Pattern Theory and Its Applications

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Ulf Grenander

- A Swedish mathematician, since 1966 with Division of Applied Mathematics at Brown University
- Highly influential research in time series analysis, probability on algebraic structures, pattern recognition, and image analysis.
- The founder of Pattern Theory

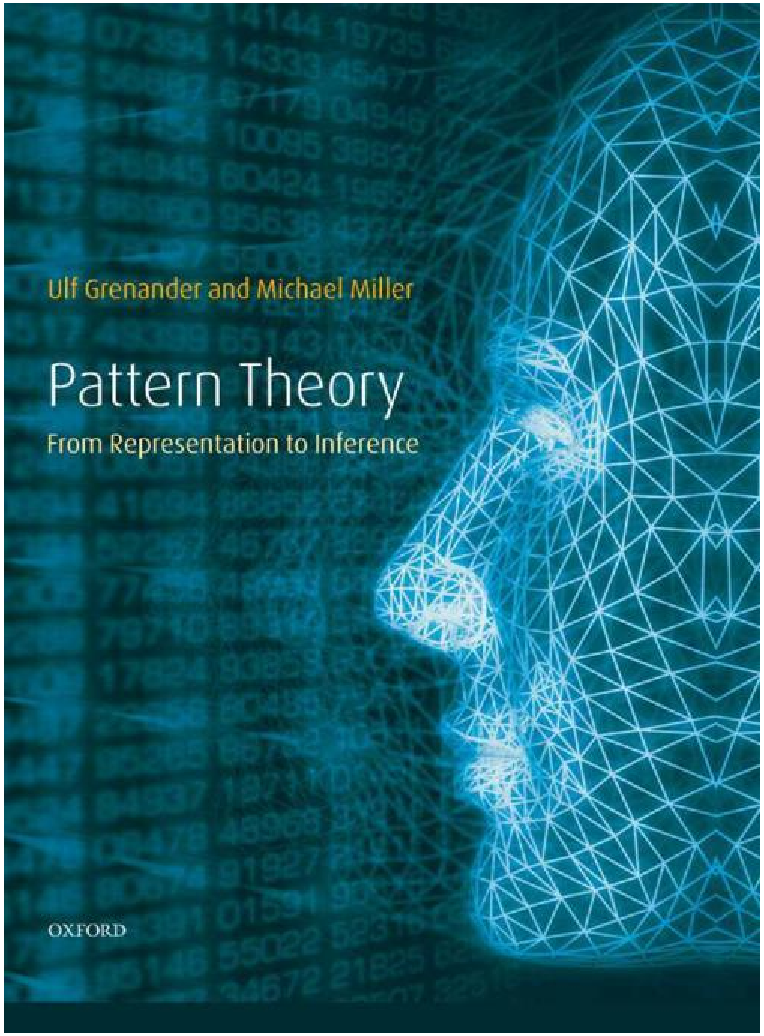
  1976, 1978, 1981: Lectures in Pattern Theory

  1993: General Pattern Theory

  1996: Elements of Pattern Theory

  2007: Pattern Theory

See initial chapters for good introduction



Further referred to as [PT07]

Ulf Grenander

Michael Miller

Agenda

1. Intuitive concepts
2. General pattern theory
3. Illustrative applications
4. Takeaway points

Why should one pay attention?

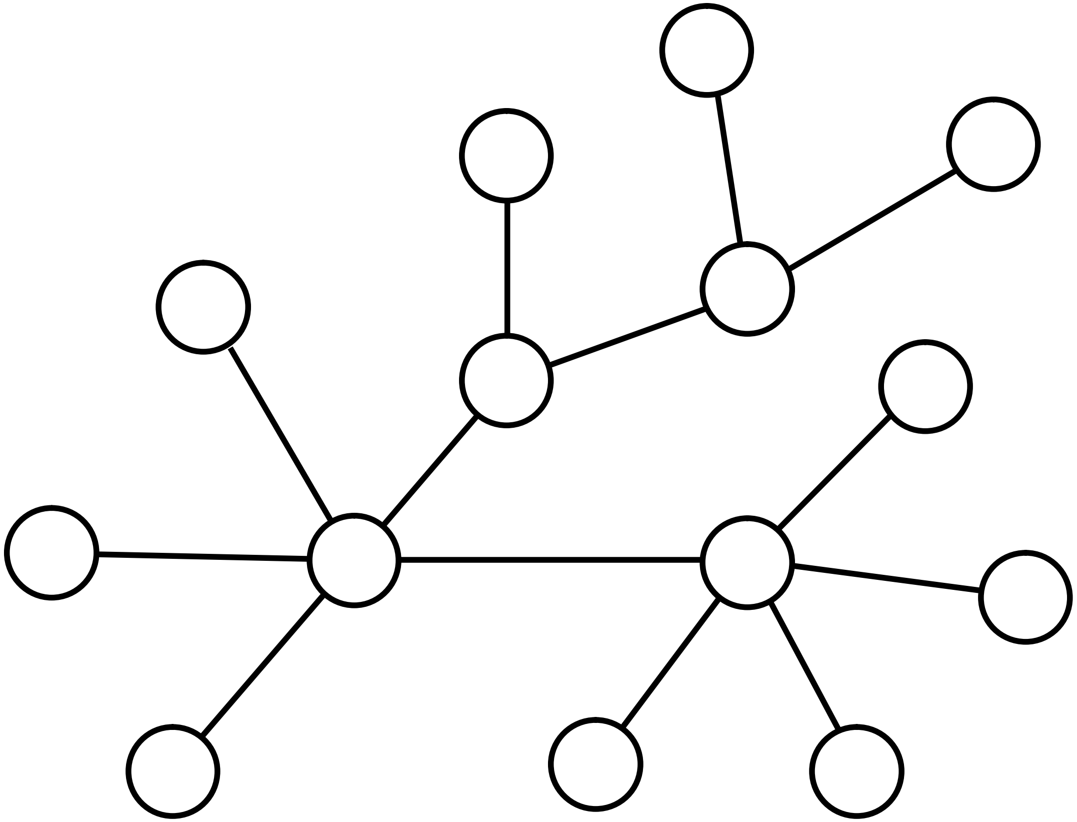
- Pattern Theory is a mathematical representation of objects with large and incompressible complexity in terms of atom-like blocks and bonds between them, similar to chemical structures.[1]
- Pattern Theory gives an algebraic framework for describing patterns as structures regulated by rules, both local and global. Probability measures are sometimes superimposed on the image algebras to account for variability of the patterns.[2]

[1] Yuri Tarnopolsky, Molecules and Thoughts, 2003

[2] Ulf Grenander and Michael I. Miller, Representations of knowledge in complex systems, 1994.

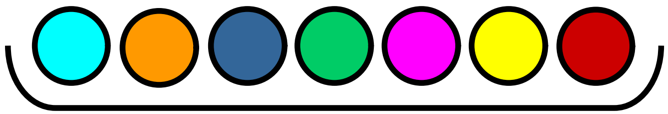
Board game analogy

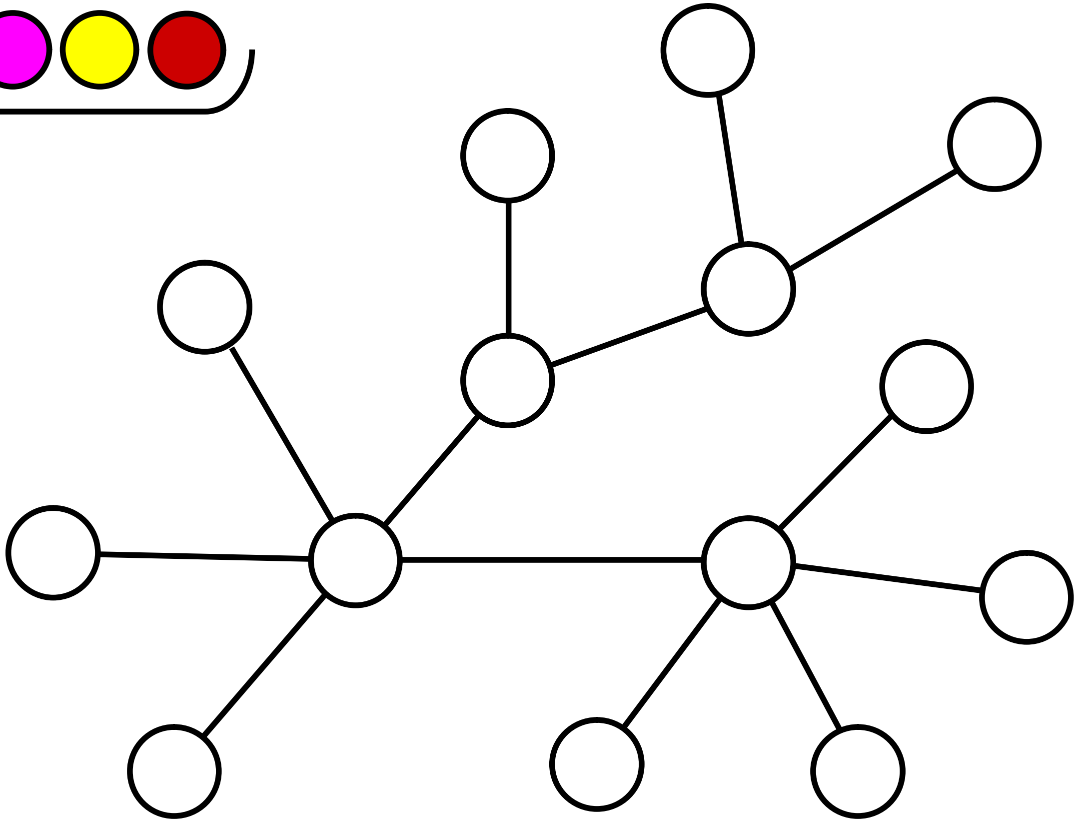
- Game board



Board game analogy

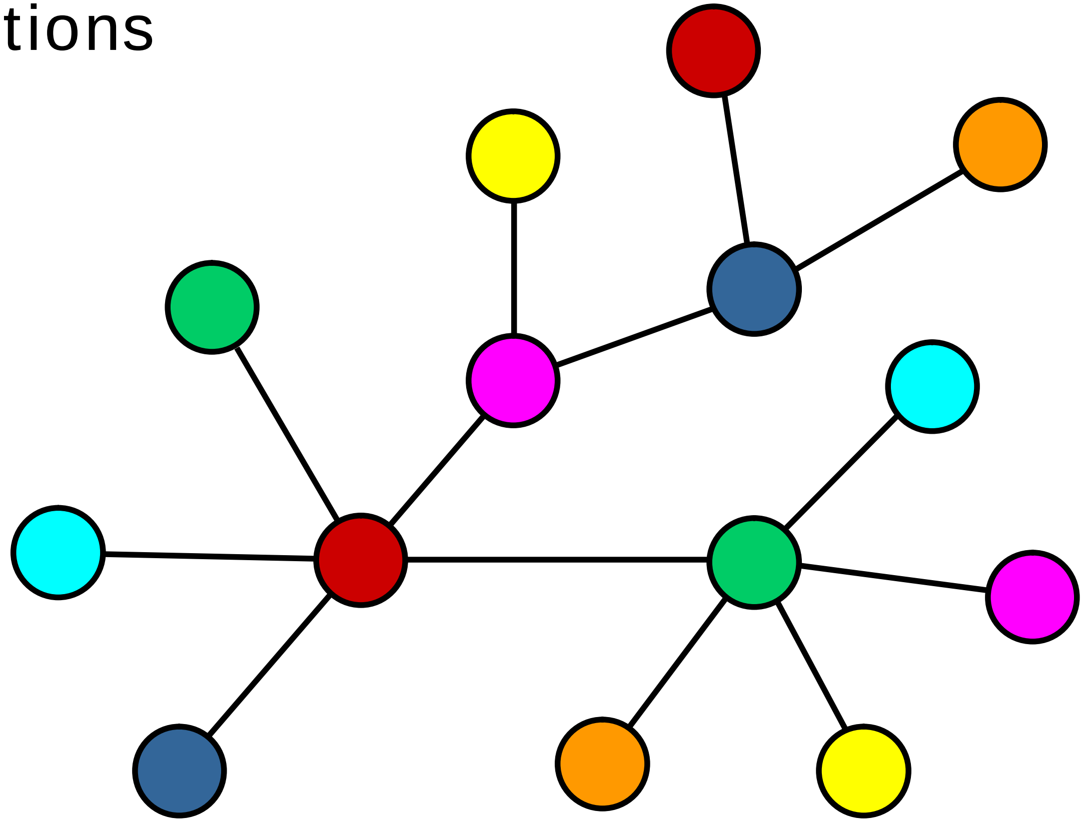
- Game board
- Game stones





Board game analogy

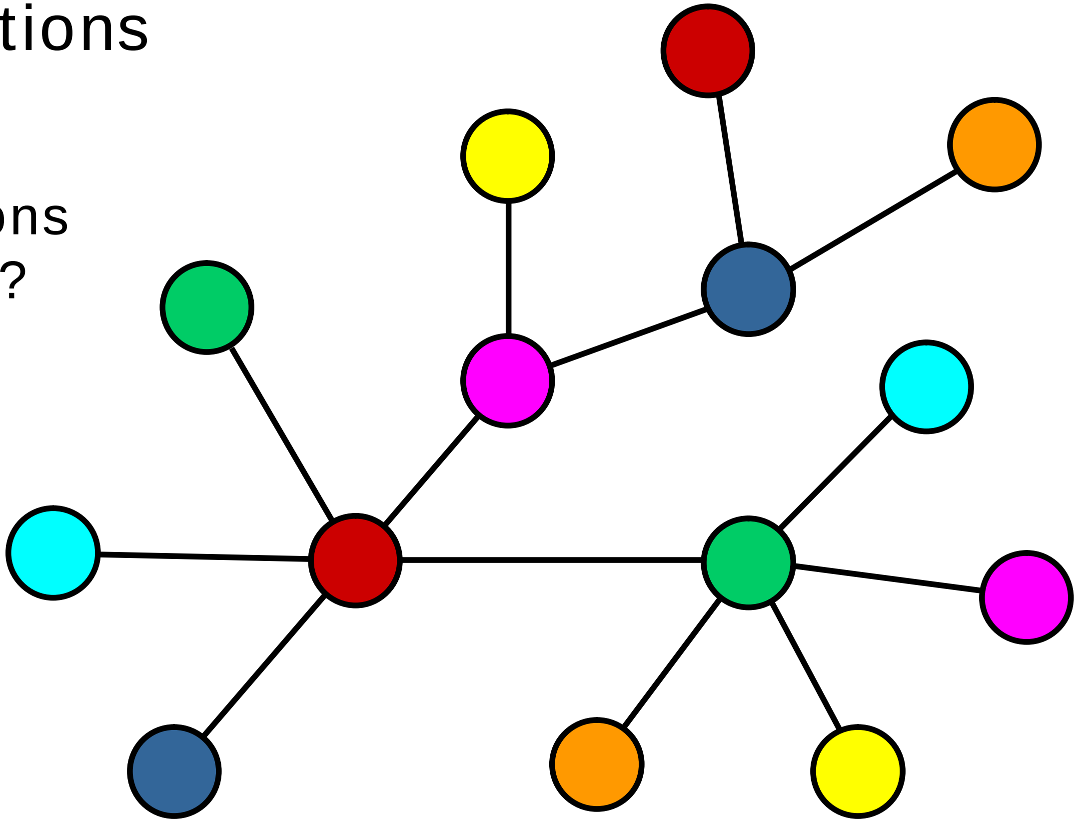
- Game board
- Game stones
- Game situations



Board game analogy

- Game board
- Game stones
- Game situations
- Game rules

Which situations
 - are allowed?
 - are likely?



Sample "games"

- Molecular geometry
- Chemical structures
- Biological shapes and anatomies
- Pixelated images and digital videos
- Visual scene representation
- Formal languages and grammars
- Economies and markets
- Human organizations
- Social networks
- Protein folding
- Historic events
- Thoughts, ideas, theories ...

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Configurations

- Configurations

$$c = \sigma(g_1, g_2, \ldots, g_n) \in \mathcal{C}(\sigma)$$

- Connector graph

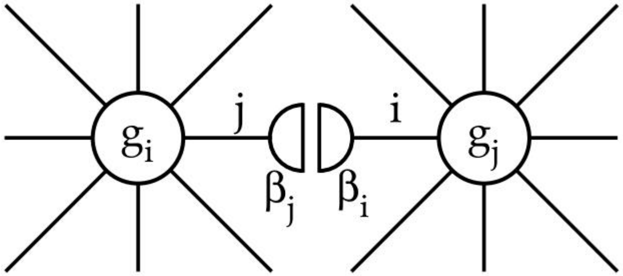
$$\sigma = (D, E)$$

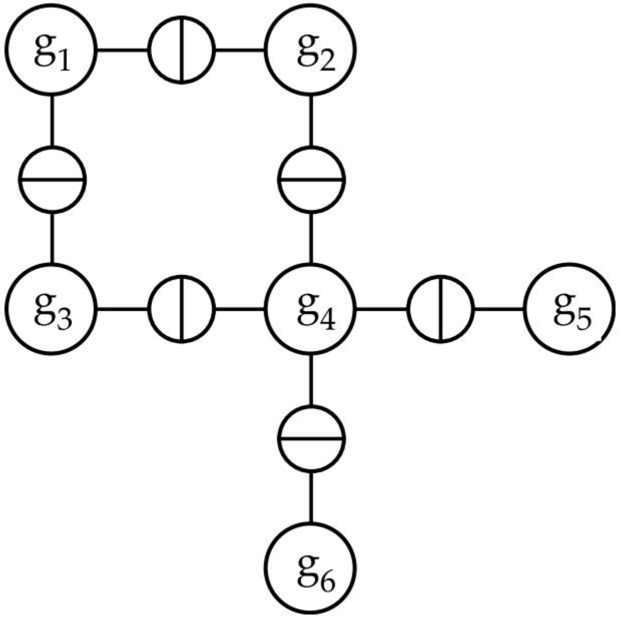
- Generators

$$g_i \in \mathcal{G},\ i = 1, 2, \ldots, n$$

- Bonds

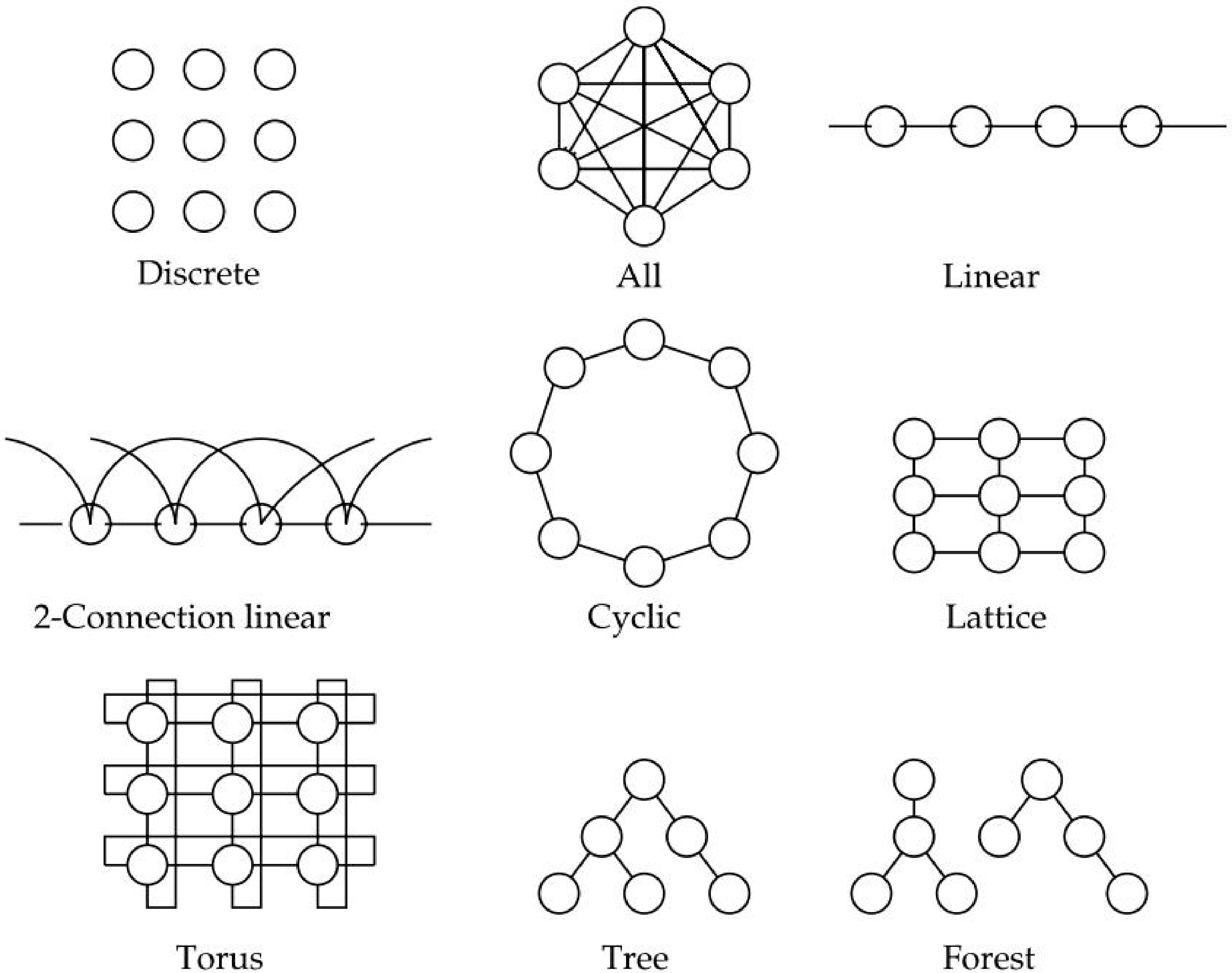
$$\beta_j(g) \in \mathcal{B},\ j = 1, 2, \ldots, \omega(g)$$





Typical connector types

Source: [PT07]



Strict regularity

- Bond function

$$\rho : \mathcal{B} \times \mathcal{B} \rightarrow \{\mathrm{TRUE}, \mathrm{FALSE}\}$$

- Regular configuration

$$\bigwedge_{(i,j) \in E} \rho(\beta_j(g_i), \beta_i(g_j)) = \mathrm{TRUE}$$

- Space of regular configurations

$$\mathcal{C}_{\mathcal{R}}(\sigma) \subset \mathcal{C}(\sigma)$$ Single graph

$$\mathcal{C}_{\mathcal{R}}(\Sigma) = \bigcup_{\sigma \in \Sigma} \mathcal{C}_{\mathcal{R}}(\sigma)$$ Multiple graphs

Example: Mitochondria shape

Source: [PT07]

- Generators

$$g_i = (x_i, x_{i+1})$$

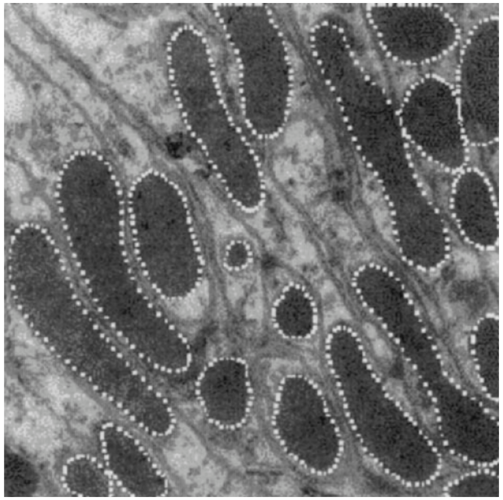
- Bonds

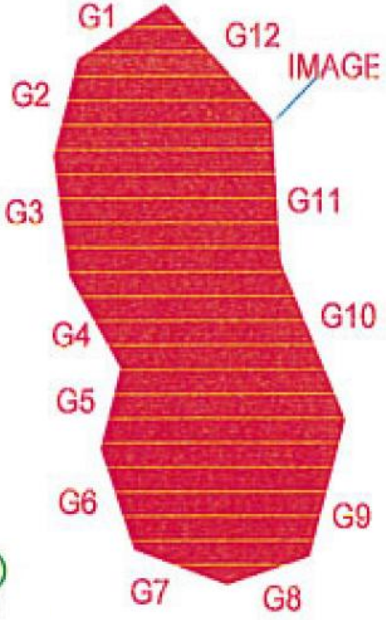
$$\beta_{\text{in}}(g_i) = x_i$$

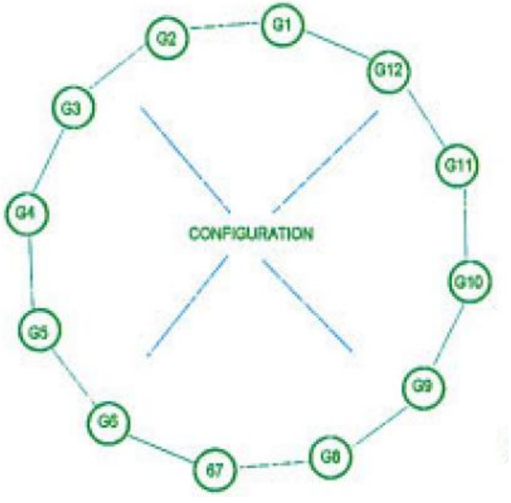
$$\beta_{\text{out}}(g_i) = x_{i+1}$$

- Regularity conditions

$$\beta_{\text{out}}(g_i) = \beta_{\text{in}}(g_{i+1})$$







Weak regularity

- Probability of configuration

$$p(c) = \frac{1}{Z} \prod_{(i,j)\in E} e^{-\Phi_{ij}(\beta_j(g_i),\, \beta_i(g_j))}$$

Matching bonds

$$= \frac{1}{Z} e^{-\sum_{(i,j)\in E} \Phi_{ij}(\beta_j(g_i),\, \beta_i(g_j))}$$

- Normalization

$$\int_{\mathcal{C}_{\mathcal{R}}} p(c)\,\mathrm{d}c = 1$$

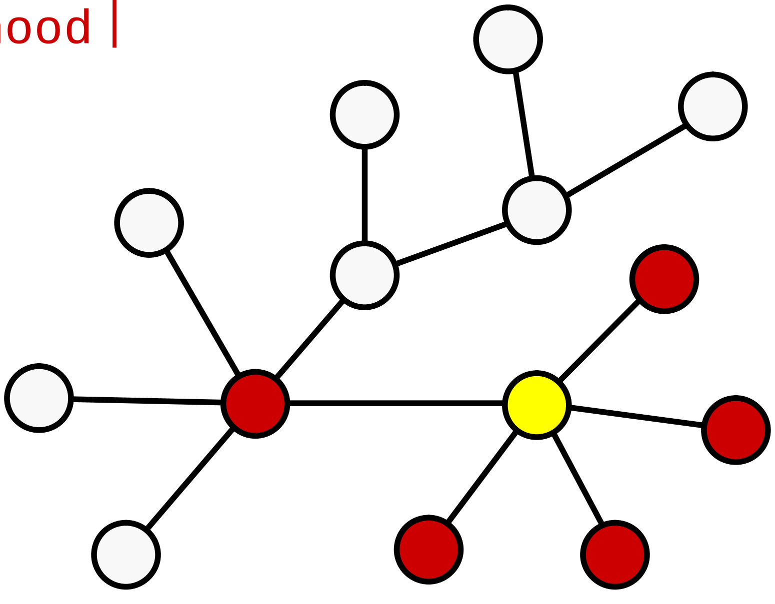
What's the motivation behind this form?

Markov random field

- A set of random variables, X, indexed over D is a Markov random field on a graph $\sigma = (D, E)$ if
  1. $p(x) > 0$ for all x
  2. $p(x_i | x_j, j \neq i) = p(x_i | x_j, j \in N_i)$ for each i

Neighborhood

Two sites are neighbors
if connected by an edge
in the connector graph.



Gibbs distribution

- Gibbs distribution

$$p(x) = \frac{1}{Z} e^{-U(x)}, \quad Z = \int e^{-U(x)}\, dx$$

Normalization constant

- Energy function

$$U(x) = \sum_{C \in \mathcal{C}} \Phi_C(x)$$

A system of cliques

- Clique potentials

$$\Phi_C(x) = \Phi_C(x_C)$$

Only sites in the clique

A set of sites forms a clique if every two distinct sites in the set are neighbors.

Hammersley-Clifford theorem

- Markov random field

$$p(x) > 0 \text{ for all } x$$

$$p(x_i | x_j, j \neq i) = p(x_i | x_j, j \in N_i) \text{ for each } i$$

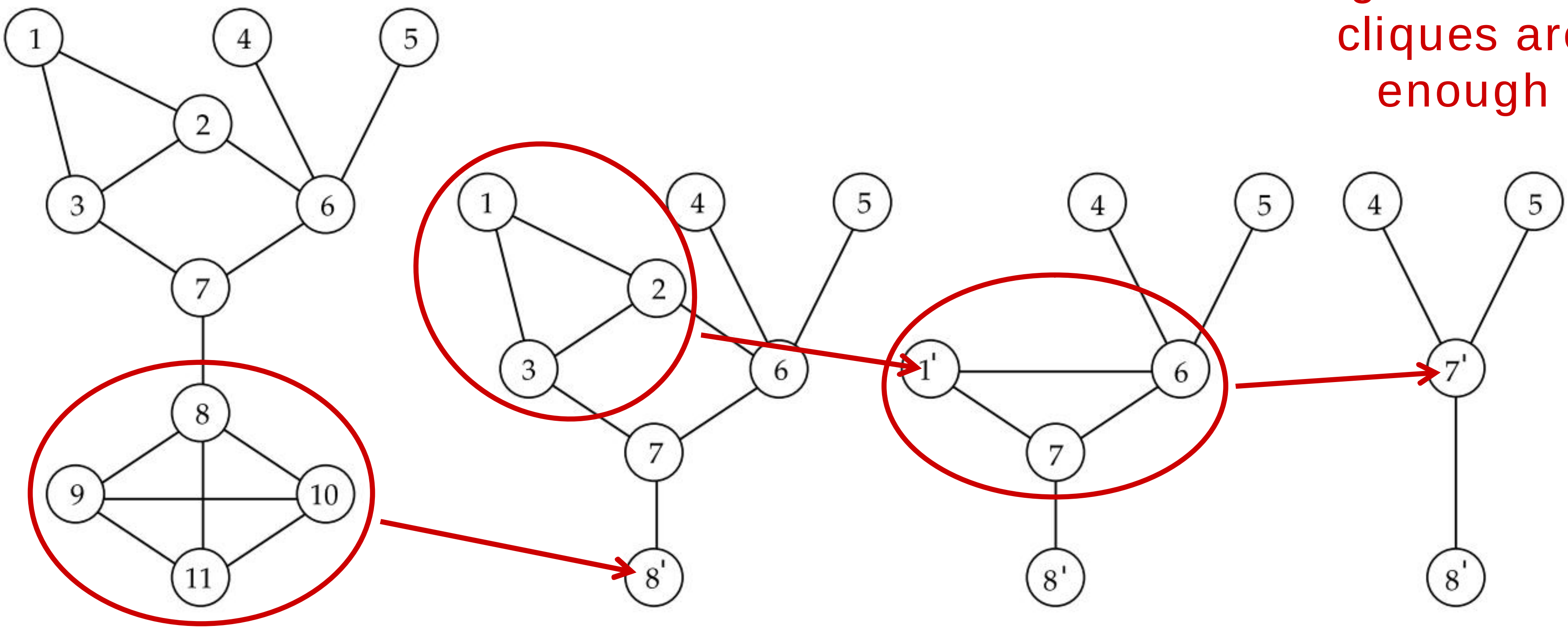
Equivalent properties

- Gibbs distribution

$$p(x) = \frac{1}{Z} e^{-U(x)}, \; Z = \int e^{-U(x)} \mathrm{d}x$$

$$U(x) = \sum_{C \in \mathcal{C}} \Phi_C(x), \; \Phi_C(x) = \Phi_C(x_C)$$

Canonical representation



Pairwise generator cliques are enough

Source: [PT07]

Weak regularity

- Gibbs distribution

$$p(c) = \frac{1}{Z} \prod_{(i,j)\in E} e^{-\Phi_{ij}(\beta_j(g_i),\, \beta_i(g_j))}$$

Pairwise potentials

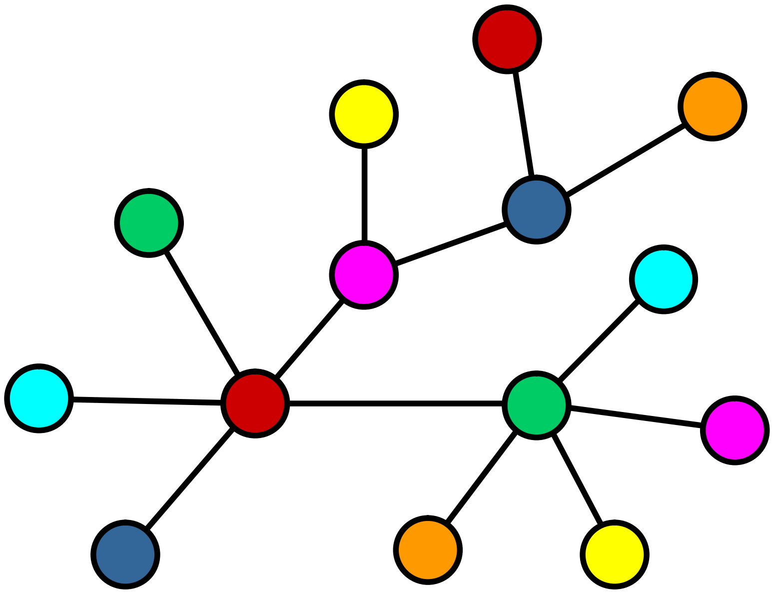
$$= \frac{1}{Z} e^{-\underbrace{\sum_{(i,j)\in E} \Phi_{ij}(\beta_j(g_i),\, \beta_i(g_j))}_{\text{Energy function}}}$$

- Normalization

$$\int_{\mathcal{C}_{\mathcal{R}}} p(c)\, \mathrm{d}c = 1$$

Board game analogy

- Connector graph
  - ~ Game board
- Generators and bonds
  - ~ Game stones
- Regularity conditions
  - ~ Game rules
- Regular configurations
  - ~ Permitted game situations
  - ~ Likely game situations



Images and deformed images

- Identification rule

$$R : c \mapsto I$$

The concept of image formalizes the idea of an observable

- Image algebra (quotient space)

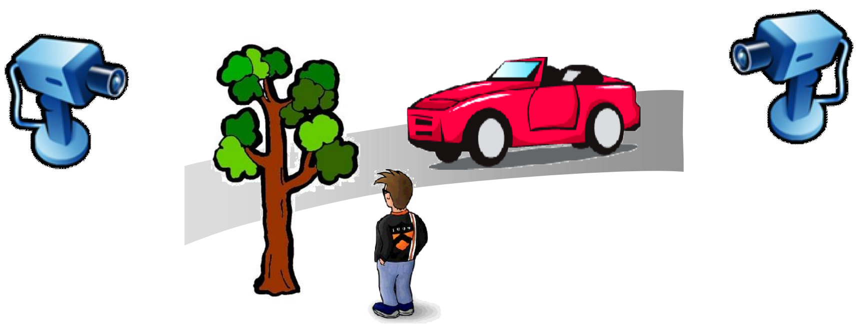
$$\mathcal{I} = \mathcal{C}_{\mathcal{R}}/R$$

- Image (equiv. class)

$$I = [c]_R$$

- Deformed image

$$I^{\mathcal{D}} = \mathcal{D}I$$



Think of 3D scene by means of 2D images

Patterns

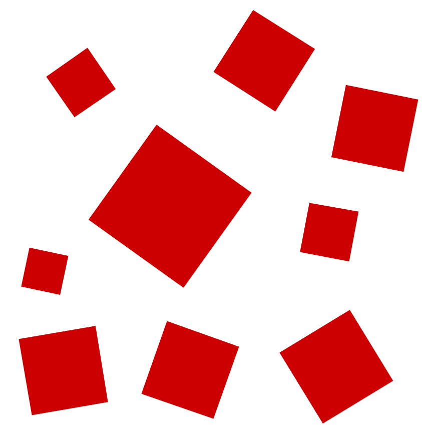
- Similarity group

$$s : \mathcal{G} \leftrightarrow \mathcal{G},\ s \in \mathcal{S}$$

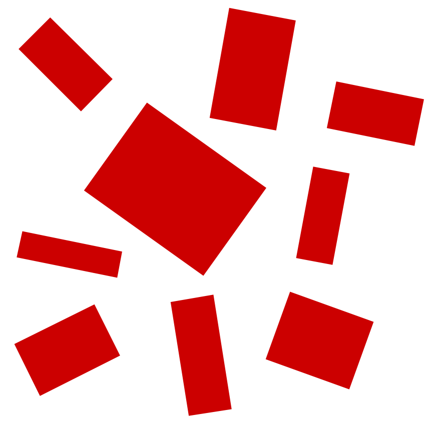
- Pattern family (quotient space)

$$\mathcal{P} = \mathcal{I} / \mathcal{S}$$

- Patterns are thought of as images modulo the invariances represented by the similarity group $\mathcal{S}$



"Square" pattern



"Rectangle" pattern

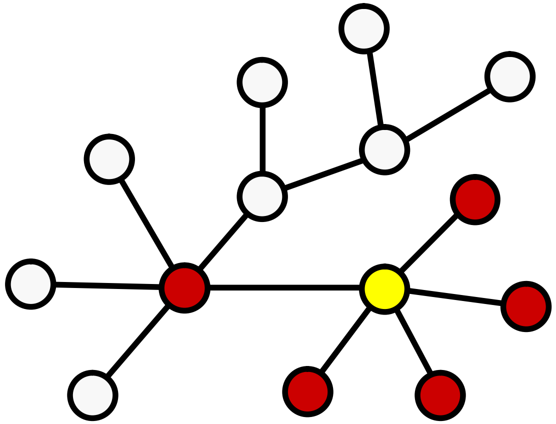
Bayesian inference

- Bayes rule

$$p(I|I^{\mathcal{D}}) \propto p(I)\,p(I^{\mathcal{D}}|I)$$

$$p(c|I^{\mathcal{D}}) \propto p(c)\,p(I^{\mathcal{D}}|c)$$

- Numerical implementation
  - Markov chain Monte Carlo methods
    - Gibbs sampler
    - Langevin sampler
    - Jump-diffusion dynamics
  - Mean field approximation
  - Renormalization group



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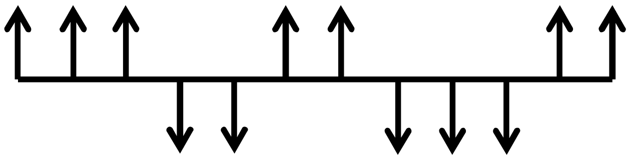
Ising model

- Energy of configuration

A row of binary elements



+1 for positive spin
−1 for negative spin

$$U(c) = -J\sum_{i\sim j}\beta(g_i)\,\beta(g_j) - m\,H\sum_{i}\beta(g_i)$$

Neighbors

Configur-
ation

>0 if attractive
<0 if repulsive

External magnetic field

Property of material, >0

- Probability of configuration

$$p(c) = \frac{1}{Z}\,e^{-\frac{1}{kT}U(c)}$$

Configurations that require high energy are less likely

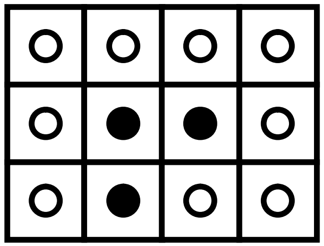
Normalizing constant

Temperature

Universal constant

Pixelated image model

- Black-and-white image



s, t : sites
$s \sim t$: neighbors
g_s : original image pixels
I_s : noisy image pixels

- Energy of configuration given a noisy image

$$U(c;I) = -w \sum_{s \sim t} \beta(g_s)\beta(g_t) - \sum_{s} \beta(g_s)\beta(I_s)$$

Smoothing factor, >0

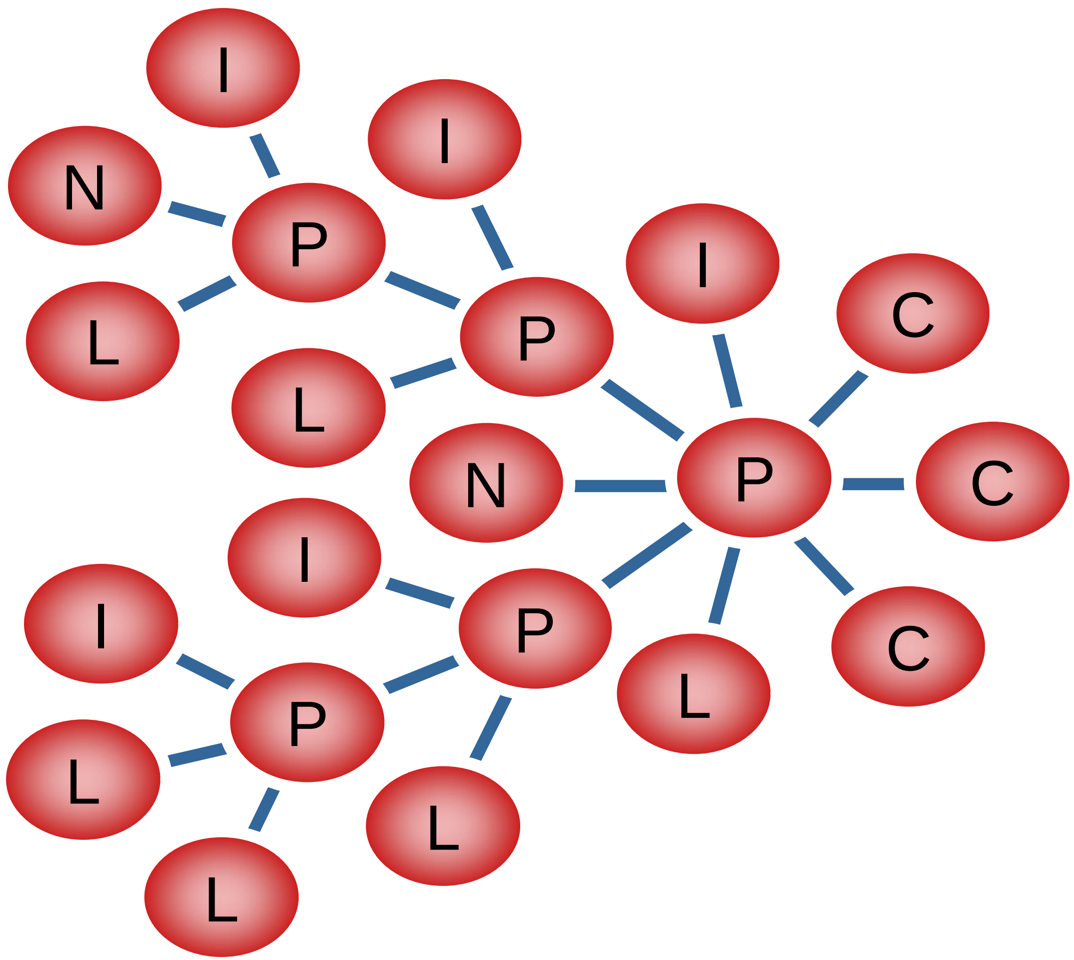
Neighbors

+1 for black
–1 for white

- Probability of configuration

$$p(c|I) = \frac{1}{Z} e^{-U(c;I)}$$

Market network model



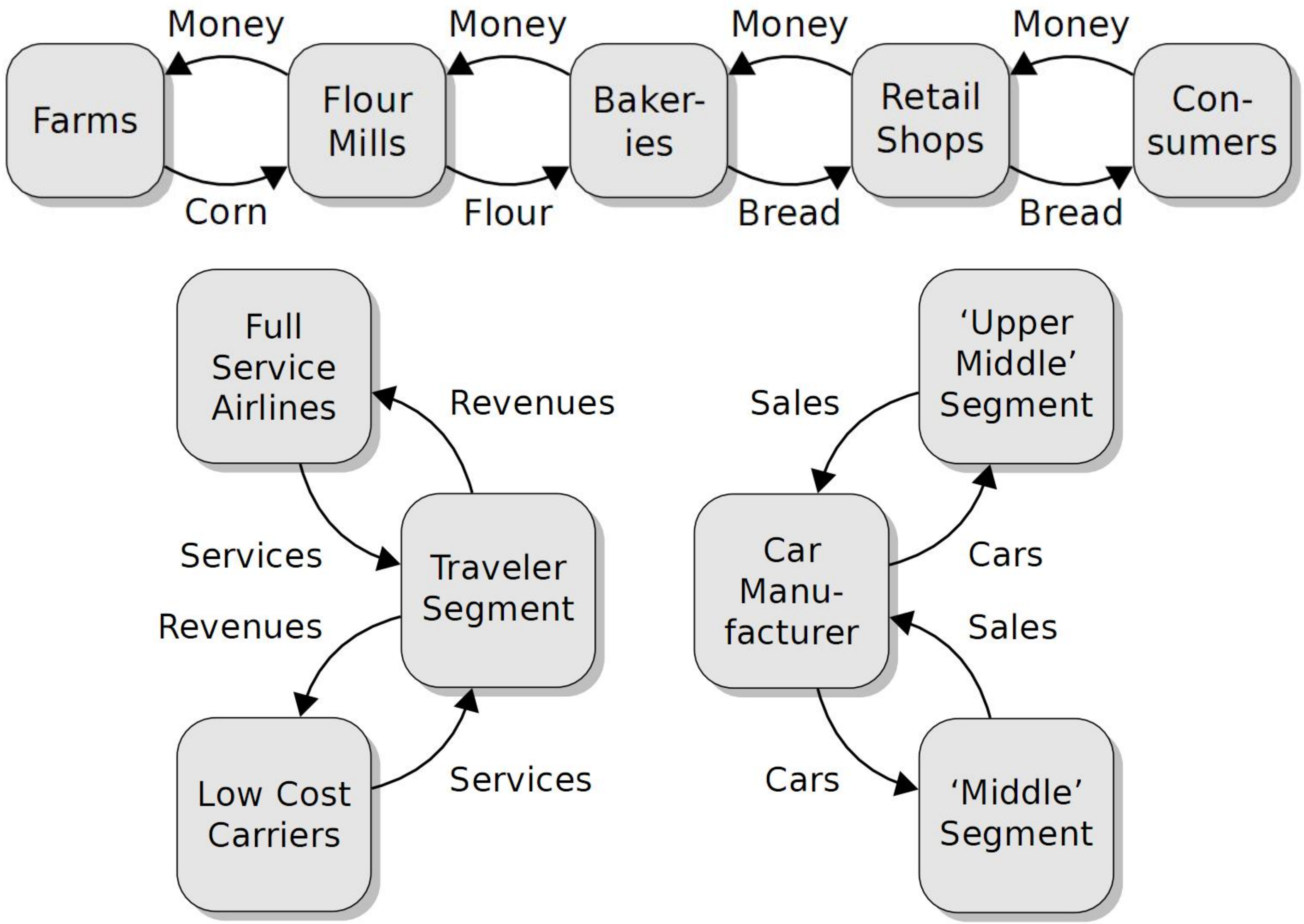
Actors

Consumer (C)
Producer (P)
Labor (L)
Investor (I)
Nature (N)

Exchanges

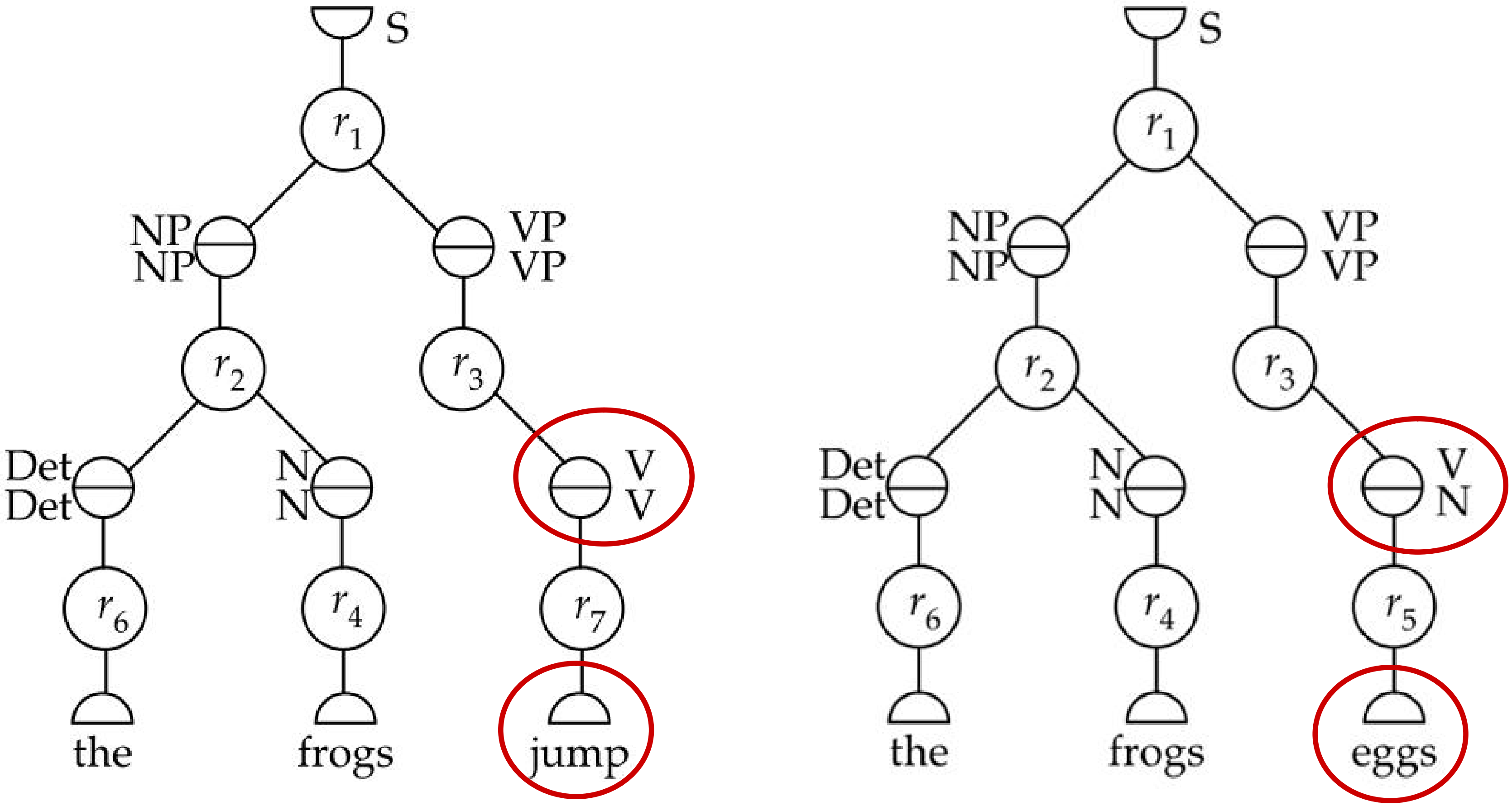
Consumers’ goods
or goods of the 1st order
Producers’ goods
or goods of higher order
or factors of production

Sample markets



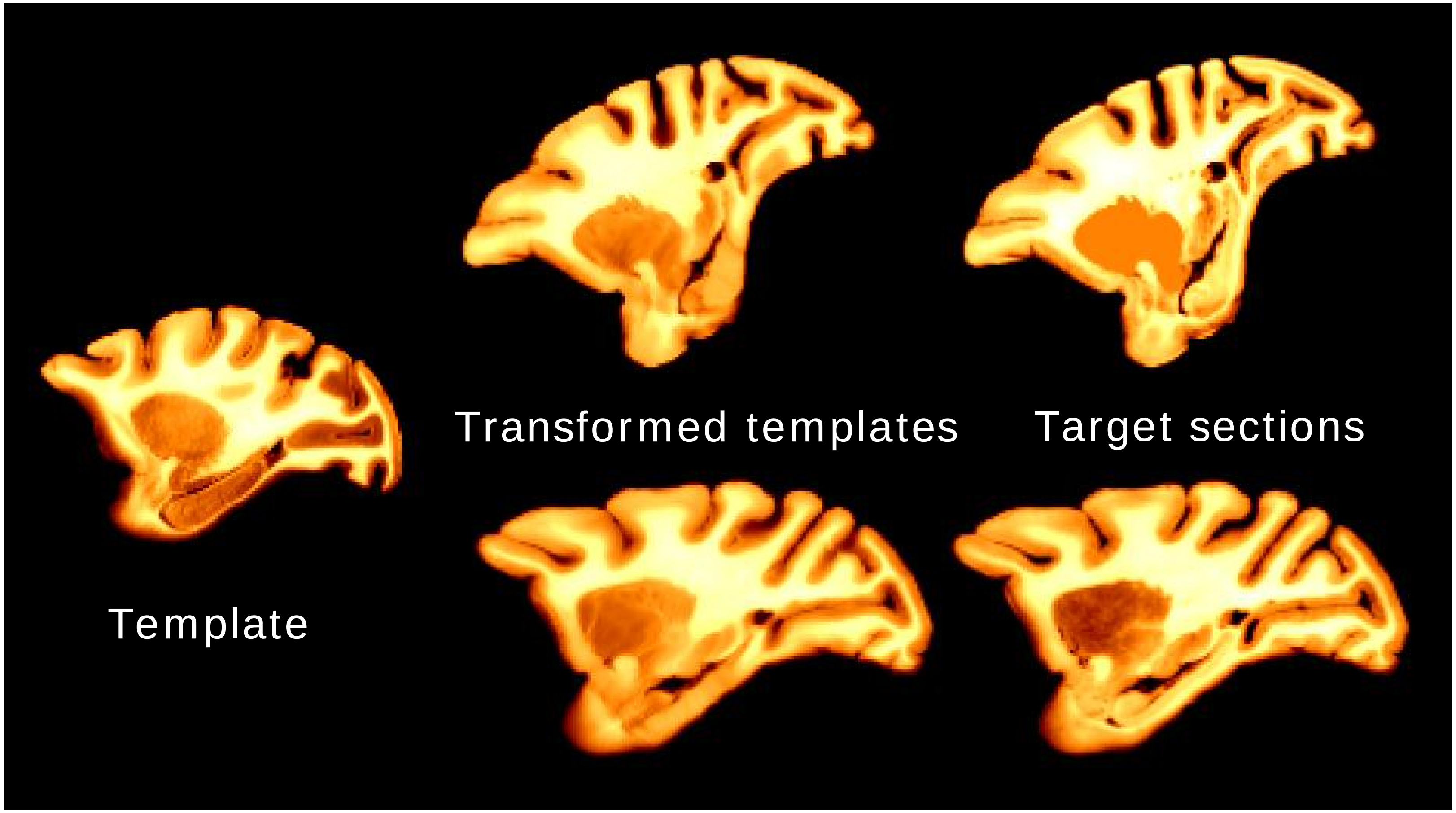
Phrase structure grammars

Source: [PT07]



MRI images

Whole brain maps in the macaque monkey



Source: Ulf Grenander and Michael I. Miller,
Computational anatomy: an emerging discipline, 1996

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When to think of Pattern Theory

- Representations of complex systems composed of atomic blocks interacting locally
- Representations that accommodate structure and variability simultaneously
- Typical applications include
  - Speech recognition
  - Computational linguistics
  - Image analysis
  - Computer vision
  - Target recognition

Further reading

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Further reading (contd.)

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