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a Young Generation Viewpoint

MILLMAN'S FORMULA IN DATA FUSION

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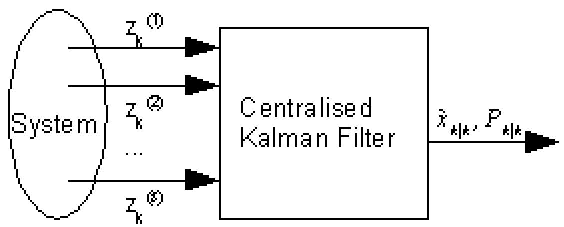
Introduction:

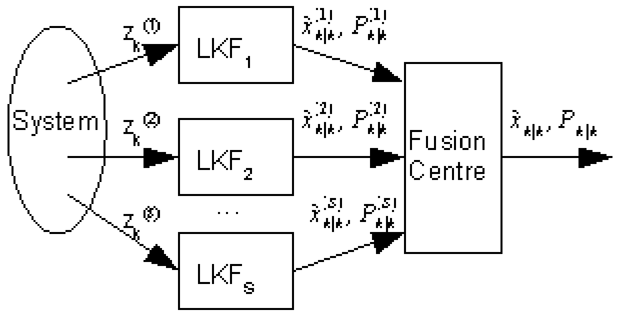
Data fusion deals with combining data from multiple sources. The paper focuses on state estimation problems. The aim is to interpret current estimations problems as data fusion ones.

Several data fusion examples will be shown. Numerical example that compares several fusion techniques is also included.

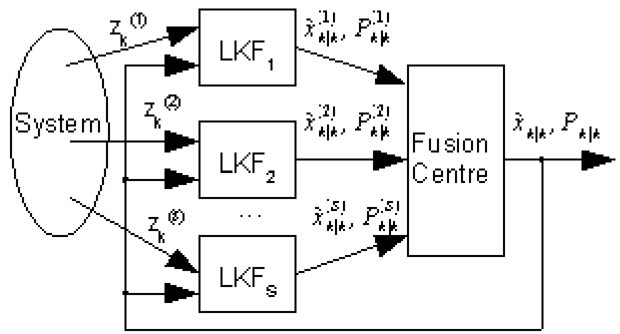
a) centralised fusion



b) multisensor fusion



c) hierarchical fusion



System description:

Let the linear discrete-time stochastic system be described by

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}\,\boldsymbol{x}_k + \boldsymbol{G}\,\boldsymbol{w}_k\,,$$
$$\boldsymbol{z}_k^{(j)} = \boldsymbol{H}^{(j)}\,\boldsymbol{x}_k + \boldsymbol{v}_k^{(j)}\,,$$

where $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$ is the immeasurable system state, $\boldsymbol{z}_k^{(j)} \in \mathbb{R}^{n_z}$ is j-th sensor measurement, $j = 1, ..., S$ is the sensor number, $k = 0, 1, ..., t$ is a time index, $\{\boldsymbol{w}_k\}$, $\{\boldsymbol{v}_k^{(j)}\}$ are zero mean white Gaussian processes with covariances $\boldsymbol{Q}$, $\boldsymbol{R}^{(j)}$, $\mathrm{E}(\boldsymbol{w}_k\, \boldsymbol{v}_l^{(j)T}) = 0$ for all k, l, j, and they are independent of the system initial state described by the Gaussian pdf $N(\bar{\boldsymbol{x}}_0, \boldsymbol{P}_0)$. The matrices $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are known.4

Generalised Millman's formula:

Linear data fusion is considered

$$\hat{\boldsymbol{x}} = \sum_{i=1}^{N} \boldsymbol{c}_i \hat{\boldsymbol{x}}_i, \quad \sum_{i=1}^{N} \boldsymbol{c}_i = \boldsymbol{I}_n,$$

where $\hat{\boldsymbol{x}}_i$, $i = 1, 2, \ldots, N$, are the estimates to be fused and their covariance matrices are

$$\boldsymbol{P}_{ij} = cov(\boldsymbol{x} - \hat{\boldsymbol{x}}_i, \boldsymbol{x} - \hat{\boldsymbol{x}}_j), \quad i, j = 1, 2, \ldots, N.$$

Applying the MSE criterion

$$J(\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_N) = E\left(\left\| \boldsymbol{x} - \sum_{i=1}^{N} \boldsymbol{c}_i \hat{\boldsymbol{x}}_i \right\|^2\right),$$

the optimal coefficients $\boldsymbol{c}_i$, $i = 1, 2, \ldots, N$, are obtained by the generalized Millman's formula (GMF).

Generalised Millman's formula cont'd:

The coefficients $\boldsymbol{c}_i$ of the GMF are given by equations

$$\sum_{i=1}^{N-1} \boldsymbol{c}_i (\boldsymbol{P}_{ij} - \boldsymbol{P}_{iN}) + \boldsymbol{c}_N (\boldsymbol{P}_{Nj} - \boldsymbol{P}_{NN}) = 0,$$

$$j = 1, 2, \ldots, N-1,$$

$$\sum_{i=1}^{N} \boldsymbol{c}_i = \boldsymbol{I}_N .$$

The Millman's formula (a special case of the GMF for two independent estimates) can be expressed in a closed form

$$\hat{\boldsymbol{x}} = \boldsymbol{P} (\boldsymbol{P}_{11}^{-1} \hat{\boldsymbol{x}}_1 + \boldsymbol{P}_{22}^{-1} \hat{\boldsymbol{x}}_2),$$

$$\boldsymbol{P} = \left(\boldsymbol{P}_{11}^{-1} + \boldsymbol{P}_{22}^{-1} \right)^{-1}.$$

MF in filtering and smoothing:

filtering:

Analysing the Kalman filter update equation, the filtering estimate can be understood as a fusion of two independent estimates, the prior information estimate $\hat{\boldsymbol{x}}_{k|k-1} = E(\boldsymbol{x}_k | \boldsymbol{z}^{(j)k-1}_{0}) = E(\boldsymbol{x}_k | \boldsymbol{z}_0^{(j)}, \ldots, \boldsymbol{z}_{k-1}^{(j)})$ and the maximum likelihood estimate $\hat{\boldsymbol{x}}_{ML} = \arg\max_{\boldsymbol{x}_k} p(\boldsymbol{z}^{(j)}{}_k | \boldsymbol{x}_k)$, which is given by $\hat{\boldsymbol{x}}_{ML} = (\boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{z}_k^{(j)}$ for the Gaussian measurement error,

$$\boldsymbol{P}_{k|k}^{-1} \hat{\boldsymbol{x}}_{k|k} = \boldsymbol{P}_{k|k-1}^{-1} \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{P}_{ML}^{-1} \hat{\boldsymbol{x}}_{ML},$$
$$\boldsymbol{P}_{k|k}^{-1} = \boldsymbol{P}_{k|k-1}^{-1} + \boldsymbol{P}_{ML}^{-1},$$

where $\boldsymbol{P}_{k|k-1}, \boldsymbol{P}_{ML}$ are corresponding covariance matrices.

smoothing:

The Millman's formula is used to combine forward $\hat{\boldsymbol{x}}_{k|k}=E(\boldsymbol{x}_k|\boldsymbol{z}_0^k)$ and backward $\hat{\boldsymbol{x}}_{k|k+1}=E(\boldsymbol{x}_k|\boldsymbol{z}_{k+1}^t)$ state estimates in the smoothing problem. The fusion relations are

$$\boldsymbol{K}_k=\boldsymbol{P}_{k|k}\boldsymbol{S}_{k|k+1}\left(\boldsymbol{I}_n+\boldsymbol{P}_{k|k}\boldsymbol{S}_{k|k+1}\right)^{-1},$$
$$\boldsymbol{P}_k=(\boldsymbol{I}_n-\boldsymbol{K}_k)\boldsymbol{P}_{k|k},$$
$$\hat{\boldsymbol{x}}_k=(\boldsymbol{I}_n-\boldsymbol{K}_k)\hat{\boldsymbol{x}}_{k|k}+\boldsymbol{P}_k\hat{\boldsymbol{y}}_{k|k+1},$$

where

$$\hat{\boldsymbol{y}}_{k|k+1}=\boldsymbol{S}_{k|k+1}\hat{\boldsymbol{x}}_{k|k+1}, \quad \boldsymbol{S}_{k|k+1}=\boldsymbol{P}_{k|k+1}^{-1}$$

is the information form of the backward estimate $\hat{\boldsymbol{x}}_{k|k+1}$ and $\boldsymbol{P}_{k|k}=cov(\boldsymbol{x}_k|\boldsymbol{z}_0^k)$, $\boldsymbol{P}_{k|k+1}=cov(\boldsymbol{x}_k|\boldsymbol{z}_{k+1}^t)$ are covariance matrices that correspond to the estimates.

smoothing cont'd:

The smoothing relations can be expressed in an alternative form

$$\boldsymbol{P}_k = \left(\boldsymbol{P}_{k|k}^{-1} + \boldsymbol{P}_{k|k+1}^{-1}\right)^{-1},$$
$$\hat{\boldsymbol{x}}_k = \boldsymbol{P}_k \boldsymbol{P}_{k|k}^{-1} \hat{\boldsymbol{x}}_{k|k} + \boldsymbol{P}_k \boldsymbol{P}_{k|k+1}^{-1} \hat{\boldsymbol{x}}_{k|k+1}$$

which better shows the use of the Millman's formula.

$\Rightarrow$ The filtering and smoothing problems were interpreted as one-sensor data fusion problems.

GMF in state estimation:

multisensor fusion with fusion centre:

The GMF can be used to combine the local Kalman filter estimates $\hat{\boldsymbol{x}}_{k|k}^{(j)} = E\left(\boldsymbol{x}_k | \boldsymbol{z}_0^{(j)k}\right)$ at the fusion centre.

For independent local estimates, the fused estimate has a simple closed form

$$\hat{\boldsymbol{x}}_{k|k} = \boldsymbol{P}_{k|k} \sum_{j=1}^{S} \boldsymbol{P}_{k|k}^{(j)-1} \hat{\boldsymbol{x}}_{k|k}^{(j)},$$

$$\boldsymbol{P}_{k|k}^{-1} = \sum_{j=1}^{S} \boldsymbol{P}_{k|k}^{(j)-1},$$

where $\boldsymbol{P}_{k|k}^{(j)}$ are covariance matrices that correspond to the local estimates.

hierarchical fusion:

The fusion centre can send fused estimates to local filters. The measurement information is extracted from local filtering and predictive estimates. The fusion relations are

$$\boldsymbol{P}_{k|k}^{-1}\hat{\boldsymbol{x}}_{k|k}=\boldsymbol{P}_{k|k-1}^{-1}\hat{\boldsymbol{x}}_{k|k-1}+\sum_{j=1}^{S}\left\{\boldsymbol{P}_{k|k}^{(j)-1}\hat{\boldsymbol{x}}_{k|k}^{(j)}-\boldsymbol{P}_{k|k-1}^{(j)\;-1}\hat{\boldsymbol{x}}_{k|k-1}^{(j)}\right\},$$

$$\boldsymbol{P}_{k|k}^{-1}=\boldsymbol{P}_{k|k-1}^{-1}+\sum_{j=1}^{S}\left\{\boldsymbol{P}_{k|k}^{(j)-1}-\boldsymbol{P}_{k|k-1}^{(j)\;-1}\right\}.$$

The terms under the sums can be understood as a product of the reverse use of the update equation in the prior information and measurement data fusion problem and they are so-called equivalent measurements.

Numerical illustration:

Consider a 1-D tracking problem. The constant velocity model is used,

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \boldsymbol{w}_k,$$

where $\boldsymbol{x}_{1,k}$ means position on an object and $\boldsymbol{x}_{2,k}$ its velocity at time-step k and T_s is the sampling period. The state noise has a covariance matrix $\boldsymbol{Q}$.
The object position is measured by three sensors,

$$\boldsymbol{z}_k^{(j)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}_k + v_k^{(j)},$$

with the measurement variance is $R = 1$.

For time horizon of K=20 steps, M=10^4 Monte-Carlo runs were done. The results of several fusion alternatives were compared by mean square error and mean Mahalanobis distance.

The mean square error of the object position/ velocity at time-step k across the Monte-Carlo runs is defined by

$$MSE(x_{i,k}) = \frac{\sum_{m=1}^{M} \left(x_{i,k,m} - \hat{x}_{i,k|k,m}\right)^2}{M}$$

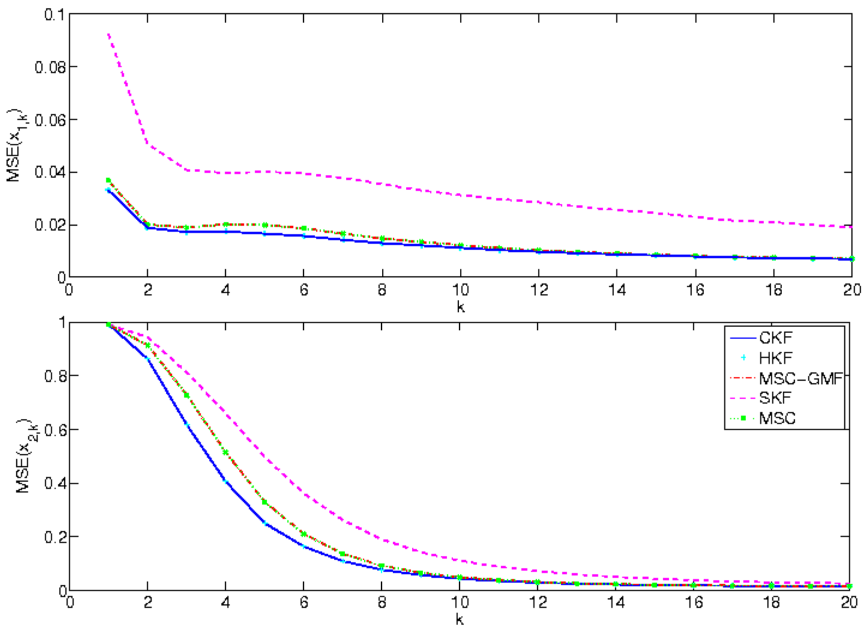
and the overall mean square error is

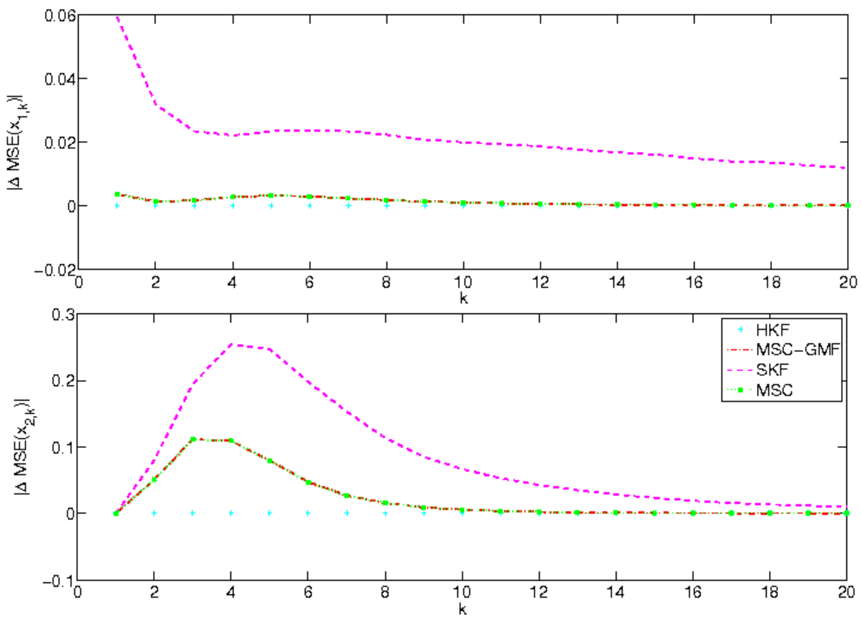
$$MSE = \frac{\sum_{i=1}^{2} \sum_{k=0}^{K} MSE(x_{i,k})}{2(K+1)}.$$

The next abbreviations are used:

CKF	centralised Kalman filter
HKF	hierarchical Kalman filter
MSC-GMF	multisensor fusion with the GMF
MSC	approximated multisensor fusion
SKF	single Kalman filter

The figures show time evolution of the *MSE*($x_{i,k}$) and its differences to the *MSE* of the optimal centralised Kalman filter.





estimator	CKF	HKF	MSC-GMF	MSC	SKF
MSE	0.1010	0.1010	0.1134	0.1134	0.1528

The Mahalanobis distance is defined by

$$D_M(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k}, \boldsymbol{P}_{k|k}) = \sqrt{(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k})^T \boldsymbol{P}_{k|k}^{-1} (\boldsymbol{x}_k - \hat{\boldsymbol{x}}_{k|k})}$$

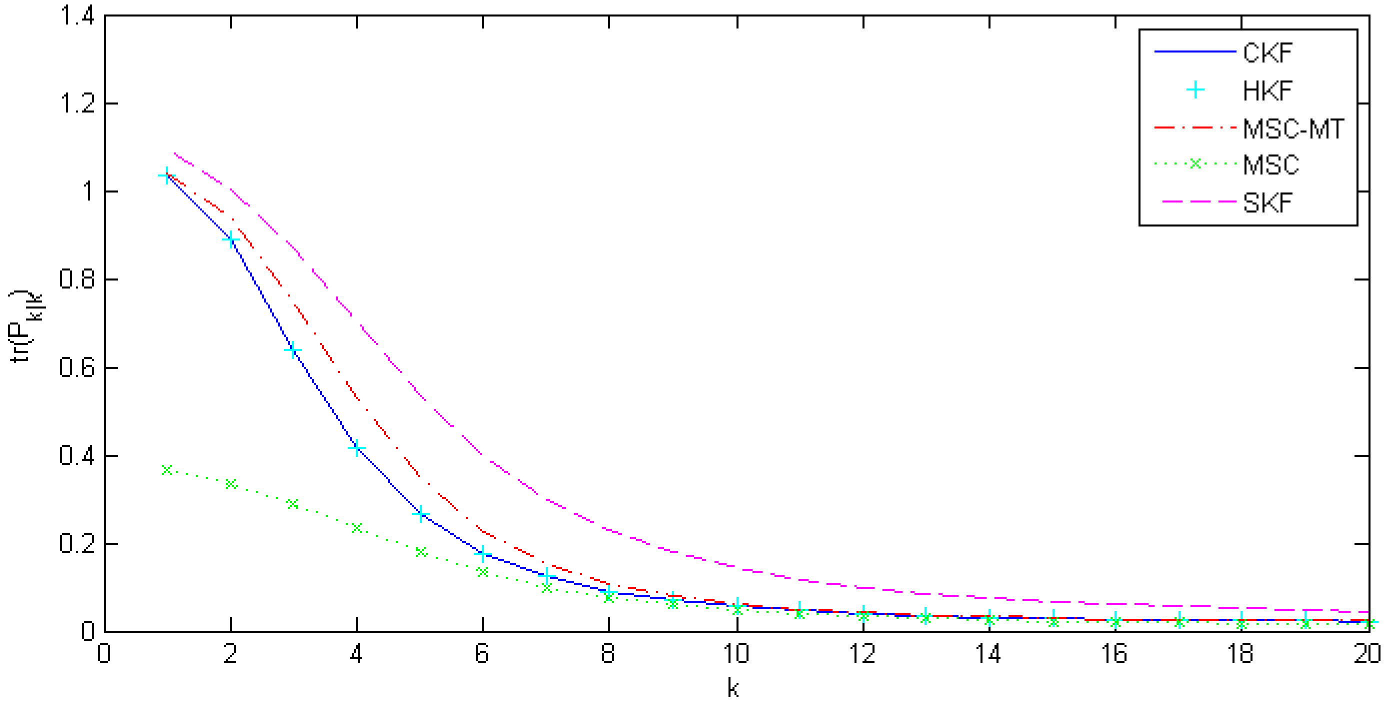
and the mean Mahalanobis distance is obtained by

$$D_M = \frac{\sum_{k=0}^{K} \sum_{m=1}^{M} D_M(\boldsymbol{x}_k, \hat{\boldsymbol{x}}_{k|k})}{(K+1)M}.$$

The table shows the results. It can be seen that the MSC estimator is over-confident.

estimator	CKF	HKF	MSC-GMF	MSC	SKF
D_M	1.2512	1.2512	1.2536	1.5204	1.2550

The over-confidence of the MSC estimator can also be seen by plotting the traces of the covariances of the estimates.



Conclusion:

The paper dealt with data fusion for linear stochastic dynamic systems.

Estimation problems (filtering, smoothing) were interpreted as data fusion problems.

Several fusion approaches were discussed and compared by means of a numerical example.

It was shown that the hierarchical fusion gives results which are equivalent to the centralised fusion, the application of the generalised Millman's formula gives suboptimal estimates only, neglecting the cross-correlations leads to over-confident estimates.

Thank you for the attention.