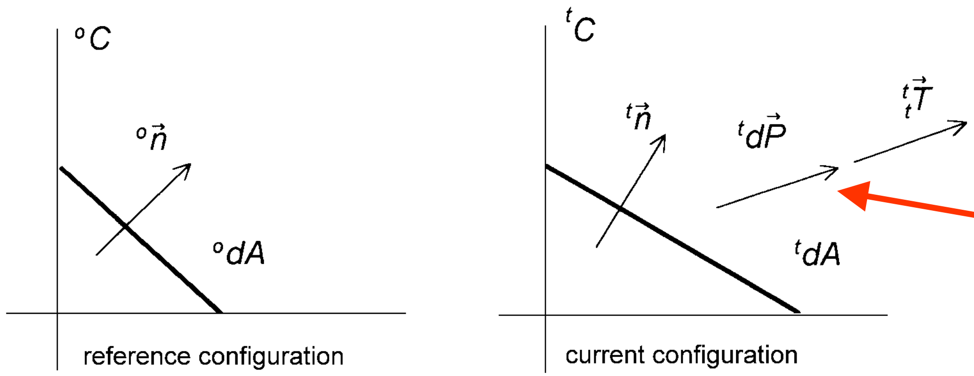
Continuum Mechanics, part 4

Stress_1

- Engineering and Cauchy
- First and second Piola-Kirchhoff
- Example for 1D strain and stress
- Green-Naghdi stress rate

Stress measures, notation and terminology



The elementary force responsible for the deformation

Stress vectors

Cauchy (true)

$$ {}^t_t\vec{T} = \lim_{\mathrm{d}^tA\to 0} \frac{\mathrm{d}^t\vec{P}}{\mathrm{d}^tA} $$

Engineering

$$ {}^t_0\vec{T} = \lim_{\mathrm{d}^0A\to 0} \frac{\mathrm{d}^t\vec{P}}{\mathrm{d}^0A} $$

Cauchy formula provides their relation to stress tensor

$$ {}^t_tT_i = {}^t_t\sigma_{ji}\;{}^tn_j $$

$$ {}^t_0T_i = {}^t_0\sigma_{ji}\;{}^0n_j $$

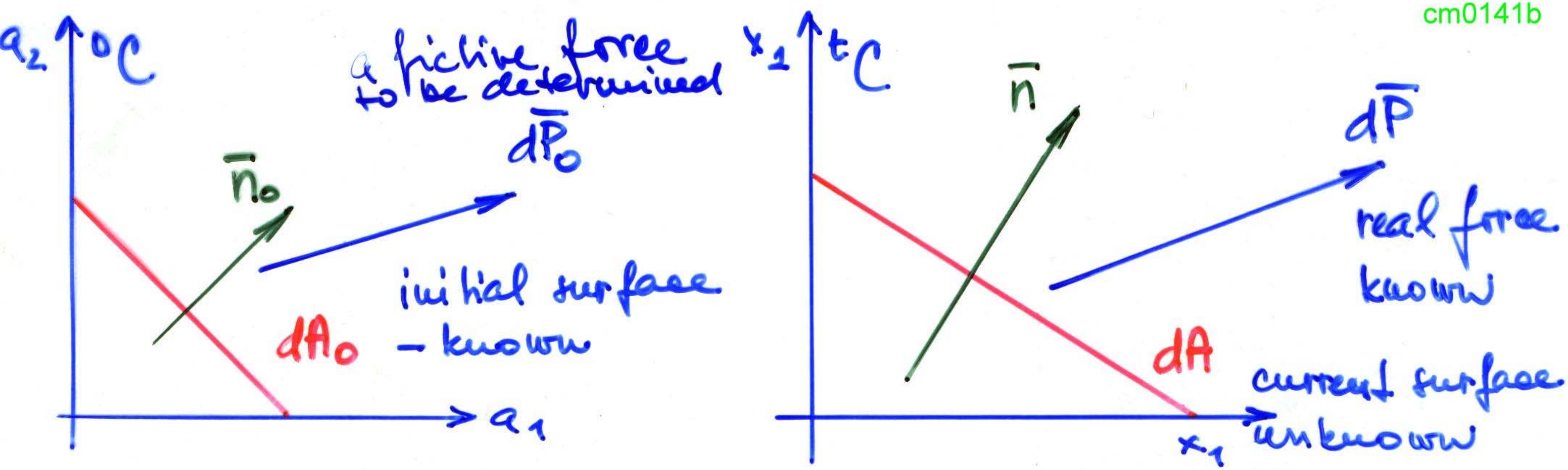
In deriving a ‘proper’ stress measure we require its independence of rigid body motion.

The engineering stress does not posses this property.

STRESS TENSORS

In linear elasticity the stress is defined as a limiting ratio of an elementary force $d\vec{P}$ and an elementary surface dA. It should be reminded that using this approach the elem. force belongs to the deformed configuration ${}^{t}C$ while the force to the initial one, ie ${}^{0}C$.

In finite deformation theory we have to take into account the change of geometry and the stresses should be properly related to strain.



Let's define unit vectors $\bar{n}_0$ and $\bar{n}$ perpendicular to elementary surfaces dA_0 and dA in initial and reference configurations ^0C and ^tC respectively.

The <u>stress vectors</u> are defined as limiting ratios

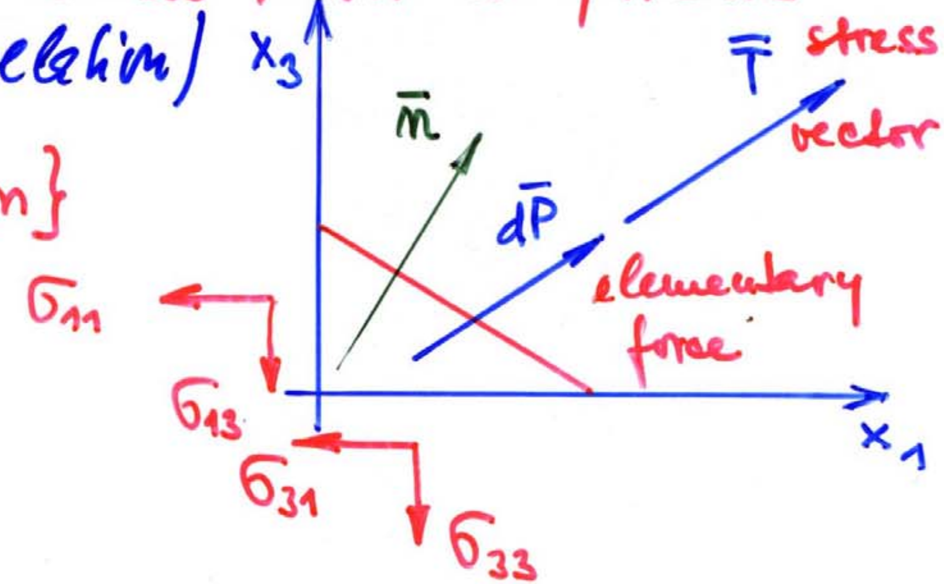
$$\bar{T}_0 = d\bar{P}_0 / dA_0 \qquad \bar{T} = d\bar{P}/dA$$

In linear continuum mechanics we take $d\bar{P}/dA_0$

– ENGINEERING STRESS

From the elementary linear elasticity we have already learned that stress vectors can be expressed by stress tensor components followingly (by Cauchy relation)

$$\{T\} = [\sigma]^T \{n\}$$



stress tensor

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yz} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = [\sigma]$$

'vector' of stress tensor components (computationally ... array)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \{\sigma\}$$

DIMENSION SIGMA(6)

NOT VECTOR IN TENSORIAL SENSE

THE TRANSFORMATION LAW IS NOT OBEYED

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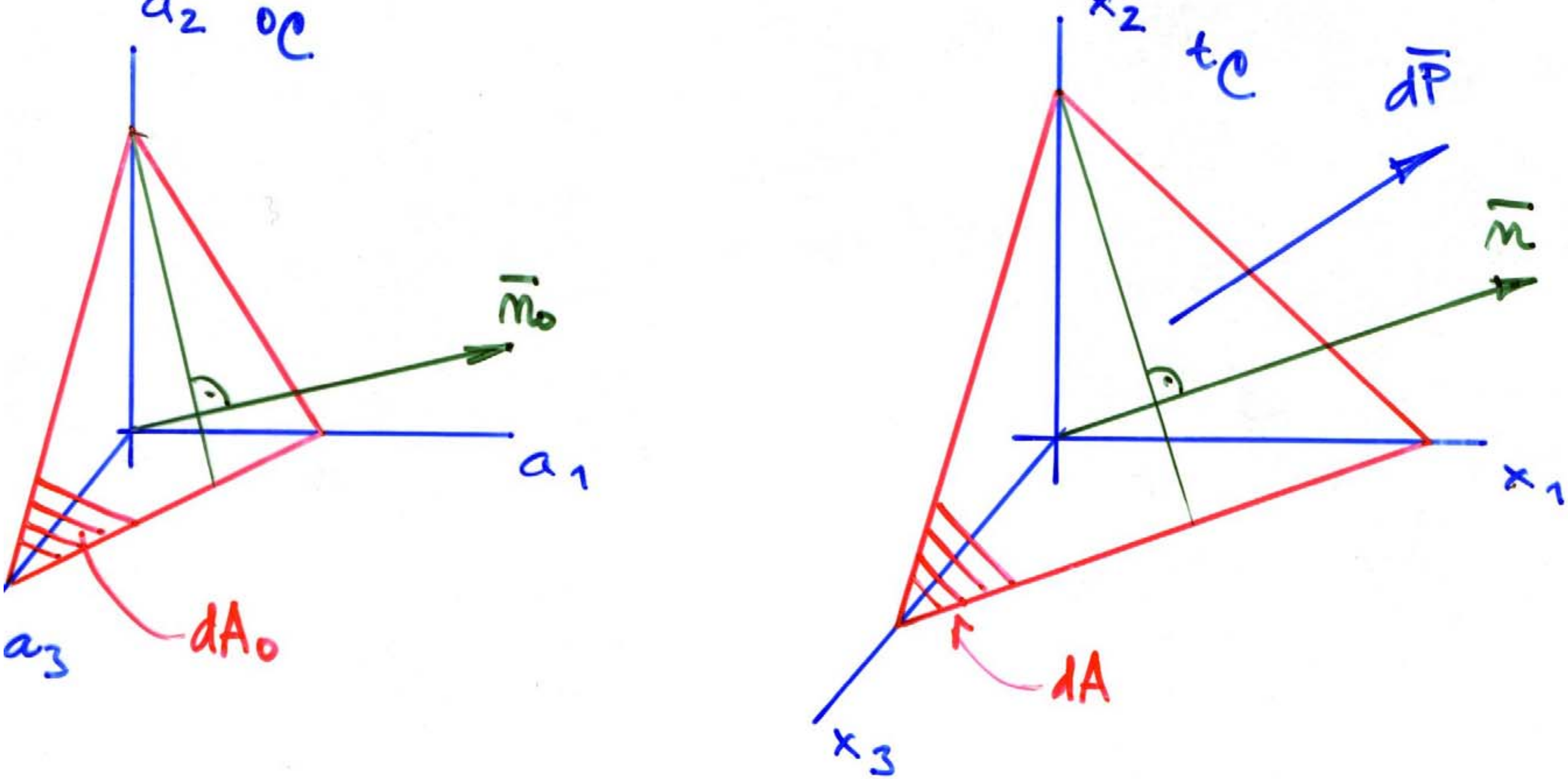

Let $d\bar{P}$ is an elementary force responsible for the change of an elementary volume from ^0C to ^tC. In the current configuration we have

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$\{dP\} = \{T\}\, dA$ and $\{T\} = [\sigma]^T \{n\}$

$$\Rightarrow \quad \boxed{\{dP\} = [\sigma]^T dA \{n\} \qquad (*)}$$

where $[\sigma]$ is the Cauchy stress tensor (related to current configuration).

ALSO CALLED TRUE STRESS

Now, we are looking for a fictive force $d\bar{P}_0$, acting in ${}^0C$, which is related in a consistent way to dP (the actual force acting in ${}^tC$)

At first assume that $\boxed{d\bar{P}_0 = dP}$

This choice will lead to the first Piola-Kirchhoff stress tensor

Motivation for the definition of a fictive force in reference configuration

In the current configuration we know the applied forces, but the deformed geometry is unknown. The stress measure is clearly defined (Cauchy or true stress) but cannot be directly computed.

Inventing a fictive force ‘acting’ in the reference configuration (related in a systematic way to the actual force in the current configuration) allows to define a new suitable measure of stress in the reference configuration, (e.g. Piola-Kirchhoff) to compute it and relate it back to the the true stress. Such a stress should be independent of rigid body motion and of the choice of coordinate system.

The true stress is the only measure which is of final interest from engineering point of view.

Other measures are just useful tools to get the true stress.

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the acting force can be related to original configuration as well (but it is a fiction)

$$\{dP_0\} = \{dP\} = \{T_0\}\, dA_0 \qquad \{T_0\} = [\tau]^T \{m_0\}$$

first Piola-Kirchhoff or Lagrange stress tensor

$$\Rightarrow \{dP\} = [\tau]^T dA_0 \{m_0\} \qquad (**)$$

Comparing (*) and (**) we get

$$[\sigma]^T dA \{m\} = [\tau]^T dA_0 \{m_0\} \qquad (***)$$

In order to find the relation between $[\sigma]$ and $[\tau]$ it is necessary to find relation between dA and dA_0.

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Comparing volumes before (dV_0) and after (dV) the deformation and taking into account the **law of conservation of mass** we can write

$$\rho_0 \, dV_0 = \rho \, dV$$

the initial volume is

$$dV_0 = \frac{1}{6} \, da_1 \, da_2 \, da_3 =$$

$$= \frac{1}{9} \left(\underbrace{\frac{1}{2} \, da_2 \, da_3}_{dA_{01}} \, da_1 + \underbrace{\frac{1}{2} \, da_1 \, da_3}_{dA_{02}} \, da_2 + \underbrace{\frac{1}{2} \, da_1 \, da_2}_{dA_{03}} \, da_3 \right) =$$

$$= \frac{1}{9} \, dA_{0i} \, da_i = \frac{1}{9} \{dA_0\}^T \{da\}$$

Notice $\overline{dA}_0 = dA_{0i} \, n_{0i}$ or $\{dA_0\} = dA_0 \{n_0\}$ $\quad dA_0 = \{dA_0\}^T \{n_0\}$ §

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So $dV_0 = \frac{1}{9}\{dA_0\}^T\{da\}$ and similarly

$$dV = \frac{1}{9}\{dA\}^T\{dx\}$$

initial and final volumes

Now we can conclude

$$\rho_0\, dV_0 = \rho\, dV$$

$$\rho_0 \{dA_0\}^T\{da\} = \rho \{dA\}^T\{dx\}$$

$\{dx\} = [F]\{da\}$

Since $\{da\}$ is arbitrary

$$\rho_0 \{dA_0\}^T = \rho\{dA\}^T[F]$$

Using ⑤ we get

$$\rho_0\, dA_0 \{n_0\}^T = \rho\, dA\, \{n\}^T[F]$$

$$dA_0 \{n_0\}^T = \frac{\rho}{\rho_0}\, dA\, \{n\}^T[F]$$

substituting the left-hand side to (***) we can write cm0145b

$$[\sigma]^T \underline{dA\,\{n\}} = [\tau]^T \frac{\rho}{\rho_0} \underline{dA}\; [F]^T \underline{\{n\}}$$

this equation must be valid for any $dA\{n\}$

we can finally write (after transposition)

$$\boxed{\begin{aligned} [\sigma] &= \frac{\rho}{\rho_0}\,[F]\,[\tau] \\ [\tau] &= \frac{\rho_0}{\rho}\,[F]^{-1}[\sigma] \end{aligned}}$$

$$J = \det [F] = \frac{\rho_0}{\rho}$$

The first Piola-Kirchfoff is not symmetric

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there is another possibility how to relate the stress vector to original configuration, namely

$$\{dP_0\} = [F]^{-1}\{dP\} \quad \text{as} \quad \{da\} = [F]^{-1}\{dx\}$$

A new measure is introduced here, the second Piola-Kirchhoff

$$\{dP_0\} = [S]^T dA_0 \{n_0\}$$

$$\{dP\} = [\sigma]^T dA \{n\}$$

As before we get the second Piola-Kirchhoff defined by

$$[S] = \frac{\rho_0}{\rho}[F]^{-1}[\sigma][F]^{-T} \qquad ((\;)^{-1})^T$$

$$[\sigma] = \frac{\rho}{\rho_0}[F][S][F]^T \qquad J = \det[F] = \frac{\rho_0}{\rho}$$

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The second Kirchhoff-Piola stress tensor is a more suitable measure of state of stress than the first K-P. It is symmetric.

It should be noted that the second Piola-Kirchhoff stress tensor has little physical meaning. It is just a suitable stress measure which is energetically conjugate with green-lagrange strain tensor. Which means that their product corresponds to work or energy.

Example for 1D strain and stress

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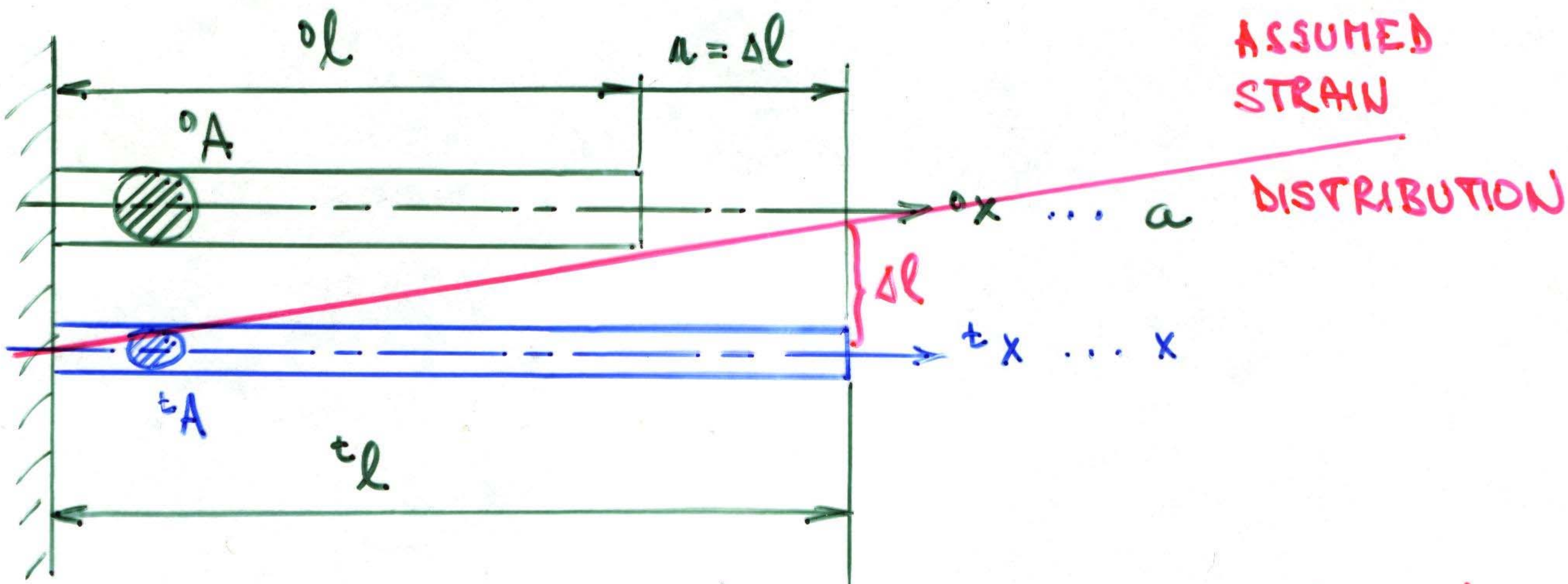
GREEN-LAGRANGE STRAIN TENSOR FOR 1D PROBLEM



Relation between coordinate systems $\quad {}^tx = {}^tx({}^0x, t)$

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In this case: $^t x = \frac{^t l}{^0 l}\, {^0 x} = \frac{^0 l + \Delta l}{^0 l}\, {^0 x} = \left(1 + \frac{\Delta l}{^0 l}\right) {^0 x}$

$$^t x = (1 + \varepsilon_x)\, {^0 x} \quad ; \quad \varepsilon_x = \frac{\Delta l}{^0 l}$$

ENGINEERING STRAIN
CAUCHY
INFINITESIMAL

Let's assume $\quad u_x = const\ {^t x}$

The unknown constant can be found from $\Delta l = const\ {^t l}$

$const = \Delta l / {^t l}$

So the distribution of displacement is

$$u_x = \frac{\Delta l}{^t l}\, {^t x} = \frac{\Delta l}{^t l}\, \frac{^t l}{^0 l}\, {^0 x} = \frac{\Delta l}{^0 l}\, {^0 x} = \varepsilon_x\, {^0 x}$$

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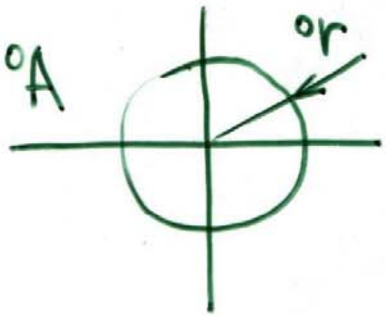
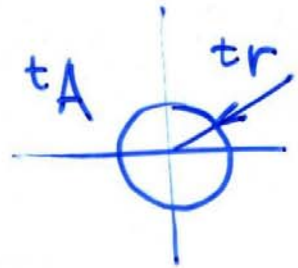
AND WHAT ABOUT CROSS-SECTION QUANTITIES





$$^{t}r = {}^{0}r + \Delta r \; ; \; \varepsilon_r = \frac{\Delta r}{^{0}r}$$

GEOMETRY : $$\frac{^{t}A}{^{0}A} = \left(\frac{^{t}r}{^{0}r}\right)^2 = \left(\frac{^{0}r + \Delta r}{^{0}r}\right)^2 = \left(1 + \frac{\Delta r}{^{0}r}\right)^2 = \left(1 + \varepsilon_r\right)^2$$

RELATIONS BETWEEN COORDINATE SYSTEMS

$$^{t}y = \text{const}\ {}^{0}y \qquad {}^{t}z = \text{const}\ {}^{0}z$$

$$^{t}r = \text{const}\ {}^{0}r$$

$$\text{const} = \frac{^{t}r}{^{0}r} = \left(\frac{^{t}A}{^{0}A}\right)^{1/2} = \left(1 + \varepsilon_r\right)$$

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So:

$$ {}^{t}r = (1+\varepsilon_r)\,{}^{o}r \qquad {}^{t}y = (1+\varepsilon_r)\,{}^{o}y $$

$$ {}^{t}z = (1+\varepsilon_r)\,{}^{o}z $$

$$ \Rightarrow \quad {}^{t}y - {}^{o}y = \varepsilon_r\,{}^{o}y \qquad {}^{t}z - {}^{o}z = \varepsilon_r\,{}^{o}z $$

$$ u_y = \varepsilon_r\,{}^{o}y \qquad u_z = \varepsilon_r\,{}^{o}z $$

DISPLACEMENT DISTRIBUTION

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SO THE KINEMATIC RELATION IN THIS CASE IS

$$ {}^{t}x_i = {}^{t}x\left({}^{0}x, t\right) $$

$$ {}^{t}x = \left(1+\varepsilon_x\right)\,{}^{0}x \qquad {}^{t}x_1 = \left(1+\varepsilon_x\right)\,{}^{0}x_1 $$

$$ {}^{t}y = \left(1+\varepsilon_r\right)\,{}^{0}y \qquad {}^{t}x_2 = \left(1+\varepsilon_r\right)\,{}^{0}x_2 $$

$$ {}^{t}z = \left(1+\varepsilon_r\right)\,{}^{0}z \qquad {}^{t}x_3 = \left(1+\varepsilon_r\right)\,{}^{0}x_3 $$

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AND THE DEFORMATION GRADIENT IS

$$F_{ij} = \frac{\partial\, {}^t x_i}{\partial\, {}^0 x_j} = \begin{bmatrix} 1+\varepsilon_x & 0 & 0 \\ 0 & 1+\varepsilon_r & 0 \\ 0 & 0 & 1+\varepsilon_r \end{bmatrix}$$

AND ITS DETERMINANT

$$J = \det[F] = (1+\varepsilon_x)(1+\varepsilon_r)^2 = \left(1+\frac{\Delta \ell}{{}^0\ell}\right)\frac{{}^tA}{{}^0A} =$$

$$= \frac{{}^0\ell + \Delta\ell}{{}^0\ell}\,\frac{{}^tA}{{}^0A} = \frac{{}^t\ell}{{}^0\ell}\,\frac{{}^tA}{{}^0A}$$

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GREEN - LAGRANGE STRAIN TENSOR

$$[E] = \frac{1}{2}\left([F]^T[F] - I\right) =$$

$$= \frac{1}{2}\begin{bmatrix} 1+\varepsilon_x+\varepsilon_x+\varepsilon_x^2-1 & & \\ & 1+\varepsilon_r+\varepsilon_r+\varepsilon_r^2-1 & \\ & & 1+\varepsilon_r+\varepsilon_r+\varepsilon_r^2-1 \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_x+\frac{1}{2}\varepsilon_x^2 & & \\ & \varepsilon_r+\frac{1}{2}\varepsilon_r^2 & \\ & & \varepsilon_r+\frac{1}{2}\varepsilon_r^2 \end{bmatrix}$$

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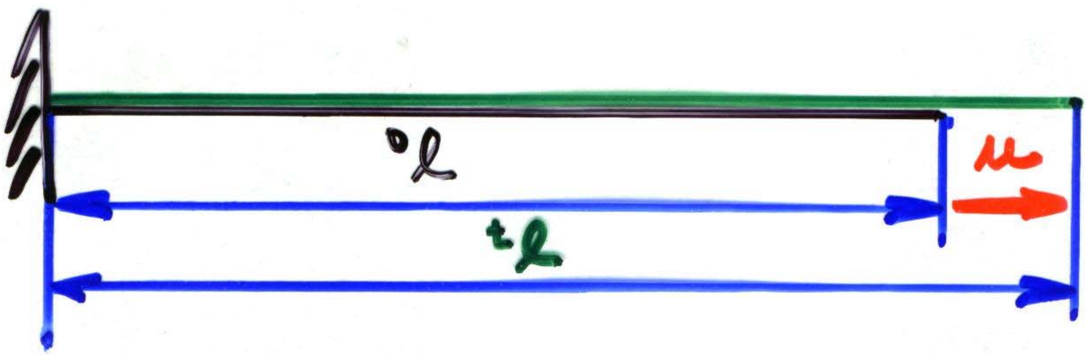
IN THE TEXT THAT FOLLOWS WE WILL USE

$$\varepsilon_{11} = \varepsilon_G = \varepsilon_x + \frac{1}{2}\varepsilon_x^2 = \frac{{}^t l^2 - {}^0 l^2}{2\,{}^0 l^2} = \frac{1}{2}\left(\xi^2 - 1\right)$$

ENGINEERING STRAIN $\qquad \varepsilon_x = \dfrac{\Delta l}{{}^0 l}$

STRETCH $\qquad \xi = \dfrac{{}^t l}{{}^0 l}$

DIFFERENT STRAIN MEASURES FOR A 'THIN' BAR



STRETCH

$$\xi = {}^{t}l \, / \, {}^{0}l$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) = e_G$$

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) = e_A$$

11 components only!

cm0155b

For 1D continuum

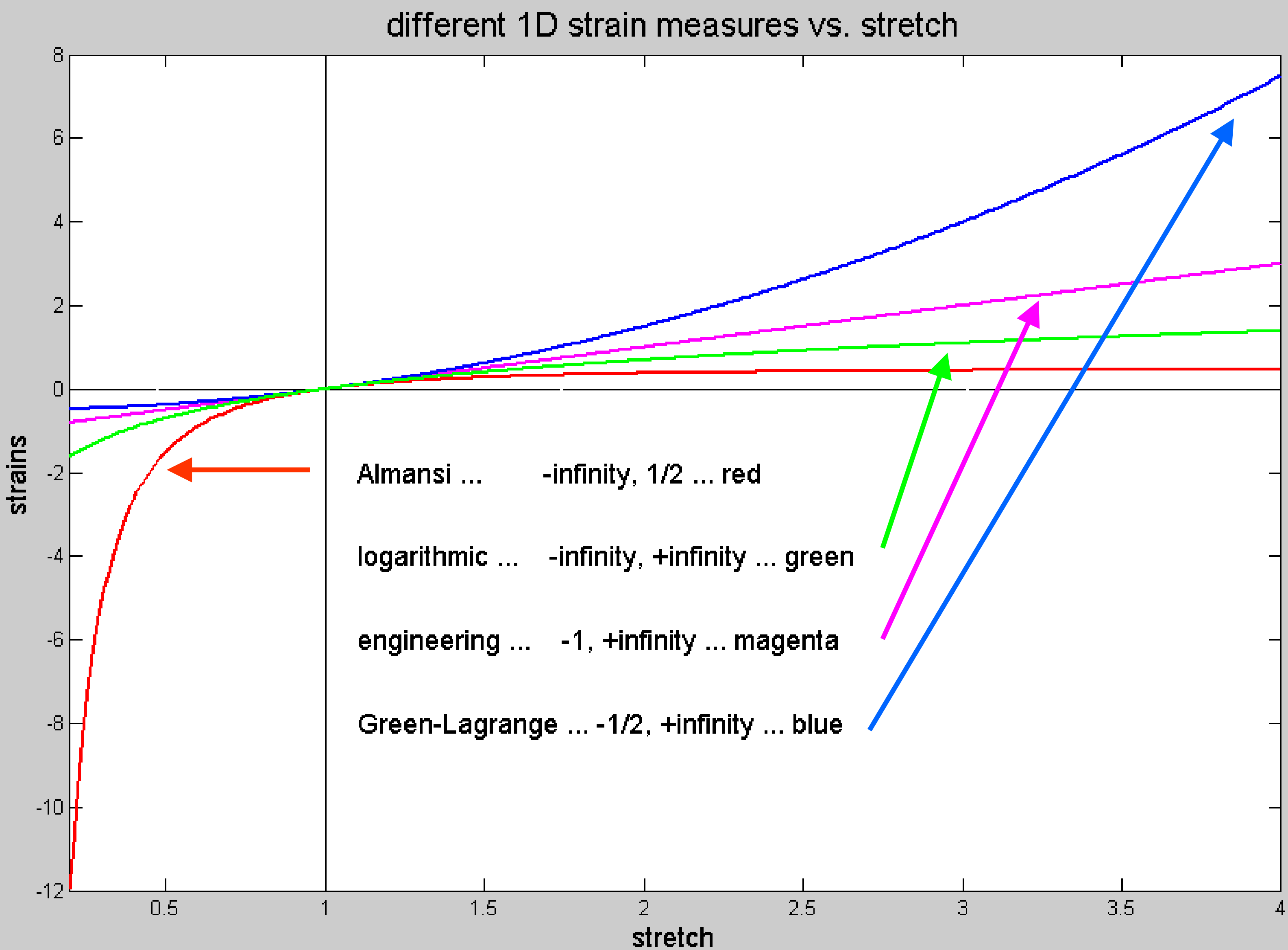
$$e_G = \frac{1}{2}\,\frac{{}^t\ell^2 - {}^0\ell^2}{{}^0\ell^2}$$ Green-Lagrange

$$e_A = \frac{1}{2}\,\frac{{}^t\ell^2 - {}^0\ell^2}{{}^t\ell^2}$$ Almansi

$$e_E = \frac{{}^t\ell - {}^0\ell}{{}^0\ell}$$ Engineering

$$e_L = \lg\left(\frac{{}^t\ell}{{}^0\ell}\right)$$ Logarithmic

The same physical phenomenon – different strain measures?



THE TRUE (CAUCHY) STRESS

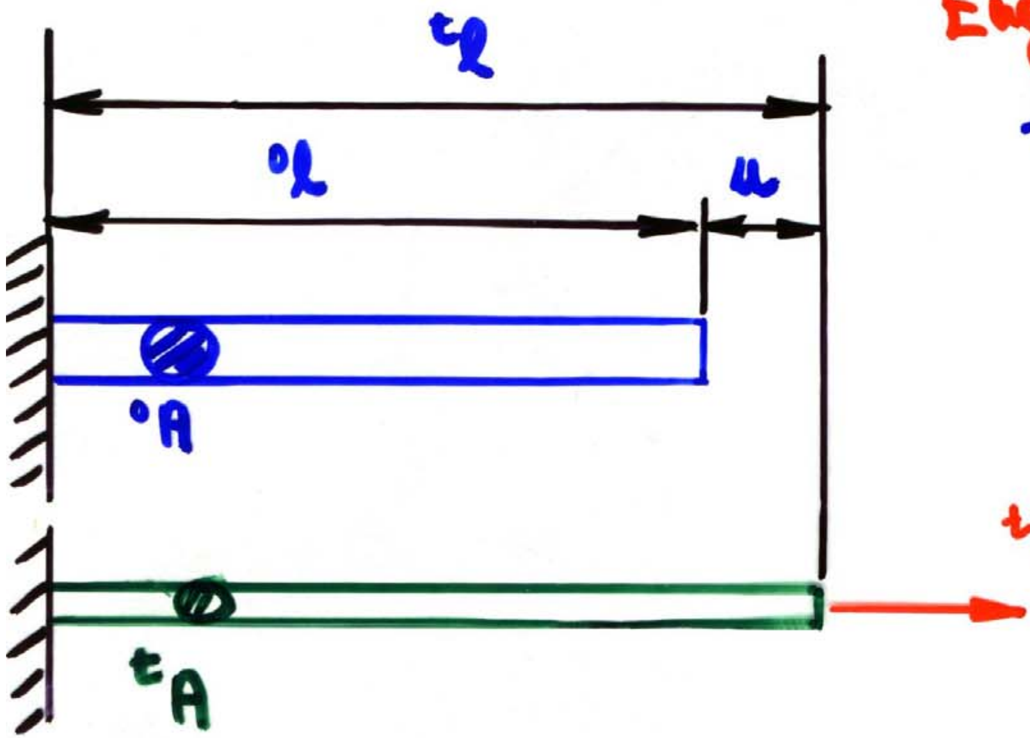
$$[\sigma] = \begin{bmatrix} {}^tP/{}^tA & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

THE SECOND PIOLA KIRCHHOFF STRESS

$$[S] = J\,[F]^{-1}[\sigma][F]^{-T} =$$

$$= \frac{{}^t\ell}{{}^0\ell}\,\frac{{}^tA}{{}^0A} \begin{bmatrix} {}^0\ell/{}^t\ell & & \\ & 1/(1+\varepsilon_r) & \\ & & 1/(1+\varepsilon_r) \end{bmatrix} \begin{bmatrix} {}^tP/{}^tA & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} {}^0\ell/{}^t\ell & & \\ & 1/(1+\varepsilon_r) & \\ & & 1/(1+\varepsilon_r) \end{bmatrix}$$

$$= \frac{P^t}{{}^0A}\,\frac{{}^0\ell}{{}^t\ell} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Engineering strain

$$P_E = \sigma_E \, {}^0A$$

actual force responsible for the change of configuration

PRINCIPLE OF VIRTUAL WORK

Green-Lagrange strain

$$\int_{{}^0V} \sigma_G \, \delta\varepsilon_G \, d\,{}^0V = P_G \, \delta\, {}^t\ell \qquad \varepsilon_G = \frac{{}^t\ell^2 - {}^0\ell^2}{2\,{}^0\ell^2}$$

$$P_G = \sigma_G \frac{{}^t\ell}{{}^0\ell} \, {}^0A \qquad \Rightarrow \delta\varepsilon_G$$

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Logarithmic strain

$$\int_{{}^tV} \sigma_L \, \delta\varepsilon_L \, d\,{}^tV = P_L \, \delta\,{}^t\ell$$

$$\varepsilon_L = \lg\left(\frac{{}^t\ell}{{}^0\ell}\right)$$

$$\Rightarrow \delta\varepsilon_L$$

$$P_L = \sigma_L \, {}^tA$$

$$\Rightarrow \sigma_L = \sigma \ldots \quad \text{true stress} = {}^tP / {}^tA$$

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but ${}^{t}A = {}^{0}A \left(\frac{{}^{0}\ell}{{}^{t}\ell} \right)^{2\nu}$ and if $\nu = 0.5$

then $P_L = \sigma_L \, {}^{0}A \, {}^{0}\ell \, / \, {}^{t}\ell$.

All the forces are identical (it is the same physical phenomenon), so

$${}^{t}P = P_E = P_\sigma = P_L$$

and from it follows that

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$$\sigma = \frac{{}^{0}A}{{}^{t}A} \frac{{}^{t}l}{{}^{0}l} \sigma_c$$

which is an equivalent of

$$[\sigma] = \frac{1}{J} [F][S][F]^T$$

shown before.

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In discussing the stress and stress rate at large deformation we will examine three different measures

1) the symmetric second Piola-Kirchhoff stress tensor $[S]$

2) the "true" Cauchy stress $[\sigma]$

3) and unrotated Cauchy stress $[\tilde{\sigma}]$

the "true" Cauchy stress $[\sigma]$ is related to the second P.-K. stress $[S]$ by the relation which was already shown

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$$[\sigma] = \frac{1}{J}[F][S][F]^T \quad \text{where} \quad J = \det[F] = \frac{\rho_0}{\rho}$$

The unrotated Cauchy stress $[\tilde{\sigma}]$ can be expressed by

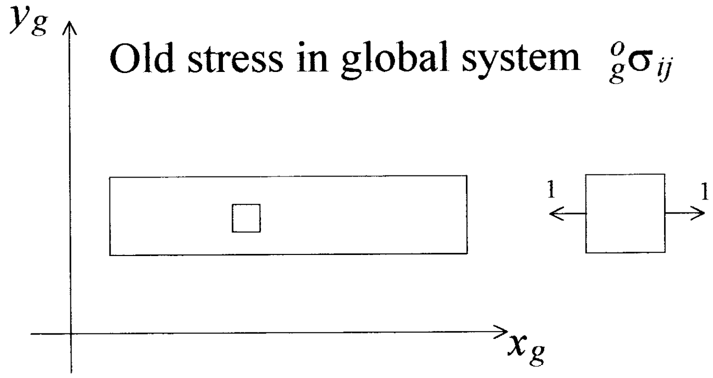
$$[\tilde{\sigma}] = [R]^T[\sigma][R] \quad \text{where } [R] \text{ comes from } [F] \to [R][U]$$

polar decomposition

One should note that unrotated Cauchy stress $[\tilde{\sigma}]$ is the true stress associated with stretching $[U]$ alone. Since $[R]$ is a proper orthogonal tensor the principal invariants of $[\sigma]$ and $[\tilde{\sigma}]$ are identical.

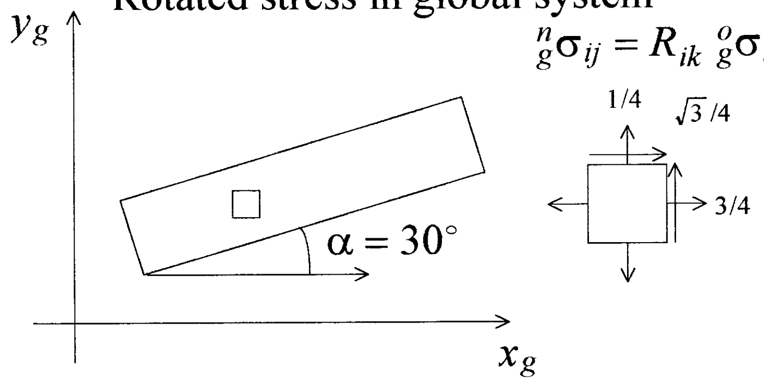
Understanding corotational stress

Old stress in global system ${}^o_g\sigma_{ij}$



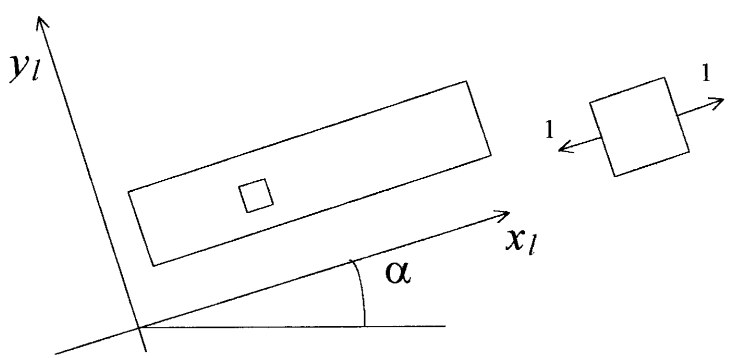
Rotated stress in global system

$${}^n_g\sigma_{ij} = R_{ik}\,{}^o_g\sigma_{km} R_{jm}$$



Unrotated (corotational) stress
in a local system rotating with the body

$${}^n_l\sigma_{ij} = {}^o_g\sigma_{ij} = R_{ki}\,{}^n_g\sigma_{km} R_{mj}$$



Green-Naghdi stress rate tensor

transformation $\tau = R^T \sigma R$ — R is from $F \rightarrow RU$ (polar decomposition)

$\uparrow$ Cauchy (σ)

R is orthogonal

$\uparrow$ unrotated Cauchy (τ)

$$R^T R = I$$

$$R^T = R^{-1}$$

rates:

$$\dot{\tau} = \dot{R}^T \sigma R + R^T \dot{\sigma} R + R^T \sigma \dot{R}$$

It holds : $\Omega = \dot{R} R^T \Rightarrow \dot{R} = \Omega R$

- rate of rotation tensor (rate of rigid body rotation of a material particle)

$$\Downarrow$$

$$\dot{R}^T = R^T \Omega^T$$

From it follows

$$\dot{\tau} = R^T \Omega^T \sigma R + R^T \dot{\sigma} R + R^T \sigma \Omega R \qquad / R^T \text{ from right}$$

$$\dot{\tau} R^T = R^T \Omega^T \sigma + R^T \dot{\sigma} + R^T \sigma \Omega \qquad / R \text{ from left}$$

$$R \dot{\tau} R^T = \Omega^T \sigma + \dot{\sigma} + \sigma \Omega$$

$$\Omega \text{ je antisymmetric} \Rightarrow \Omega^T = -\Omega$$

$$\boxed{\left[\overset{o}{\sigma}\right]^{GN} = R \dot{\tau} R^T = \dot{\sigma} - \Omega \sigma + \sigma \Omega}$$

This is Green - Naghdi .If Ω is used instead of W then we have so called Jaumann stress rate spin tensor W is antisymmetric part of L (velocity gradient) and represents rate of rotation of principal axis of D (rate of deformation)

$$\left[\overset{o}{\sigma}\right]^{Ja} = \dot{\sigma} - W \sigma + \sigma W$$

$$W = \frac{1}{2}(L - L^T) \; ; \quad L_{ij} = \frac{\partial v_i}{\partial^t x_j}$$