Affine Moment Invariants Generated by Automated Solution of the Equations

Tomáš Suk *and* Jan Flusser
Institute of Information Theory and Automation of the ASCR
Pod vodárenskou věží 4, 182 08 Praha 8, Czech Republic
E-mail: suk@utia.cas.cz, flusser@utia.cas.cz

Abstract

Pattern recognition of objects on affinely distorted images based on moments is important task researched for many years. Affine moment invariants are significant tool for it. They can be generated by a few methods. Automated direct solution of the Cayley – Aronhold differential equation has not been referred yet. It makes possible to generate invariants with much higher weights than other methods, but we must pay more attention to the numerical stability of the computation. The method is demonstrated on examples.

1. Introduction

Affine moment invariants are useful features for recognition of objects on affinely distorted images. They are studied for many years, but there are still problems with generation of the invariants of high weights. They can be derived as a solution of corresponding equations. A pioneer work on this field was done independently by Reiss [3] and Flusser and Suk [1]. Manual solving these equations is laborious, particularly for higher orders, but it can be avoided, if we generate the invariants automatically. The topic of this contribution is automated direct solution of the equations by the computer.

The competitive approach is tensor method [3] or graph method [5]. They are easy for programming, but their disadvantage is relative slowness. Another approach, called normalized moments, transforms the image into some standard position and moments are computed in it (e.g. [4] or [6] among others). They can relatively easily reach high orders, but if we write complete formulas for them, they are quite complicated. Their main disadvantage is less stability in recognition of symmetric objects.

2. Derivation of Affine Moment Invariants

The *affine transform* is a general linear transformation of spatial coordinates. It can be decomposed into horizontal and vertical translations, scaling, stretching, horizontal and vertical skews and mirror reflection. If the Jacobian of the affine transform is positive, then the mirror reflection is excluded from the decomposition. If some function is invariant to all these seven transforms, it is invariant to the affine transform, and vice versa. Other decompositions are also possible; this one is advantageous for derivation of the direct equations.

The translation invariance is achieved by using central moments instead of geometric moments, the scaling invariance can be obtained by normalization $\nu_{pq} = \mu_{pq}/\mu_{00}^{(p+q+2)/2}$, but an invariant is often written in form of polynomial of μ_{pq}'s with scaling normalization at the end. The product of moments, where the sum of the first indices equals the sum of the second indices, is invariant to the stretching

$$\Phi = \prod_{k=1}^{r} \mu_{p_k q_k} \text{ , where} \sum_{k=1}^{r} p_k = \sum_{k=1}^{r} q_k = w \text{ .} \quad (1)$$

From the invariance to the horizontal skew, we can derive (the derivative with respect to the parameter of the skew must be zero [1]) the equation

$$\sum_p \sum_q p\mu_{p-1,q+1} \frac{\partial I}{\partial \mu_{pq}} = 0 \text{ .} \quad (2)$$

In the theory of algebraic invariants, this is called Cayley – Aronhold differential equation. We can derive a similar equation from the vertical skew. The condition of symmetry can be derived from the mirror reflection, it implies that terms with interchanged first and second indices have identical (even w) or opposite (odd w) coefficients. Then we need not solve the equation from the

978-1-4244-2175-6/08/$25.00 ©2008 IEEE

vertical skew separately and we can reduce the number of unknowns.

The invariant has various attributes. It is

- *weight* w – the sum of the first indices of moments in one term
- *order* s – the maximum moment order
- *degree* r – the number of moments in one term
- *structure* $(k_2, k_3, ..., k_s)$ – each term is the product of k_2 moments of the 2nd order, k_3 moments of the 3rd order, ..., k_s moments of the s-th order
- *the number of terms*
  - theoretical number of terms n_t – the number of the terms Φ of the given structure
  - actual number of terms n_r – the number of terms in the result with nonzero coefficients.

3. Automation of the Algorithm

The input parameter of the algorithm is the structure of the desired invariant. If the structure is $(k_2, k_3, ..., k_s)$, then $r = k_2 + k_3 + ... + k_s$ and $w = (2k_2 + 3k_3 + ... + sk_s)/2$. First, we need to generate moment indices of all possible terms of the invariant; it means to generate all possible partitions of the number w to sums of k_2 integers from 0 to 2, k_3 integers from 0 to 3 up to k_s integers from 0 to s. It can be done in two stages. The first stage defines distribution among orders and the second one defines distributions inside the orders. Now we use the members of the partitions as first indices of the moments. The number of partitions is n_t.

Then moments in a term and terms in an invariant are sorted and symmetric terms are searched. The number of different coefficients after applying the condition of symmetry is n_s. Now the members of the Cayley – Aronhold equation are computed. Let us label the number of derivatives s_o and the number of occurrences of μ_{pq} in j-th term $n(p, q, j)$. If the form of the invariant is

$$I = (\sum_{j=1}^{n_t} c_j \prod_{\ell=1}^{r} \mu_{p_{j\ell}, q_{j\ell}})/\mu_{00}^{r+w} , \tag{3}$$

then the Cayley – Aronhold equation has form

$$(\sum_{i=1}^{s_o} \sum_{j=1}^{n_t} c_j \; p_i \; n(p_i, q_i, j) \frac{\mu_{p_i-1,q_i+1}}{\mu_{p_i,q_i}} \prod_{\ell=1}^{r} \mu_{p_{j\ell}, q_{j\ell}})/\mu_{00}^{r+w} = 0 . \tag{4}$$

Now we can construct the system of linear equations for unknown coefficients. The Cayley – Aronhold equation must hold for all values of the moments, therefore the sum of coefficients at identical terms of the derivatives must equal zero. The dimension of the space of the solutions equals the number of the invariants. The base of the solution can be found by singular value decomposition (SVD) of the matrix, the number of invariants equals the number of zero singular values. Many coefficients yielded by SVD are non-integers, but we know from the theory they can be integers. Therefore it is suitable to use reduced row echelon form (RREF). If there are still some non-integer coefficients, we look for their maximum fractional part $0 \le d \le 0.5$. If there are fractional parts $0.5 < d \le 1$, we use $1 - d$. Then we divide all coefficients by this maximum, it is repeated until all coefficients become integers.

4. Example with the Structure (2,0,2)

We need to generate all partitions of the number 6 to a sum of 2 numbers from 0 to 2 and 2 numbers from 0 to 4. They are in Tab. 1, $n_t = 14$. The moments in the terms are then sorted (4th column), the terms are also sorted and symmetric counterparts are searched. The unknown coefficients in the 3rd column are numbered according to it, $n_s = 10$.

Table 1. All possible terms of the invariant

1st stage	2nd stage	coefficient	term
6+0	4+2+0+0	c_1	$\mu_{02}^2\mu_{40}\mu_{22}$
	3+3+0+0	c_2	$\mu_{02}^2\mu_{31}^2$
5+1	4+1+1+0	c_3	$\mu_{11}\mu_{02}\mu_{40}\mu_{13}$
	3+2+1+0	c_4	$\mu_{11}\mu_{02}\mu_{31}\mu_{22}$
4+2	4+0+2+0	c_5	$\mu_{20}\mu_{02}\mu_{40}\mu_{04}$
	4+0+1+1	c_8	$\mu_{11}^2\mu_{40}\mu_{04}$
	3+1+2+0	c_6	$\mu_{20}\mu_{02}\mu_{31}\mu_{13}$
	3+1+1+1	c_9	$\mu_{11}^2\mu_{31}\mu_{13}$
	2+2+2+0	c_7	$\mu_{20}\mu_{02}\mu_{22}^2$
	2+2+1+1	c_{10}	$\mu_{11}^2\mu_{22}^2$
3+3	3+0+2+1	c_3	$\mu_{20}\mu_{11}\mu_{31}\mu_{04}$
	2+1+2+1	c_4	$\mu_{20}\mu_{11}\mu_{22}\mu_{13}$
2+4	2+0+2+2	c_1	$\mu_{20}^2\mu_{22}\mu_{04}$
	1+1+2+2	c_2	$\mu_{20}^2\mu_{13}^2$

Now, we need to put together the Cayley – Aronhold differential equation. The term with the derivative with respect to moment μ_{13} is an example

$$D(\mu_{13}) = 2c_2\mu_{20}^2\mu_{13}\mu_{04} + c_4\mu_{20}\mu_{11}\mu_{22}\mu_{04}+ c_6\mu_{20}\mu_{02}\mu_{31}\mu_{04} + c_9\mu_{11}^2\mu_{31}\mu_{04} + c_3\mu_{11}\mu_{02}\mu_{40}\mu_{04}.$$

The matrix derived from the equation

$$D(\mu_{13}) + D(\mu_{22}) + D(\mu_{31}) + D(\mu_{40}) + D(\mu_{11}) + \\ + D(\mu_{20}) = 0 \tag{5}$$

is in Tab. 2. SVD yields 8 nonzero and 2 zero singular values, the solution after RREF is in the last two rows of Tab. 2, i.e. we have two solutions

Table 2. The matrix of the system of linear equations for the coefficients. The solution is in the last two rows.

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
2	2	0	0	0	0	0	0	0	0
4	0	3	1	0	0	0	0	0	0
0	0	1	0	4	1	0	0	0	0
0	0	2	0	0	0	0	4	1	0
0	0	1	0	2	0	0	2	0	0
0	4	0	2	0	0	0	0	0	0
0	0	0	1	0	3	4	0	0	0
0	0	0	2	0	0	0	0	3	4
0	0	4	2	0	2	0	0	2	0
2	0	1	0	0	0	0	0	0	0
0	0	0	3	0	0	2	0	0	2
4	6	0	1	0	0	0	0	0	0
1	-1	-2	2	0	2	-2	1	0	-1
0	0	0	0	1	-4	3	-1	4	-3

$$I_1 = (\mu_{20}^2\mu_{22}\mu_{04} - \mu_{20}^2\mu_{13}^2 - 2\mu_{20}\mu_{11}\mu_{31}\mu_{04} + \\ 2\mu_{20}\mu_{11}\mu_{22}\mu_{13} + 2\mu_{20}\mu_{02}\mu_{31}\mu_{13} - 2\mu_{20}\mu_{02}\mu_{22}^2 + \\ \mu_{11}^2\mu_{40}\mu_{04} - \mu_{11}^2\mu_{22}^2 - 2\mu_{11}\mu_{02}\mu_{40}\mu_{13} + \\ 2\mu_{11}\mu_{02}\mu_{31}\mu_{22} + \mu_{02}^2\mu_{40}\mu_{22} - \mu_{02}^2\mu_{31}^2)/\mu_{00}^{10},$$

$$I_2 = (\mu_{20}\mu_{02}\mu_{40}\mu_{04} - 4\mu_{20}\mu_{02}\mu_{31}\mu_{13} + \\ 3\mu_{20}\mu_{02}\mu_{22}^2 - \mu_{11}^2\mu_{40}\mu_{04} + 4\mu_{11}^2\mu_{31}\mu_{13} \\ -3\mu_{11}^2\mu_{22}^2)/\mu_{00}^{10}.$$

If we are interested in the invariants with the structure (2,0,2) only, then these invariants are independent, but if we have computed all simpler invariants, we can find that I_2 is product of two other invariants $I_{2a} = (\mu_{20}\mu_{02} - \mu_{11}^2)/\mu_{00}^4$ and $I_{2b} = (\mu_{40}\mu_{04} - 4\mu_{31}\mu_{13} + 3\mu_{22}^2)/\mu_{00}^6$.

5. Additional Remarks

The computation over all possible structures can be automated, too. The generation of invariants of all structures of weight w and degree r means to compute all partitions of the number $2w$ into r integers from 2 to w. The algorithm is similar to that for generation of the terms of one invariant in the second stage. The products of other invariants (e.g. $I_2 = I_{2a}I_{2b}$) and the linearly dependent invariants should be eliminated.

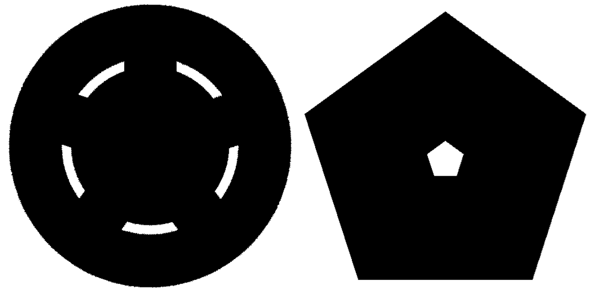

Figure 1. The five-rung wheel and the holey pentagon adjusted so their 2nd- and 4th-order moments are the same.

An example of problems with non-integer coefficients is the structure (1,1,1,3). It gives $w = 12$, $s = 5$, $r = 6$, $n_t = 330$ and $n_s = 165$. SVD yields 156 nonzero and 9 zero (under-threshold) singular values. After RREF, 4 from 9 solutions include non-integers. E.g. the third solution has $n_r = 244$. The biggest fractional part of them is 0.3333 and when we divide them by it, we can round them to integers. If the threshold for this rounding is improper, it could cause errors.

We use magnitude normalization to the growing order $p + q$ and to the growing degree r

$$\hat{\nu}_{pq} = \pi^{\frac{p+q}{2}} \left(\frac{p+q}{2} + 1 \right) \nu_{pq} \ , \ \hat{I} = \mathrm{sign}(I)|I|^{\frac{1}{r}} \ . \tag{6}$$

6. Numerical Experiment

The experiment should illustrate using invariants with high weights. An important problem in pattern recognition is recognition of symmetric objects. We know from the theory [2] that if we have an object with n-fold rotational symmetry, i.e. it repeats n-times itself during rotation by 360°, then there are some dependencies among the moments. The higher n, the more such dependencies. An example of two objects (500×500 pixels) with 5-fold rotational symmetry adjusted so their 2nd- and 4th-order moments are identical (3rd-order moments are zero) is on Fig. 1.

The complete and independent system of affine moment invariants up to the 4th order has 9 invariants with structures (2), (0,4), (1,2), (3,2), (0,0,2), (0,0,3), (2,0,1), (2,0,2) and (0,4,1). They were generated by the graph method [5]. Their values in Tab. 3 are very similar; the deviations are caused by sampling error.

Table 3. Values of the invariants up to the fourth order.

	I_{4a}	I_{4b}	I_{4c}	I_{4d}	I_{4e}	I_{4f}	I_{4g}	I_{4h}	I_{4i}
wheel	0.5259	-0.0000	0.0004	-0.0103	0.4797	0.2770	0.6742	0.5398	0.0001
pentagon	0.5261	0.0003	-0.0035	-0.0301	0.4799	0.2771	0.6744	0.5399	0.0011

For recognition of such symbols, we need invariants of at least 5th order. The complete and independent set of homogeneous[1] invariants of the 5th order contains 3 invariants with weights 10, 20 and 30; the latter two are extremely expensive when using the graph method. Let us label them I_{5a}, I_{5b} and I_{5c}, their structures are (0,0,0,4), (0,0,0,8) and (0,0,0,12), $n_t = 12$, 73 and 252 and $n_s = 9$, 44 and 140, respectively. As a by-product, we obtain dependent invariants $I_{5a}^2 - 8I_{5b}$, $I_{5a}I_{5b} + 17I_{5c}$ and $I_{5a}^3 - 12I_{5a}I_{5b} - 156I_{5c}$. The values of the invariants are in Tab. 4.

Table 4. Values of the invariants of the fifth order.

	I_{5a}	I_{5b}	I_{5c}
wheel	-0.0013	0.0004	-0.0003
pentagon	-0.0628	0.0091	-0.0048

We carried out an experiment with random affine transforms. Both symbols were successively deformed by 100 random affine transforms and normally distributed zero-mean random noise with gradually increased standard deviation was added to the objects. When we use I_{5a}, I_{5b} and I_{5c} as features for recognition, the symbols were recognized correctly up to SNR 0.2 dB. The example of recognized symbols is on Fig. 2. When we used I_{4a},...,I_{4i}, first errors appeared at SNR 30 dB.

For comparison, the normalized moments [6] of the 5th order with optimal threshold of non-zero moments reached SNR 0.6 dB without an error.

[1] Homogeneous invariants contain moments of one order only

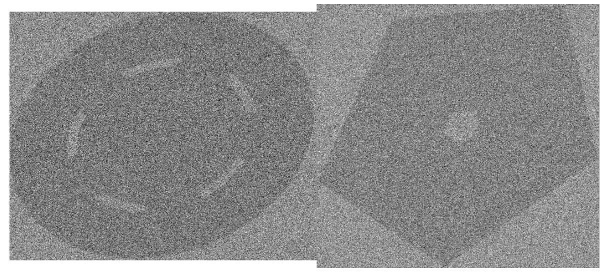

Figure 2. The affinely distorted noisy objects.

7. Conclusion

There are a few methods of automatic generation of affine moment invariants, automated direct solution of Cayley – Aronhold differential equation is one of them. Its main advantage is approximately polynomial computing complexity, while that of the method of tensors is exponential, therefore this method is suitable in case of relatively high weights, but we must pay attention to the correct processing of the non-integer coefficients.

8. Acknowledgement

This work has been supported by the grant No. 102/08/1593 of the Grant Agency of the Czech Republic.

References

[1] J. Flusser and T. Suk. Pattern recognition by affine moment invariants. *Pattern Recognition*, 26:167–174, 1993.

[2] J. Flusser and T. Suk. Moment invariants for recognizing symmetric objects. In *CAIP 2005, Computer Analysis of Images and Patterns*, pages 9–16. Springer, 2005.

[3] T. H. Reiss. The revised fundamental theorem of moment invariants. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 13:830–834, 1991.

[4] T. W. Shen, D. P. K. Lun, and W. C. Siu. On the efficient computation of 2-D image moments using the discrete Radon transform. *Pattern Recognition*, 31:115–120, 1998.

[5] T. Suk and J. Flusser. Graph method for generating affine moment invariants. In *ICPR 2004, 17th International Conference on Pattern Recognition*, pages 192–195. IEEE Computer Society, 2004.

[6] T. Suk and J. Flusser. Affine normalization of symmetric objects. In *Acivs 2005, Advanced Concepts for Intelligent Vision Systems*, pages 100–107. Springer, 2005.