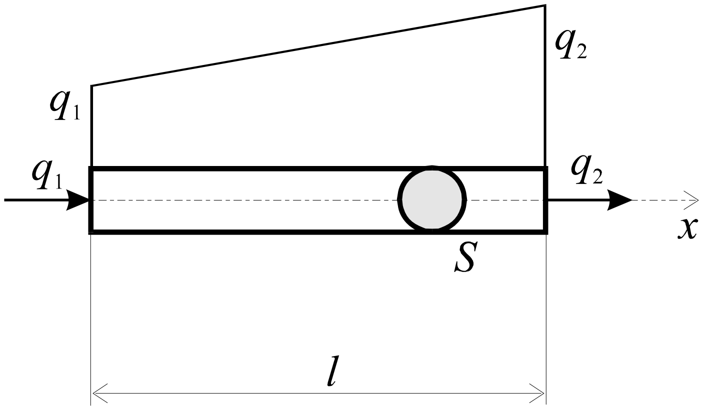
1D element for large strains and large deformations

Linear case

Non-linear case

Bar element, small strains, small displacements, linear material



Approximation of displacements $\{u\} = [A]\{q\}$ has the form

$$u_{\text{approx}} = u = c_1 + c_2 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = [U]\{c\}$$

and must be valid at nodes as well $u|_{x=0} = q_1$ a $u|_{x=l} = q_2$.

Substituting we get $\{q\} = [S]\{c\}$,

where $$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}, \quad [S] = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}, \quad \{c\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}.$$

If the length of element is greater than zero, then $\{c\} = [S]^{-1}\{q\}$, kde $[S]^{-1} = \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix}$.

So the approximation of displacements is $\{u\} = [U]\{c\} = [U][S]^{-1}\{q\} = [A]\{q\}$.

where $$[A] = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/l & 1/l \end{bmatrix} = \begin{bmatrix} 1 - x/l & x/l \end{bmatrix} = \begin{bmatrix} a_1(x) & a_2(x) \end{bmatrix}.$$

Approximation of strains $\{\varepsilon\} = [B]\{q\}$

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left([A]\{q\}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[1 - x/l \quad x/l\right]\{q\} = \left[-1/l \quad 1/l\right]\{q\},$$

where $$[B] = \left[-1/l \quad 1/l\right].$$

The mass and stiffness matrices are

$$[m] = \rho\int_V [A]^{\mathrm{T}}[A]\mathrm{d}V = \rho S\int_0^l [A]^{\mathrm{T}}[A]\mathrm{d}l = \frac{\rho S l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$[k] = \int_V [B]^{\mathrm{T}}[C][B]\mathrm{d}V = \int_0^l \begin{bmatrix} -1/l \\ 1/l \end{bmatrix} E\left[-1/l \quad 1/l\right] S\,\mathrm{d}x = \frac{ES}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$[C] = E$ – the Young's modulus.

Bar element, large strains, large displacements, non-linear material

Displacements in reference configuration $\{{}^{t}u\} = [{}_{0}A]\{q\}$.

Shape functions $$[A] = \left[1 - {}^{0}x/{}^{0}l \quad {}^{0}x/{}^{0}l\right] = \left[a_1({}^{0}x) \quad a_2({}^{0}x)\right].$$

Derivatives of shape functions $$\left[{}_{0}A,_{{}^{0}x}\right] = \left[-1/{}^{0}l \quad 1/{}^{0}l\right] = \{r\}^{\mathrm{T}},$$

Material displacement gradient $$Z = {}_{0}^{t}Z_{11} = \frac{\partial\, {}^{t}u_1}{\partial\, {}^{0}x_1} = \left[-1/{}^{0}l \quad 1/{}^{0}l\right]\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{r\}^{\mathrm{T}}\{q\},$$

Its increment $$\Delta Z = {}_{0}\Delta Z_{11} = \{r\}^{\mathrm{T}}\{\Delta q\}.$$

Green Lagrange strain tensor, its linear and non-linear parts

$$\Delta E = \tfrac{1}{2}(\Delta Z + \Delta Z^{\mathrm{T}}) + \tfrac{1}{2}(\Delta Z^{\mathrm{T}} Z + Z^{\mathrm{T}} \Delta Z) + \tfrac{1}{2}\Delta Z^{\mathrm{T}} \Delta Z = \Delta E^{\mathrm{L1}} + \Delta E^{\mathrm{L2}} + \Delta E^{\mathrm{N}} .$$

The first linear part

$$\Delta E^{\mathrm{L1}} = \tfrac{1}{2}\left(\{r\}^{\mathrm{T}}\{\Delta q\} + \{\Delta q\}^{\mathrm{T}}\{r\}\right) = \{r\}^{\mathrm{T}}\{\Delta q\} .$$

$$[{}_0B^{\mathrm{L1}}] = \{r\}^{\mathrm{T}} = \frac{1}{{}^0l}[\,-1 \quad 1\,].$$

The second linear part

$$\Delta E^{\mathrm{L2}} = \tfrac{1}{2}\left(\{\Delta q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{q\} + \{q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{\Delta q\}\right) = \{q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{\Delta q\} .$$

$$[{}_0B^{\mathrm{L2}}] = \{q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}} = \{q_1 \quad q_2\}\begin{Bmatrix} -1/{}^0l \\ 1/{}^0l \end{Bmatrix}\{-1/{}^0l \quad 1/{}^0l\} = \frac{1}{{}^0l^2}\left[q_1 - q_2 \quad -\left(q_1 - q_2\right)\right].$$

The non-linear part and its increment

$$\Delta E^{\mathrm{N}} = \tfrac{1}{2}\Delta Z^{\mathrm{T}}\Delta Z = \tfrac{1}{2}\{\Delta q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{\Delta q\},$$

$$\delta\Delta E^{\mathrm{N}} = \{\delta\Delta q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{\Delta q\}.$$

Recall

$${}_0^tS_{ij}\,\delta\mathit{\Delta}E_{ij}^{\mathrm{N}} = \{\delta\mathit{\Delta}\tilde{E}^{\mathrm{N}}\}^{\mathrm{T}}[{}_0^t\tilde{S}]\{\mathit{\Delta}\tilde{E}^{\mathrm{N}}\},$$

$${}_0^tS_{ij}\,\delta\mathit{\Delta}E_{ij}^{\mathrm{N}} = {}_0^tS\{\delta\mathit{\Delta}q\}^{\mathrm{T}}\{r\}\{r\}^{\mathrm{T}}\{\mathit{\Delta}q\}.$$

Comparing the above relations we get

$$\{\Delta\tilde{E}^{\mathrm{N}}\} = [B^{\mathrm{N}}]\{\Delta q\},$$

So

$$\{\delta\Delta\tilde{E}^{\mathrm{N}}\}^{\mathrm{T}}[{}_0^t\tilde{S}]\{\Delta\tilde{E}^{\mathrm{N}}\} = \{\delta\Delta q\}^{\mathrm{T}}[B^{\mathrm{N}}]^{\mathrm{T}}\,{}_0^tS[B^{\mathrm{N}}]\{\delta\Delta q\}$$

where

$$[B^{\mathrm{N}}] = \{r\}^{\mathrm{T}}.$$

The linear and non-linear incremental stiffness matrices and the vector of internal forces are

$$[k^{\mathrm{L}}] = \frac{{}^{0}A\,{}_{0}E}{{}^{0}l^{3}}\left({}^{0}l^{2} + 2q_{21}\,{}^{0}l + q_{21}^{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \text{where } q_{21} = q_2 - q_1 .$$

$$[k^{\mathrm{N}}] = \frac{{}^{0}A\,{}_{0}^{t}S}{{}^{t}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{{}^{t}P}{{}^{t}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \text{where } {}_{0}^{t}S = \frac{{}^{t}P\,{}^{0}l}{{}^{0}A\,{}^{t}l} .$$

$$\{F\} = \frac{{}_{0}^{t}S\,{}^{0}A\,{}^{t}l}{{}^{0}l}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = {}^{t}P\begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

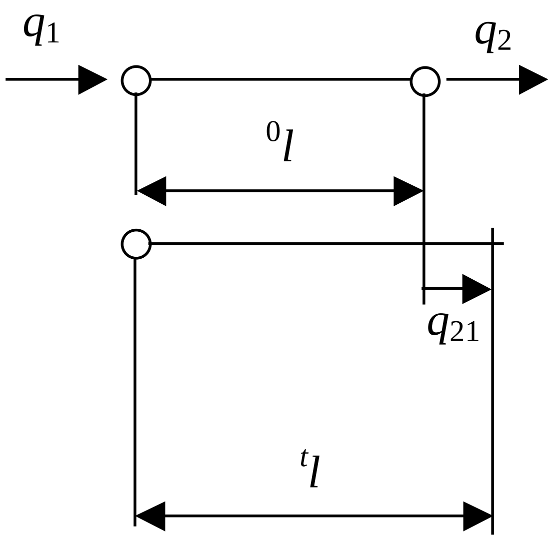
where ${}^{t}P$ is axial force in ${}^{t}C$

Summary

$$k^{\mathrm{L}} = \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\left({}^{0}l^{2} + 2q_{21}\,{}^{0}l + q_{21}^{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} =$$

$$= \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\left({}^{0}l + q_{21}\right)^{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{{}^{0}A\,{}_{0}C}{{}^{0}l^{3}}\,{}^{t}l^{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{{}^{0}A\,{}_{0}C\xi^{2}}{{}^{0}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

where



$$q_{21} = q_2 - q_1$$

$${}^{0}l + q_{21} = {}^{t}l$$

$$\xi = {}^{t}l \,/\, {}^{0}l$$

$$k^{\mathrm{N}} = \frac{{}^{0}A\,{}_{0}^{t}S}{{}^{0}l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Assuming 1D stress

$$\sigma_{11} = \frac{{}^{t}\rho}{{}^{0}\rho} F_{11} \; {}_{0}^{t}S_{11} \; F_{11}^{T}$$

or simply with scalar quantities

$${}_{t}^{t}\sigma = {}^{t}\rho \,/\, {}^{0}\rho \,(\, F \; {}_{0}^{t}S \; F^{T} \,)$$

Uniform_deformation

$${}^{t}x = \frac{{}^{t}l}{{}^{0}l} \; {}^{0}x = \xi \; {}^{0}x$$

$$\Rightarrow F_{11} = F = \frac{\partial \, {}^{t}x}{\partial \, {}^{0}x} = \xi$$

Mass_conservation

$${}^{0}\rho \; {}^{0}l \; {}^{0}A = {}^{t}\rho \; {}^{t}l \; {}^{t}A$$

$$\frac{{}^{t}\rho}{{}^{0}\rho} = \frac{{}^{0}l}{{}^{t}l} \; \frac{{}^{0}A}{{}^{t}A} = \frac{1}{\xi} \; \frac{{}^{0}A}{{}^{t}A}$$

Thus the true stress vs. 2PK stress can be written in the form

$${}^{t}_{t}\sigma = \frac{1}{\xi}\frac{{}^{0}A}{{}^{t}A}\xi \; {}^{t}_{0}S \; \xi = \; {}^{t}_{0}S \; \frac{{}^{0}A}{{}^{t}A}\xi$$

Realizing that true stress is

$${}^{t}_{t}\sigma = \frac{{}^{t}P}{{}^{t}A}$$

and combining the last two equations we get

$$\frac{{}^{t}P}{{}^{t}A} = \; {}^{t}_{0}S \; \frac{{}^{0}A}{{}^{t}A}\xi \; ; \qquad {}^{t}P = {}^{t}_{0}S \; {}^{0}A \, \xi$$

The relation between ${}^{0}A$ and ${}^{t}A$ cannot be obtained from 1D considerations.
An assumption of type of deformation must be taken into account.
Assuming for example the isovolumetric deformation, ie. ${}^{0}V = {}^{t}V$ (typical for rubber) we get

$${}^{0}A\ {}^{0}l = {}^{t}A\ {}^{t}l\,; \qquad \frac{{}^{0}A}{{}^{t}A} = \frac{{}^{t}l}{{}^{0}l} = \xi$$

Together with above equations it gives

$${}^{t}_{t}\sigma = {}^{t}_{0}S\ \xi^{2}$$

where we have used $\left({}^{t}_{0}S = {}^{t}P /\ {}^{0}A\xi\right)$.

So $[k^{\mathrm{N}}]$ could be rewritten into

$$[k^{\mathrm{N}}]=\frac{{}^{0}A\ {}_{0}^{t}S}{{}^{0}l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{{}^{0}A}{{}^{0}l}\frac{{}^{t}P}{{}^{0}A\,\xi}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=\frac{{}^{t}P}{{}^{0}l\ \xi}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=$$

$$=\frac{{}^{t}P}{{}^{t}l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\text{with }\frac{{}^{t}P}{{}^{0}l}=\xi\ \text{ and }\ {}^{t}l={}^{0}l\ \xi\ .$$

And similarly

$$\{F\}=\frac{{}_{0}^{t}S\ \ {}^{0}A\ \ {}^{t}l}{{}^{0}l}\begin{Bmatrix}-1\\ 1\end{Bmatrix}={}^{t}P\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

Assume that the properties of the material were experimentally tested in tension and compression and that a polynomial fit was performed over experimental data

$$ {}^{t}P = c_1\, \xi^3 + c_2\, \xi^2 + c_3\, \xi + c_4 $$

(see c:\prog_c\prog\mtlnelin\gumafit.sam)

$$ c = 1e\,3\left[0.2510 \quad -1.1876 \quad 1.9991 \quad -1.0578\right] $$

We have already shown that

$$ {}^{t}P = {}_{0}^{t}S \; {}^{0}A\,\xi \quad \Rightarrow \quad {}_{0}^{t}S = \frac{{}^{t}P}{{}^{0}A\,\xi} $$

also $$ E = \frac{1}{2}\,\frac{{}^{t}l^{2} - {}^{0}l^{2}}{{}^{0}l^{2}} = \frac{1}{2}\left(\xi^{2} - 1\right) \quad \Rightarrow \quad \xi = \sqrt{2E+1} $$

$$ \frac{\mathrm{d}\xi}{\partial E} = \frac{1}{2}\,\frac{1}{\sqrt{2E+1}} \cdot 2 $$

So

$$ {}_{0}^{t}S = \frac{{}^{t}P}{{}^{0}A\,\xi} = \left(c_1\ \xi^3 + c_2\ \xi^2 + c_3\ \xi + c_4\right) = \ \frac{1}{{}^{0}A}\left(c_1\ \xi^2 + c_2\ \xi\ \ + c_3\ \ + c_4\ \xi^{-1}\right) $$

$$ \frac{\mathrm{d}\ {}_{0}^{\mathrm{t}}S}{\mathrm{d}\xi} = \frac{1}{{}^{0}A}\left(2c_1\ \xi + c_2 + 0\ \ - c_4\ \xi^{-2}\right) $$

$$ \mathrm{d}\ {}_{0}^{\mathrm{t}}S = \frac{1}{{}^{0}A}\left(2c_1\ \xi + c_2\ \ - c_4\ \xi^{-2}\right)\mathrm{d}\ \xi\ ; \quad \mathrm{d}\ \xi = \frac{1}{\sqrt{2E+1}}\,\mathrm{d}E $$

$$ \mathrm{d}\ {}_{0}^{\mathrm{t}}S = \frac{1}{{}^{0}A}\left(2c_1\ \xi + c_2\ \ - c_4\ \xi^{-2}\right)\frac{1}{\sqrt{2E+1}}\,\mathrm{d}E\ \ \underbrace{= \frac{1}{{}^{0}A}\left(2c_1\ \xi + c_2\ \xi^{-1}\ \ - c_4\ \xi^{-3}\right)}_{{}_{c}C}\,\mathrm{d}E $$

So the constitutive „constant“ appearing in

$$\mathrm{d}\ {}_{0}^{t}S = {}_{0}C\ \mathrm{d}E$$

is $${}_{0}C = \frac{1}{{}^{0}A}\left(2c_1 + c_2\ \xi^{-1} - c_4\ \xi^{-3}\right)$$

As the first step it is sufficient to show the behaviour of a single element

$$\left(\frac{{}^{0}A\ {}_{0}C\ \xi^{2}}{{}^{0}l}+\frac{{}^{0}A\ {}_{0}^{t}S}{{}^{0}l}\right)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\Delta q_1\\ \Delta q_2\end{Bmatrix}=\begin{Bmatrix}R_1\\ R_2\end{Bmatrix}-\begin{Bmatrix}F_1\\ F_2\end{Bmatrix}$$

$$\{F\}=\frac{{}_{0}^{t}S\ \ {}^{0}A\ \ {}^{t}l}{{}^{0}l}\begin{Bmatrix}-1\\ 1\end{Bmatrix}={}_{0}^{t}S\ \ {}^{0}A\ \xi\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

Let’s fix the second DOF, then we have

$$\left(\frac{{}^{0}A\ {}_{0}C\ \xi^{2}}{{}^{0}l}+\frac{{}^{0}A\ {}_{0}^{t}S}{{}^{0}l}\right)\Delta q=R-{}_{0}^{t}S\ \ {}^{0}A\ \xi$$

where $c = 1e\ 3 * [0.2510 \quad -1.1876 \quad 1.9991 \quad -1.0578]$

$l0 = 1;$

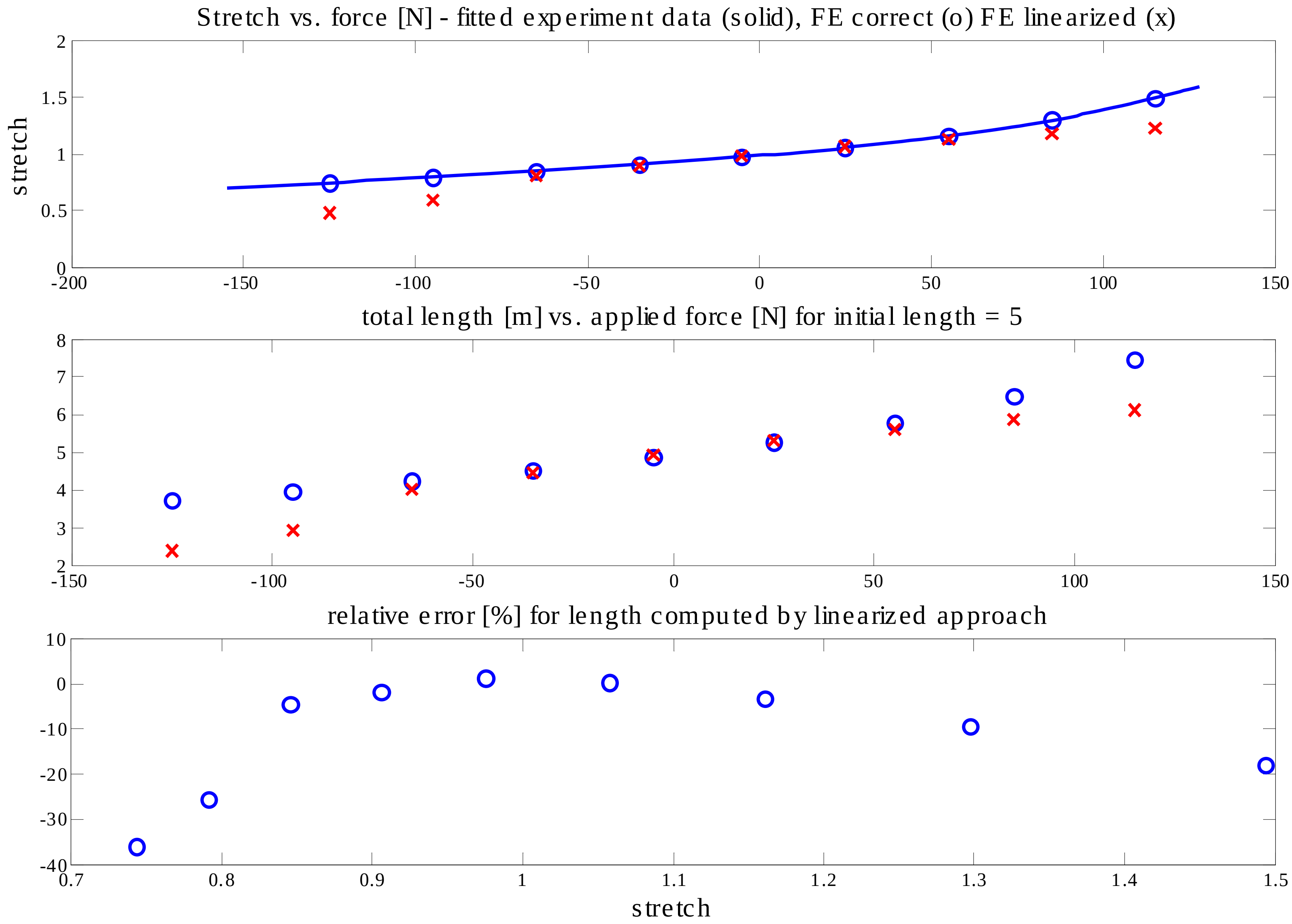
d0 = 0,0115

$a0$ = pi * d0 ^ 2 / 4;

$$\mathrm{d}_0^{\mathrm{t}} S = {}_0C \ \mathrm{d}_0 E$$

$${}_0C = \frac{1}{{}^0A}\left(2c_1 + c_2\ \xi^{-1} - c_4\ \xi^{-3}\right)$$

see telemachos c:\prog_all_backup\prog\mtlnelin\ttl_1el4.m



Stretch vs. force [N] - fitted experiment data (solid), FE correct (o) FE linearized (x)
stretch
2
1.5
1
0.5
0
-200
-150
-100
-50
0
50
100
150
total length [m] vs. applied force [N] for initial length = 5
8
7
6
5
4
3
2
-150
-100
-50
0
50
100
150
relative error [%] for length computed by linearized approach
10
0
-10
-20
-30
-40
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
stretch