Stability Analysis of Mean-CVaR Investment Model with Transaction Costs and Integer Allocations

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Content

Outline

Mean-risk model with real features
Conditional Value at Risk, transaction costs, integer allocations

Optimal investment
Multiobjective integer stochastic programming

Stability of stochastic programs
Stress testing, contamination bounds

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Investing vs. trading

- **Investing** "buy and hold" - goal is to take long term ownership of an instrument with a high level of confidence that it will continually increase in value (*investment horizon*).
- **Trading** "sell high, buy low" - goal is to buy and sell to capitalize on short term relative changes in value of an instrument (*trading frequency*).

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Mean-risk models

Markowitz (1952): expected return $\mathcal{E}(x)$, risk $\mathcal{R}(x)$ (variance)
constraints $X \subset \mathbb{R}^n$ on portfolio composition

$$\max \mathcal{E}(x) \ \& \ \min \mathcal{R}(x) : x \in X.$$

Multiobjective optimization

We are looking for efficient solutions: $\hat{x} \in X$ such that there is no element $x \in X$ with $\mathcal{R}(x) \leq \mathcal{R}(\hat{x})$ and $\mathcal{E}(x) \geq \mathcal{E}(\hat{x})$ with at least one strict inequality. There are two main approaches for solving such problems, both leading to single objective problems and under mild condition to efficient solutions: **weighted sum approach**

$$\min_{x \in X} \Big[-(1-\rho)\mathcal{E}(x) + \rho\mathcal{R}(x) \Big]$$

for some $\rho \in (0,1)$, and ε-**constraint approach**

$$\min_{x \in X} \mathcal{R}(x)$$
$$\mathcal{E}(x) \geq r^{min}$$

with r^{min} such that $\{x \in X : \mathcal{E}(x) \geq r^{min}\}$ is nonempty.

Improvements

- Transaction costs
- Indivisible assets
- Nonsymmetrical quantitation of risk

$\longrightarrow$ Stochastic integer programming problem

- Dynamics (not included :-(

Content

Conditional Value at Risk

R.T. Rockafellar, S. Uryasev (2002):
If we denote Z a general **loss random variable** with distribution function F, then α VaR is defined as

$$VaR_\alpha = \min\{z : F(z) \geq \alpha\}$$

for some level $\alpha \in (0,1)$, usually 0.95 or 0.99.
CVaR is defined as mean of losses in the α-tail distribution

$$\begin{aligned} F_\alpha(z) &= \frac{F(z) - \alpha}{1 - \alpha}, \text{ if } z \geq VaR_\alpha \\ &= 0, \text{ otherwise.} \end{aligned}$$

For application of CVaR in optimization problems, the following **minimization formula** is of crucial importance.

$$CVaR_\alpha = \min_{\eta \in \mathbb{R}} \eta + \frac{1}{1-\alpha}\mathbb{E}[Z-\eta]^+ \tag{1}$$

where $[\cdot]^+$ denotes positive part and η is a real auxiliary variable.

If the loss variable depends on decision variables, say $Z(x)$, $x \in X$, we can use **optimization shortcut**, i.e.

$$\min_{x \in X} CVaR_\alpha(x) = \min_{(\eta,x)\in\mathbb{R}\times X} \eta + \frac{1}{1-\alpha}\mathbb{E}[Z(x)-\eta]^+.$$

We will assume that the distribution of the loss random variable depends on P, i.e. $\omega \sim P$.

Objective function

$$f_\rho(\eta, x; P) = (1-\rho)\mathbb{E}_P Z(x,\omega) \tag{2}$$

$$+\rho\left(\eta + \frac{1}{1-\alpha}\mathbb{E}_P[Z(x,\omega)-\eta]^+\right), \tag{3}$$

where $\rho \in (0,1)$ is a parameter corresponding to the agregate function. If we set $\rho = 0$ we minimize expected loss without involving risk minimization. On the other hand, if we set $\rho = 1$ we are absolutely risk averse, i.e. we minimize risk only without considering mean loss (return).

Loss random variable

We denote P_i quotation of security i, f_i fixed transaction costs, c_i proportional transaction costs (not depending on investment amount), R_i random return of security i, x_i number of securities, y_i binary variables which indicate, whether the security i is bought or not. Then the loss random function depending on our decision x, y and random returns R is equal to

$$Z(x, y, R) = -\sum_{i=1}^{n} (R_i - c_i) P_i x_i + \sum_{i=1}^{n} f_i y_i$$

together with the constraints $0 \leq x_i \leq u_i y_i$ using upper bounds $u_i > 0 \; \forall i$.

We will assume that the distribution of random returns is finite discrete, i.e $P \sim D(\{p_j, r_j^P\}_{j=1}^{J^P})$ with probabilities $p_j \geq 0$ of realizations r_j^P, and $\sum_{j=1}^{J^P} p_j = 1$.

$$g_\rho(\eta, x, y; P) = (1-\rho) \sum_{j=1}^{J} p_j Z(x, y, r_j^P)$$

$$+\rho \left(\eta + \frac{1}{1-\alpha} \sum_{j=1}^{J} p_j [Z(x, y, r_j^P) - \eta]^+ \right),$$

where $\rho \in (0, 1)$.

Our investment problem is

$$\begin{array}{ll} \min & g_\rho(\eta, x, y; P) \\ \text{s.t.} & l_i y_i \le x_i \le u_i y_i, \; i = 1, \dots, n, \\ & C_l \le \sum_{i=1}^n P_i x_i \le C_u, \\ & x_i \ge 0, \; \text{integer}, \; i = 1, \dots, n, \\ & y_i \in \{0, 1\}, \; i = 1, \dots, n, \\ & \eta \in \mathbb{R}, \end{array} \tag{4}$$

where C_l and C_p are lower and upper bound on the capital available for the portfolio investment, $l_i > 0$ and $u_i > 0$ are lower and upper number of units for each security i.

Content

In general, we may consider the following **stochastic optimization problem**

$$\varphi(P) = \inf_{x \in X} g(x, P), \tag{5}$$

where X is a closed subset of $\mathbb{Z}^{n'} \times \mathbb{R}^{n-n'}$ and the KNOWN underlying probability measure P belongs to a general class of Borel probability measures $\mathcal{P}$ with support $\Xi \subseteq \mathbb{R}^m$, $g : \mathbb{R}^n \times \mathcal{P} \to \overline{\mathbb{R}}$.

However, the distribution is usually estimated or approximated. Hence, stability analysis with respect to some changes of the distribution is necessary.

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Metric stability

For $P, Q \in \mathcal{P}$

$$|\varphi(P) - \varphi(Q)| \leq L \cdot d_{MI}(P, Q) \leq L \cdot d_I(P, Q) \tag{6}$$

for some $L > 0$ and an *appropriate* minimal information d_{MI} and ideal d_I (pseudo)metrics.

Example of metric

W. Römisch, S. Vigerske (2008, 2009): for fully random two-stage mixed integer linear programming problems (Fortret-Mourier, discrepancy)

$$\zeta_{2,ph_k}(P,Q) = \sup\left\{ \left| \int_B f(\xi)(P-Q)(d\xi) \right| : f \in \mathcal{F}_2(\Xi),\ B \in \mathcal{B}_{ph_k}(\Xi) \right\},$$

where $\mathcal{B}_{ph_k}(\Xi)$ denotes the set of all polyhedra being subsets of Ξ and having at most k faces and

$$\mathcal{F}_p(\Xi) = \{F : \Xi \to \mathbb{R} : |F(\xi) - F(\tilde{\xi})| \leq c_p(\xi,\tilde{\xi})\|\xi - \tilde{\xi}\|, \forall \xi, \tilde{\xi} \in \Xi\}$$

with the growth function $c_p(\xi,\tilde{\xi}) = max\{1, \|\xi\|, \|\tilde{\xi}\|\}^{p-1}$ describing the growth of the local Lipschitz constant.

Contamination techniques in SP

J. Dupačová (1990): Let $P \in \mathcal{P}$ and $Q \in \mathcal{P}$, then the **contaminated distribution** P^t is defined for all $t \in [0,1]$ by

$$P^t = (1-t)P + tQ.$$

We denote **extreme value function** and **optimal solution set mapping** of contaminated stochastic programming problem as

$$\begin{aligned} \varphi(t) &= \inf_{x \in X} g(x, P^t), \\ \psi(t) &= \arg\min_{x \in X} g(x, P^t) = \{x \in X : \ g(x, P^t) = \varphi(t)\}. \end{aligned}$$

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The **Gateaux derivative** of the extreme value function at P in direction $Q-P$ is then defined as

$$\varphi'(P;Q-P) = \lim_{t\to 0_+} \frac{\varphi(t)-\varphi(0)}{t}$$

if the limit exists.

If we assume, that all optimal values are finite, the concavity of the objective function in the underlying distribution ensures concavity of the extreme value function. Hence, we can construct the **contamination bounds for the extreme value function** of the contaminated problem as follows

$$(1-t)\varphi(0)+t\varphi(1) \le \varphi(t) \le \varphi(0)+t\varphi'(P;Q-P), \;\; t\in[0,1].$$

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Complexity of contamination techniques

- Contamination techniques may be more computationally tractable in stochastic integer programming than the approach based on probability metrics, however theoretically less general.
- **We do not need to solve any contaminated problem which is always larger then the original and fully contaminated problem**.

Applications of contamination techniques

- **Integer stochastic programming**: P. Dobiáš (2003).
- **Conditional Value at Risk, Value at Risk:** J. Dupačová, J. Polívka (2005).
- **Bond portfolio management:** J. Dupačová, M. Bertocchi, and V. Moriggia (2008).
- ...

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Numerical example

We consider

- mean-CVaR model with indivisible assets
- 30 Czech **shares funds** (4 types: stock funds, bond funds, financial funds, mixed funds)
- Week returns from January 2005 to April 2009 downloaded from www.kurzy.cz. We used them to estimate month returns on which we based our portfolio optimization model.
- Two riskless assets (term deposits) with different guaranteed interest rates.
- Proportional transaction costs which range 0 to 2 per cent depending on concrete fund.
- budget 500 000 CZK

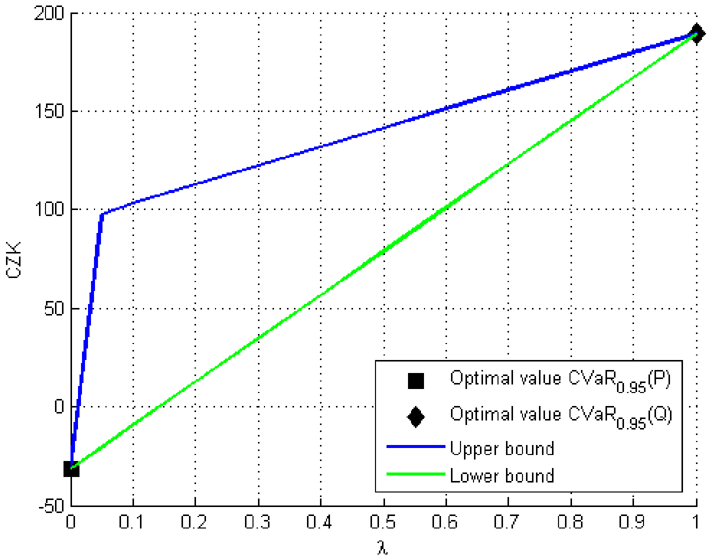
Numerical example

In the source of data we can differ two periods - before and during distress. We used the first period to construct our portfolio and the second period to the post-analysis of our results. We studied performance of our portfolios and apply the contamination techniques.

Optimal values and solutions, upper bounds for derivatives

MPL & CPLEX

Distribution	P	Q
ČPI - OPF Penezni	100 000 CZK	100 000 CZK
IKS Balancovany	50 000 CZK	0
IKS Global konzervativni	100 000 CZK	0
Term deposits	250 000 CZK	400 000 CZK
$CVaR_\alpha$	-31.5020	189.1228
φ' (upper bound)	2579.9976	-95.1548


200
150
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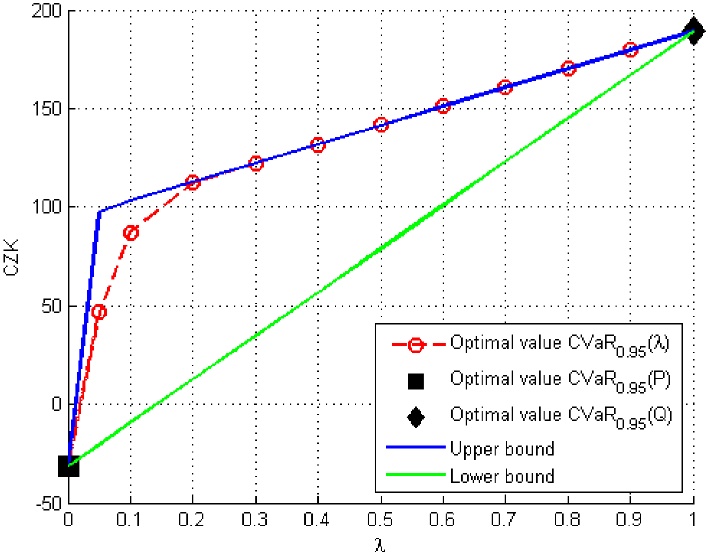
-50
CZK
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
λ
Optimal value CVaR0.95(P)
Optimal value CVaR0.95(Q)
Upper bound
Lower bound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0.9
1
CZK
λ
Optimal value CVaR0.95(λ)
Optimal value CVaR0.95(P)
Optimal value CVaR0.95(Q)
Upper bound
Lower bound

Future research

Multistage investment stochastic programming problems

- scenario tree
- decomposition algorithms
- contamination techniques

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THANK YOU FOR YOUR ATTENTION.

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