + 4\mu_{11}^2\mu_{22} - 4\mu_{11}\mu_{02}\mu_{31} + \mu_{02}^2\mu_{40})/\mu_{00}^7$$

$$\begin{aligned}
I_9 = \ & (\mu_{20}^2\mu_{22}\mu_{04} - \mu_{20}^2\mu_{13}^2 - 2\mu_{20}\mu_{11}\mu_{31}\mu_{04} + 2\mu_{20}\mu_{11}\mu_{22}\mu_{13} + \mu_{20}\mu_{02}\mu_{40}\mu_{04} \\
& - 2\mu_{20}\mu_{02}\mu_{31}\mu_{13} + \mu_{20}\mu_{02}\mu_{22}^2 + 4\mu_{11}^2\mu_{31}\mu_{13} - 4\mu_{11}^2\mu_{22}^2 - 2\mu_{11}\mu_{02}\mu_{40}\mu_{13} \\
& + 2\mu_{11}\mu_{02}\mu_{31}\mu_{22} + \mu_{02}^2\mu_{40}\mu_{22} - \mu_{02}^2\mu_{31}^2)/\mu_{00}^{10}
\end{aligned}$$

$$\begin{aligned}
I_{10} = \ & (\mu_{20}^3\mu_{31}\mu_{04}^2 - 3\mu_{20}^3\mu_{22}\mu_{13}\mu_{04} + 2\mu_{20}^3\mu_{13}^3 - \mu_{20}^2\mu_{11}\mu_{40}\mu_{04}^2 - 2\mu_{20}^2\mu_{11}\mu_{31}\mu_{13}\mu_{04} \\
& + 9\mu_{20}^2\mu_{11}\mu_{22}^2\mu_{04} - 6\mu_{20}^2\mu_{11}\mu_{22}\mu_{13}^2 + \mu_{20}^2\mu_{02}\mu_{40}\mu_{13}\mu_{04} - 3\mu_{20}^2\mu_{02}\mu_{31}\mu_{22}\mu_{04} \\
& + 2\mu_{20}^2\mu_{02}\mu_{31}\mu_{13}^2 + 4\mu_{20}\mu_{11}^2\mu_{40}\mu_{13}\mu_{04} - 12\mu_{20}\mu_{11}^2\mu_{31}\mu_{22}\mu_{04} + 8\mu_{20}\mu_{11}^2\mu_{31}\mu_{13}^2 \\
& - 6\mu_{20}\mu_{11}\mu_{02}\mu_{40}\mu_{13}^2 + 6\mu_{20}\mu_{11}\mu_{02}\mu_{31}^2\mu_{04} - \mu_{20}\mu_{02}^2\mu_{40}\mu_{31}\mu_{04} \\
& + 3\mu_{20}\mu_{02}^2\mu_{40}\mu_{22}\mu_{13} - 2\mu_{20}\mu_{02}^2\mu_{31}^2\mu_{13} - 4\mu_{11}^3\mu_{40}\mu_{13}^2 + 4\mu_{11}^3\mu_{31}^2\mu_{04} \\
& - 4\mu_{11}^2\mu_{02}\mu_{40}\mu_{31}\mu_{04} + 12\mu_{11}^2\mu_{02}\mu_{40}\mu_{22}\mu_{13} - 8\mu_{11}^2\mu_{02}\mu_{31}^2\mu_{13} + \mu_{11}\mu_{02}^2\mu_{40}^2\mu_{04} \\
& + 2\mu_{11}\mu_{02}^2\mu_{40}\mu_{31}\mu_{13} - 9\mu_{11}\mu_{02}^2\mu_{40}\mu_{22}^2 + 6\mu_{11}\mu_{02}^2\mu_{31}^2\mu_{22} - \mu_{02}^3\mu_{40}^2\mu_{13} \\
& + 3\mu_{02}^3\mu_{40}\mu_{31}\mu_{22} - 2\mu_{02}^3\mu_{31}^3)/\mu_{00}^{15}
\end{aligned}$$

$$\begin{aligned}
I_{11} = \ & (\mu_{20}\mu_{30}\mu_{12}\mu_{04} - \mu_{20}\mu_{30}\mu_{03}\mu_{13} - \mu_{20}\mu_{21}^2\mu_{04} + \mu_{20}\mu_{21}\mu_{12}\mu_{13} + \mu_{20}\mu_{21}\mu_{03}\mu_{22} \\
& - \mu_{20}\mu_{12}^2\mu_{22} - 2\mu_{11}\mu_{30}\mu_{12}\mu_{13} + 2\mu_{11}\mu_{30}\mu_{03}\mu_{22} + 2\mu_{11}\mu_{21}^2\mu_{13} \\
& - 2\mu_{11}\mu_{21}\mu_{12}\mu_{22} - 2\mu_{11}\mu_{21}\mu_{03}\mu_{31} + 2\mu_{11}\mu_{12}^2\mu_{31} + \mu_{02}\mu_{30}\mu_{12}\mu_{22} \\
& - \mu_{02}\mu_{30}\mu_{03}\mu_{31} - \mu_{02}\mu_{21}^2\mu_{22} + \mu_{02}\mu_{21}\mu_{12}\mu_{31} + \mu_{02}\mu_{21}\mu_{03}\mu_{40} \\
& - \mu_{02}\mu_{12}^2\mu_{40})/\mu_{00}^{10}
\end{aligned}$$

$$\begin{aligned}
I_{12} = \ & (\mu_{20}^3\mu_{12}^2\mu_{04} - 2\mu_{20}^3\mu_{12}\mu_{03}\mu_{13} + \mu_{20}^3\mu_{03}^2\mu_{22} - 4\mu_{20}^2\mu_{11}\mu_{21}\mu_{12}\mu_{04} \\
& + 4\mu_{20}^2\mu_{11}\mu_{21}\mu_{03}\mu_{13} + 2\mu_{20}^2\mu_{11}\mu_{12}^2\mu_{13} - 2\mu_{20}^2\mu_{11}\mu_{03}^2\mu_{31} + 2\mu_{20}^2\mu_{02}\mu_{30}\mu_{12}\mu_{04} \\
& - 2\mu_{20}^2\mu_{02}\mu_{30}\mu_{03}\mu_{13} - 2\mu_{20}^2\mu_{02}\mu_{21}\mu_{12}\mu_{13} + 2\mu_{20}^2\mu_{02}\mu_{21}\mu_{03}\mu_{22} + \mu_{20}^2\mu_{02}\mu_{12}^2\mu_{22} \\
& - 2\mu_{20}^2\mu_{02}\mu_{12}\mu_{03}\mu_{31} + \mu_{20}^2\mu_{02}\mu_{03}^2\mu_{40} + 4\mu_{20}\mu_{11}^2\mu_{21}^2\mu_{04} - 8\mu_{20}\mu_{11}^2\mu_{21}\mu_{03}\mu_{22} \\
& - 4\mu_{20}\mu_{11}^2\mu_{12}^2\mu_{22} + 8\mu_{20}\mu_{11}^2\mu_{12}\mu_{03}\mu_{31} - 4\mu_{20}\mu_{11}\mu_{02}\mu_{30}\mu_{21}\mu_{04} \\
& + 4\mu_{20}\mu_{11}\mu_{02}\mu_{30}\mu_{03}\mu_{22} + 4\mu_{20}\mu_{11}\mu_{02}\mu_{21}^2\mu_{13} - 4\mu_{20}\mu_{11}\mu_{02}\mu_{21}\mu_{12}\mu_{22} \\
& + 4\mu_{20}\mu_{11}\mu_{02}\mu_{12}^2\mu_{31} - 4\mu_{20}\mu_{11}\mu_{02}\mu_{12}\mu_{03}\mu_{40} + \mu_{20}\mu_{02}^2\mu_{30}^2\mu_{04} \\
& - 2\mu_{20}\mu_{02}^2\mu_{30}\mu_{21}\mu_{13} + 2\mu_{20}\mu_{02}^2\mu_{30}\mu_{12}\mu_{22} - 2\mu_{20}\mu_{02}^2\mu_{30}\mu_{03}\mu_{31} + \mu_{20}\mu_{02}^2\mu_{21}^2\mu_{22} \\
& - 2\mu_{20}\mu_{02}^2\mu_{21}\mu_{12}\mu_{31} + 2\mu_{20}\mu_{02}^2\mu_{21}\mu_{03}\mu_{40} - 8\mu_{11}^3\mu_{21}^2\mu_{13} + 16\mu_{11}^3\mu_{21}\mu_{12}\mu_{22}
\end{aligned}$$

$$-8\mu_{11}^3\mu_{12}^2\mu_{31}+8\mu_{11}^2\mu_{02}\mu_{30}\mu_{21}\mu_{13}-8\mu_{11}^2\mu_{02}\mu_{30}\mu_{12}\mu_{22}-4\mu_{11}^2\mu_{02}\mu_{21}^2\mu_{22}$$
$$+4\mu_{11}^2\mu_{02}\mu_{12}^2\mu_{40}-2\mu_{11}\mu_{02}^2\mu_{30}^2\mu_{13}+4\mu_{11}\mu_{02}^2\mu_{30}\mu_{12}\mu_{31}+2\mu_{11}\mu_{02}^2\mu_{21}^2\mu_{31}$$
$$-4\mu_{11}\mu_{02}^2\mu_{21}\mu_{12}\mu_{40}+\mu_{02}^3\mu_{30}^2\mu_{22}-2\mu_{02}^3\mu_{30}\mu_{21}\mu_{31}+\mu_{02}^3\mu_{21}^2\mu_{40})/\mu_{00}^{14}$$

These invariants are irreducible, i.e. they cannot be expressed as a polynomial of other invariants, but a more complicated dependency can be among them. The number of the independent invariants can be computed by so called *rule of thumb*: The number $n$ of independent invariants equals

$$n = m - p, \tag{5}$$

where $m$ is the number of independent measurements of some object, in our case it is the number of moments, and $p$ is the number of independent constraints, which must be satisfied (see e.g. [15]). Mostly it equals the number of parameters of the transformation. This formula is called "rule of thumb", because often it is not easy to find, which measurements and constraints are independent and which not. In our case, the moments are independent and if we have the moments to the second order only, then there is one dependency among the constraints, so we have 6-5=1 invariant of the second order. If we use the moments of higher orders, then the constraints are independent, and we have 10-6=4 independent invariants of the third order and 15-6=9 independent invariants of the fourth order.

## NORMALIZED MOMENTS

The moments (4) are normalized to translation and scaling. We can continue with the normalization to achieve affinely invariant normalized moments. The affine transform (1) can be decomposed into six simple one-parameter transforms[1]

$$\begin{array}{lll}
\text{Horizontal translation}: & \text{vertical translation}: & \text{Scaling}: \\
u = x - x_0 & u = x & u = \omega x \\
v = y & v = y - y_0 & v = \omega y \\
& & \\
\text{First rotation}: & \text{Stretching}: & \text{Second rotation}: \\
u = x\cos\alpha - y\sin\alpha & u = \delta x & u = x\cos\rho - y\sin\rho \\
v = x\sin\alpha + y\cos\alpha & v = \frac{1}{\delta} y & v = x\sin\rho + y\cos\rho .
\end{array} \tag{6}$$

It is advantageous to use complex moments for the normalization to the rotation. The complex moment $c_{pq}$ of order $(p+q)$ is defined as

$$c_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+iy)^p(x-iy)^q f(x,y)\,dxdy\ , \tag{7}$$

[1] This decomposition does not include the affine transforms with negative Jacobian, we would have to insert mirror reflection. The consequences of not doing it will be commented later.

where $i$ denotes imaginary unit. Each complex moment can be expressed in terms of geometric moments $m_{pq}$ as

$$c_{pq} = \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{q-j} \cdot i^{p+q-k-j} \cdot m_{k+j,p+q-k-j} \ . \tag{8}$$

If we use moments $v_{pq}$ normalized to translation and scaling instead of the $m_{pq}$ in (8), we obtain complex moments normalized to translation and scaling. The reverse formula is

$$m_{pq} = \frac{1}{2^{p+q} i^q} \sum_{k=0}^{p} \sum_{j=0}^{q} \binom{p}{k} \binom{q}{j} (-1)^{q-j} \cdot c_{k+j,p+q-k-j} \ . \tag{9}$$

In polar coordinates, (7) becomes the form

$$c_{pq} = \int_0^\infty \int_0^{2\pi} r^{p+q+1} e^{i(p-q)\theta} f(r,\theta) dr d\theta. \tag{10}$$

After the rotation by an angle $\alpha$ the complex moment becomes

$$c'_{pq} = e^{i(p-q)\alpha} \cdot c_{pq} \ . \tag{11}$$

We can use the complex moments either for the normalization or for the creation of the rotation invariants. During the normalization we choose some suitable normalizing moment $c_{p_0q_0}$ and then we ask it become real and positive. It can be done by virtual rotation of the image by the angle

$$\alpha = -\frac{1}{p_0 - q_0} \arctan\left(\frac{\Im(c_{p_0q_0})}{\Re(c_{p_0q_0})}\right), \tag{12}$$

i.e. each complex moment is multiplied as in (11), where the $\alpha$ is from (12). The $\Re(c_{p_0q_0})$ and $\Im(c_{p_0q_0})$ are real and imaginary parts of the $c_{p_0q_0}$.

It is suitable to use the complex moment $c_{20}$ for the normalization to the first rotation. The principal axis is then horizontal. The normalization to the stretching is provided so the $c_{20}$ becomes zero after it. From it we can compute the normalizing coefficient $\delta$

$$\delta = \sqrt[4]{\frac{c_{11} - \sqrt{c_{20}c_{02}}}{c_{11} + \sqrt{c_{20}c_{02}}}} \ . \tag{13}$$

We scale the image horizontally by it and vertically by the coefficient $1/\delta$. It can be performed simply in case of geometric moments

$$m'_{pq} = \delta^{p-q} m_{pq} \ , \tag{14}$$

but it is more complicated in case of complex moments. We can convert them to the geometric ones by (9), normalize them by (14) and then convert back by (8).

Now we could finish the normalization to the affine transform by the normalization to the second rotation by $c_{21}$, but it emerged that simpler equations can be obtained, when we use rotation invariants instead. The product

$$\Phi = \prod_{i=1}^{n} c_{p_i q_i}^{k_i} \text{ is invariant to rotation, if } \sum_{i=1}^{n} k_i(p_i - q_i) = 0, \tag{15}$$

where $n \geq 1$, $k_i, p_i$, and $q_i$ $(i = 1, \cdots, n)$ are non-negative integers. Some other details about rotation moment invariants can be found in [16].

During the normalization, the $c_{10}, c_{01}, c_{20}$ as well as $c_{02}$ become zero, the $c_{00}$ becomes one and rotation invariants $\Phi$ from (15) become affine invariants.

The main idea of the proof of independence and completeness of some set of affine moment invariants is following: we choose some suitable complete and independent set of the normalized moments of the same orders and if we can unambiguously compute values of a set from the other set and vice versa, then the other set is complete and independent, too.

We can compute values of the affine moment invariants directly from the complex moments by means of the following theorem.

**Theorem 1:** Let us denote the value of the invariant computed from the geometric moments $I(\mu_{pq})$ and $I(c_{pq})$ the value obtained by substitution the complex moments $c_{pq}$ instead of $\mu_{pq}$. Then there is relation between them

$$(-2i)^w I(\mu_{pq}) = I(c_{pq}) \ , \tag{16}$$

where $w$ is the weight of the invariant.

**Proof:** We use a special affine transform

$$\begin{aligned} x' &= x + iy \\ y' &= x - iy. \end{aligned} \tag{17}$$

Its Jacobian $J = -2i$. According to (7) the value of the moment after the transform

$$\mu'_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q |J| f(x,y) dx dy = |J| c_{pq} \tag{18}$$

(the coordinates of the centroid remains unchanged). From the fundamental theorem [7, 4] we have for the values of the invariants without normalization by $\mu_{00}$: $I(\mu'_{pq}) = J^w |J|^k I(\mu_{pq})$ and from (18)we have $I(\mu'_{pq}) = |J|^k I(c_{pq})$. Since $c_{00} = \mu_{00}$, it imply directly (16). □

The former normalization is a special case of an affine transform, so we can compute affine moment invariants by Theorem 1 with constraints $c_{20} = c_{02} = 0$ and $c_{00} = 1$. First, we would like to check, if the first four affine moment invariants of the 2nd and 3rd orders are complete and independent

$$\begin{aligned} (-2i)^2 I_1 &= -c_{11}^2 \\ (-2i)^6 I_2 &= -c_{30}^2 c_{03}^2 + 6c_{30}c_{21}c_{12}c_{03} - 4c_{30}c_{12}^3 - 4c_{21}^3 c_{03} + 3c_{21}^2 c_{12}^2 \\ (-2i)^4 I_3 &= -c_{11}c_{30}c_{03} + c_{11}c_{21}c_{12} \\ (-2i)^6 I_4 &= 2c_{11}^3 c_{30}c_{03} + 6c_{11}^3 c_{21}c_{12}. \end{aligned} \tag{19}$$

Solving these equations we obtain

$$\begin{array}{lllll} c_{11} & = & 2\sqrt{I_1}, & c_{21}c_{12} & = \frac{1}{\sqrt{I_1}}(2I_3 - \frac{I_4}{I_1}), \\ c_{30}c_{03} & = & -\frac{1}{\sqrt{I_1}}(6I_3 + \frac{I_4}{I_1}), & \Re(c_{30}c_{12}^3) & = 8I_2 - 12\frac{I_3^2}{I_1} + \frac{I_4^2}{I_1^3}. \end{array} \tag{20}$$

The set $c_{11}$, $c_{21}c_{12}$, $c_{30}c_{03}$ and $\Re(c_{30}c_{12}^3)$ is used as the other set of invariants. This system of rotation invariants is independent and complete except the sign of $\Im(c_{30}c_{12}^3)$. It relates with fact that the set $I_1, I_2, I_3, I_4$ cannot distinguish two objects differing by mirror reflection. The general affine transform include the mirror reflection, therefore the set $I_1, I_2, I_3, I_4$ is complete from this point of view, obversely, we should use absolute values of the invariants with odd weights. If we insert $I_5$ there, we can compute $\Im(c_{30}c_{12}^3) = 4I_5/(\sqrt{I_1})^3$, but we can compute absolute value of $\Im(c_{30}c_{12}^3)$ also as $|\Im(c_{30}c_{12}^3)| = \sqrt{c_{30}c_{03}(c_{21}c_{12})^3 - \Re^2(c_{30}c_{12}^3)}$ , therefore the set $I_1, I_2, I_3, I_4, I_5$ is complete, but dependent.

Now, we need to insert invariants of the 4th order. If we add the invariants $I_6$, $I_7$, $I_8$, $I_9$ and $I_{10}$, we obtain

$$\begin{array}{lll} (-2i)^4 I_6 & = & c_{40}c_{04} - 4c_{31}c_{13} + 3c_{22}^2 \\ (-2i)^6 I_7 & = & c_{40}c_{22}c_{04} - c_{40}c_{13}^2 - c_{31}^2 c_{04} + 2c_{31}c_{22}c_{13} - c_{22}^3 \\ (-2i)^4 I_8 & = & 4c_{11}^2 c_{22} \\ (-2i)^6 I_9 & = & 4c_{11}^2 c_{31}c_{13} - 4c_{11}^2 c_{22}^2 \\ (-2i)^9 I_{10} & = & -4c_{11}^3 c_{40}c_{13}^2 + 4c_{11}^3 c_{31}^2 c_{04} \,. \end{array} \tag{21}$$

Solving these equations we obtain

$$\begin{array}{lllllll} c_{22} = \frac{I_8}{I_1}, & c_{31}c_{13} & = & \frac{I_8^2}{I_1^2} - 4\frac{I_9}{I_1}, & \Re(c_{40}c_{13}^2) & = & 32I_7 + 8\frac{I_6 I_8}{I_1} - 12\frac{I_8 I_9}{I_1^2} + \frac{I_8^3}{I_1^3}, \\ & c_{40}c_{04} & = & 16I_6 + \frac{I_8^2}{I_1^2} - \frac{I_9}{I_1}, & \Im(c_{40}c_{13}^2) & = & \frac{8I_{10}}{(\sqrt{I_1})^3}. \end{array} \tag{22}$$

We can see the absolute value of $\Im(c_{40}c_{13}^2)$ can be computed by two ways, from

$$|\Im(c_{40}c_{13}^2)| = \sqrt{c_{40}c_{04}(c_{31}c_{13})^2 - \Re^2(c_{40}c_{13}^2)} \tag{23}$$

and from the last equation in (22), while the phase of the $c_{31}c_{12}^2$ cannot be computed. The set $I_1, \ldots, I_4,\ I_6, \ldots, I_{10}$ is neither complete nor independent.

From the graph method, we have 66 irreducible invariants up to the fourth order. There is no space to present them here, we can show the most interesting examples. The simplest equations can be obtained, when we substitute the $I_{10}$ in the set by the $I_{12}$

$$(-2i)^8 I_{12} = -8c_{11}^3 c_{21}^2 c_{13} + 16c_{11}^3 c_{21}c_{12}c_{22} - 8c_{11}^3 c_{12}^2 c_{31} \,. \tag{24}$$

The last equation in (22) is then

$$\Re(c_{31}c_{12}^2) = \frac{1}{\sqrt{I_1}}\left(2\frac{I_3 I_8}{I_1} - \frac{I_4 I_8}{I_1^2} - 2\frac{I_{12}}{I_1}\right) \,. \tag{25}$$

The set $c_{11}$, $c_{21}c_{12}$, $c_{30}c_{03}$, $\Re(c_{30}c_{12}^3)$, $c_{22}$, $c_{31}c_{13}$, $c_{40}c_{04}$, $\Re(c_{40}c_{13}^2)$ and $\Re(c_{31}c_{12}^2)$ is complete and independent and therefore the set $I_1, I_2, I_3, I_4, I_6, I_7, I_8, I_9, I_{12}$ is complete and independent, too.

We may want to use $I_{11}$ instead of $I_{12}$ because it is simpler, then the last equation from (21) becomes

$$\begin{aligned}(-2i)^6 I_{11} &= -2c_{11}c_{30}c_{12}c_{13} - 2c_{11}c_{21}c_{03}c_{31} + 2c_{11}c_{30}c_{03}c_{22} - 2c_{11}c_{21}c_{12}c_{22}\\ &\quad +2c_{11}c_{21}^2c_{13} + 2c_{11}c_{12}^2c_{31}\end{aligned} \tag{26}$$

We can arrange it in the form

$$32\frac{I_{16}}{c_{11}} = 2\Re(c_{31}c_{12}^2)\Re(\frac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1) + 2\Im(c_{31}c_{12}^2)\Im(\frac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1) - c_{30}c_{03}c_{22} + c_{21}c_{12}c_{22}\ . \tag{27}$$

If we substitute $|\Im(c_{31}c_{12}^2)| = \sqrt{c_{31}c_{13}(c_{21}c_{12})^2 - \Re^2(c_{31}c_{12}^2)}$, we obtain a quadratic equation

$$\begin{aligned}&\Re^2(c_{31}c_{12}^2)(\Re^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1) + \Im^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1))\\ &-\Re(c_{31}c_{12}^2)\Re(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1)(32\tfrac{I_{16}}{c_{11}} + c_{30}c_{03}c_{22} - c_{21}c_{12}c_{22})\\ &+\tfrac{1}{4}(32\tfrac{I_{16}}{c_{11}} + c_{30}c_{03}c_{22} - c_{21}c_{12}c_{22})^2 - \Im^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1)(c_{31}c_{13}c_{21}^2c_{12}^2) = 0\ .\end{aligned} \tag{28}$$

It has two solutions

$$\Re(c_{31}c_{12}^2) = \frac{\begin{aligned}&\Re(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1)(32\tfrac{I_{16}}{c_{11}} + c_{30}c_{03}c_{22} - c_{21}c_{12}c_{22})\\ &\pm\Im(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1)\sqrt{\begin{aligned}&(32\tfrac{I_{16}}{c_{11}} + c_{30}c_{03}c_{22} - c_{21}c_{12}c_{22})^2\\ &+4c_{31}c_{13}c_{21}^2c_{12}^2(\Re^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1) + \Im^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1))\end{aligned}}\end{aligned}}{2(\Re^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1) + \Im^2(\tfrac{c_{30}c_{12}^3}{c_{21}^2c_{12}^2} - 1))}\ . \tag{29}$$

You can see that a real solution always exists, i.e. the set set $I_1, I_2, I_3, I_4, I_6, I_7, I_8, I_9, I_{11}$ is independent, but also increasing complexity of the equations. Using of some other invariants, similarly simple as $I_{11}$ leads the final equation to be quartic. It leads to effort for simplification of the equations. If we need not have general formulas, but only dependence test, the solution in some points might be satisfactory. We obtain simple equations, if we choose the values of the affine invariants so the values of rotation invariants would be 1, but it cannot be kept, when it could lead to a zero denominator.

In our case, if we choose $I_1 = 1/4,\ I_2 = 0,\ I_3 = 0,\ I_4 = -1/8$, then we obtain $c_{11} = 1,\ c_{21}c_{12} = 1,\ c_{30}c_{03} = 1,\ c_{30}c_{12}^3 = 1$. Similarly, if $I_6 = 0,\ I_7 = 0,\ I_8 = 1/4,\ I_9 = 0$, then $c_{22} = 1,\ c_{31}c_{13} = 1,\ c_{40}c_{04} = 1,\ c_{40}c_{13}^2 = 1$. Now, if we use $I_{10}$, then the last equation becomes $\Im(c_{40}c_{13}^2) = 64I_{10}$ , but from the previous equations we have $\Im(c_{40}c_{13}^2) = 0$ , therefore we cannot choose $I_{10}$ freely, it is dependent. If we use $I_{12}$ instead of $I_{10}$, then the last equation becomes $\Re(c_{31}c_{12}^2) = 1 - 16I_{12}$ . We can choose $I_{12}$ freely, if $I_{12} = 0$, then

**TABLE 1.** The numbers of errors from 1000 cases for various sets of invariants. The first 8 invariants was always $I_1$, $I_2$, $I_3$, $I_4$, $I_6$, $I_7$, $I_8$ and $I_9$, the 9th invariant is in the first row.

| 9th invariant | - | $I_{10}$ | $I_{11}$ | $I_{12}$ |
|---|---|---|---|---|
| number of errors | 259 | 259 | 209 | 209 |

$c_{31}c_{12}^2 = 1$. The $I_{12}$ is independent. If we use $I_{11}$, then the last equation becomes $I_{11} = 0$. It looks like the $I_{11}$ would be dependent, but if we change e.g. $I_2 = -\frac{1}{4}$, then $c_{30}c_{12}^3 = -1$ and we obtain $\Re(c_{31}c_{12}^2) = -8I_{11}$ . The $I_{11}$ can be chosen freely, it is independent. If we look at the general formula (29), we found that the values of the affine invariants must be chosen so the denominator would be non-zero. This is the biggest problem with limited solution in one point. There are singular points in the space of invariants and if we choose such a point, the dependency test fails in spite of the invariants are independent.

## NUMERICAL EXPERIMENT

To show differences in recognizing abilities of various sets of invariants, the following numerical experiment was carried out. The images of ten digits from 0 to 9 were created, then each digit was 100 times affinely deformed and additive random noise was inserted to the rectangle circumscribed each digit. The random affine transform was composed from first rotation by uniformly distributed angle, scaling and stretching with coefficients with mean one and standard deviation 0.1 and second rotation by uniformly distributed angle. Too extreme scaling and stretching with parameters less than 0.5 or greater than 2 were refused. The noise had zero mean and standard deviation 1/20 of the range from black to white. The results are in Tab. 1, the example of the original digit and the deformed digit are in Figs. 1a and 1b respectively.

The noise was chosen so heavy the number of errors to be significant. The values of the invariants were normalized by their standard deviation all over the whole set of the images so the invariants to have the same range of values. First, the incomplete set of 8 invariants $I_1$, $I_2$, $I_3$, $I_4$, $I_6$, $I_7$, $I_8$ and $I_9$ was used, then the dependent invariant $I_{10}$ was inserted. The error rate did not change. Then the $I_{10}$ was substituted by the independent invariant $I_{11}$. The error rate significantly decreased. When the $I_{11}$ was substituted by another independent invariant $I_{12}$, the error rate stayed decreased.

## CONCLUSION

The affine moment invariants are important tool for recognition of geometrically deformed images for many years. Nevertheless, the proof of independence of a chosen set of them is still problem. This contribution presents a new approach to these proofs by comparison with the normalized moments. The proof for invariants up to the fourth order is presented directly, the proof for higher orders can be more complicated, but it can be simplified, if we compute the solution not in the whole space of the feature values, but in some suitably chosen specific values.

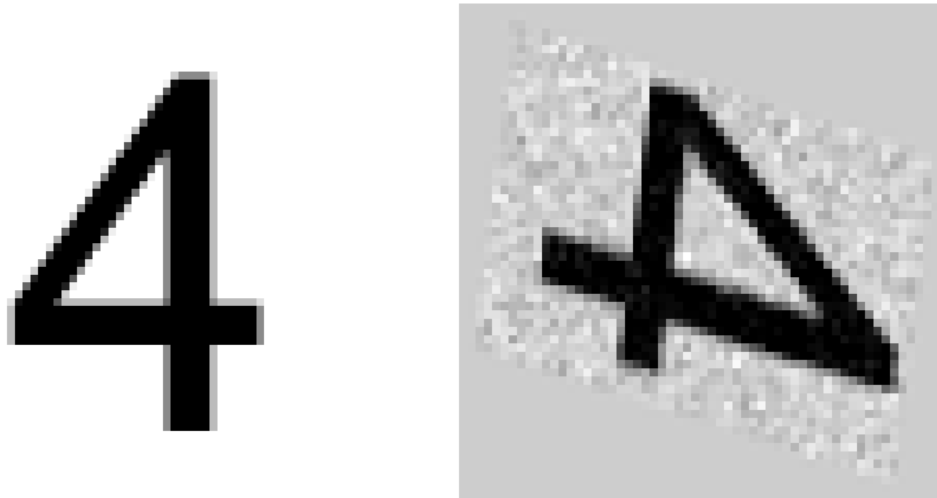

**FIGURE 1.** a) The example of the original digit, b) The example of the affinely deformed noisy digit

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