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Finitely fibered Rosenthal compacta and trees

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Abstract

We study some topological properties of trees with the interval topology. In particular, we characterize trees which admit a 2-fibered compactification and we present two examples of trees whose one-point compactifications are Rosenthal compact with certain renorming properties of their spaces of continuous functions.

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Contents

1 Introduction

A compact space is *n-fibered* if it has an at most n-to-1 continuous map onto a metric space. Obvious variations of the above definition give the notions of *finitely/metrizably fibered* spaces. It is clear that every metrizably fibered compact is first countable. A famous open question, attributed to Fremlin (see [1]), asks whether it is consistent with the usual axioms of set theory that every perfectly normal compact is 2-fibered.

Another motivation for studying metrizably fibered compacta is a result proved by Todorčević [12]: every hereditarily separable Rosenthal compact is 2-fibered. It seems to be unknown whether every Rosenthal compact is a continuous image of a finitely fibered (or at least metrizably fibered) compact space.

Finitely fibered compacta are in some sense close to metric spaces, so it is natural to ask whether their spaces of continuous functions have good renorming properties. We show that this is not the case. Namely, there exists a 2-fibered scattered Rosenthal compact K for which $\mathcal{C}(K)$ fails to have a Kadec renorming. We also show that the existence of a Kadec renorming for $\mathcal{C}(K)$, with K Rosenthal, does not imply that K is a continuous image of a 2-fibered compact. The remaining question is whether spaces of continuous functions over 2-fibered compacta have rotund renormings. One has to mention a recent positive result in this direction [3]: $\mathcal{C}(K)$ has a locally uniformly rotund renorming whenever K is a separable Rosenthal compact which, with respect to a suitable representation, consists of functions with countably many points of discontinuity.

Our examples are built from trees with the interval topology. We rely on the results of Haydon [2], where renormings of spaces of continuous functions over one-point compactifications of trees were studied. Our inspiration was the work of Todorčević [11], where it is proved that the Alexandrov compactification of a certain well-known tree is Rosenthal compact.

2 Preliminaries

By a *space* we mean a Hausdorff topological space. Let $n > 0$ be a natural number. A compact space K is *n-fibered* if there is a continuous map $f\colon K \to S$ such that S is a second countable space and $|f^{-1}(y)| \leqslant n$ for every $y \in S$. We shall say that K is *n-determined* if there are a second countable space S and an upper semicontinuous map $\Phi\colon S \to [K]^n$ such that $K = \bigcup_{s\in S} \Phi(s)$. This class of spaces was denoted by $L\Sigma(\leqslant n)$ in [4]. It is well known and not hard to prove (see e.g. [4]) that a regular space K is n-determined if there are a cover $\mathcal{C} \subseteq [K]^n$ and a countable family of closed sets $\mathcal{N}$ which forms a *network* for $\mathcal{C}$, i.e. for every $C \in \mathcal{C}$ and an open set $V \supseteq C$ there is $N \in \mathcal{N}$ with $C \subseteq N \subseteq V$. If K is compact, it is enough to require that for every $C \in \mathcal{C}$ there is $\mathcal{N}_C \subseteq \mathcal{N}$ such that $C = \bigcap \mathcal{N}_C$. Note that every continuous image of a compact n-fibered space is n-determined. The converse is false, see [4]. Various classes and topological properties of metrizably determined spaces were studied earlier in [9, 10] and later in [6, 7, 8].

In this paper we are interested in n-fibered and n-determined compact spaces, where $n \leqslant 3$.

A *Rosenthal compact* is a compact space homeomorphic to a subspace of the space of Baire class one functions $\mathrm{B}_1(P)$ endowed with the pointwise topology, where P is a Polish space. We shall use the fact that the characteristic function 1_F of a set $F \subseteq P$ is of Baire class one iff F is at the same time F_σ and G_δ.

2.1 Alexandrov-type compactifications

Let X be a locally compact space which is not compact. The well known *Alexandrov compactification* of X is the space $\alpha X = X \cup \{\infty\}$, where $\infty \notin X$ and a neighborhood of ∞ is of the form $\alpha X \setminus F$, where $F \subseteq X$ is compact. This construction can be naturally generalized to obtain more complex compactifications of X. Namely, fix a continuous map $f \colon X \to K$, where K is a compact space. Denote by τ_X and τ_K the topologies of X and K respectively. For technical reasons, assume that $X \cap K = \emptyset$. We claim that there exists a unique compact topology τ on $X \cup K$ which extends the topologies of X and K and for which the map $r \colon X \cup K \to K$, defined by conditions

$$r \restriction X = f \quad \text{and} \quad r \restriction K = \mathrm{id}_K,$$

is continuous.

Let us first see uniqueness. The continuity of r implies that $U \cup f^{-1}[U] \in \tau$ for every $U \in \tau_K$. Further, $\tau_X \subseteq \tau$, because K is closed in $X \cup K$. Finally, $(X \cup K) \setminus F \in \tau$ whenever $F \subseteq X$ is compact. Now, using the local compactness of X, it is straight to check that the family

$$\mathcal{B} = \tau_X \cup \{(U \cup f^{-1}[U]) \setminus F \colon U \in \tau_K,\ F \subseteq X \text{ compact}\}$$

induces a Hausdorff topology on $X \cup K$. By compactness, $\mathcal{B}$ must be a basis of τ. Finally, it is an easy exercise to show that $\langle X \cup K, \tau \rangle$ is indeed compact.

The space $\langle X \cup K, \tau \rangle$ will be denoted by $\alpha_f X$. Note that $\alpha_f X = \alpha X$ when $f \colon X \to \{\infty\}$ is the constant map. If X is a discrete space and $f \colon X \to K$ is any one-to-one map into a compact space, then $\alpha_f X$ is the well known Alexandrov duplicate of $f[X]$ in K.

We shall need the following property of $\alpha_f X$.

Lemma 2.1. *Let X be a locally compact non-compact space and let f be a continuous map from X into a compact space K. If both αX and K are Rosenthal compact then so is $\alpha_f X$.*

Proof. Let P_0, P_1 be disjoint Polish spaces such that $\alpha X \subseteq \mathrm{B}_1(P_0)$ and $K \subseteq \mathrm{B}_1(P_1)$. We may assume that $\infty \in \alpha X$ corresponds to the constant zero function 0_{P_0} in $\mathrm{B}_1(P_0)$. Let P be the disjoint topological sum of P_0, P_1. Then P is a Polish space. We identify $x \in \mathrm{B}_1(P_1)$ with $x \cup 0_{P_0} \in \mathrm{B}_1(P)$. By this way $K \subseteq \mathrm{B}_1(P)$. Define $j \colon X \to \mathrm{B}_1(P)$ by setting

$$j(x) \restriction P_0 = x \quad \text{and} \quad j(x) \restriction P_1 = f(x) \restriction P_1.$$

Clearly, j is well defined, one-to-one and $j[X] \cap K = \emptyset$, because $x \neq 0_{P_0}$ for $x \in X$. Using the continuity of f, we conclude that j is a homeomorphic embedding. It suffices to check that $L = j[X] \cup K$ is closed in $[-\infty, +\infty]^P$. Since there is only one compact topology on L extending $j[X]$ and K for which $f \circ j^{-1}$ is continuous, this will ensure that L is homeomorphic to $\alpha_f X$.

Define $r_1(x) = x \cdot 1_{P_1}$. That is, $r_1(x) \restriction P_1 = x \restriction P_1$ and $r_1(x) \restriction P_0 = 0$. Clearly, $r_1 \colon [-\infty, +\infty]^P \to [-\infty, +\infty]^P$ is a continuous map.

Fix $v \in [-\infty, +\infty]^P \setminus L$. Let $v_0 = v \restriction P_0$. If $v_0 = 0_{P_0}$ then $r_1(v) = v \notin K$ and, using the continuity of r_1, we can easily find a neighborhood of v disjoint from L. So assume $v_0 \neq 0_{P_0}$. Now, if $v_0 \notin X$ then, using the compactness of $X \cup \{0_{P_0}\} = \alpha X$, we again find a neighborhood of v disjoint from L. It remains to consider the case that $v_0 =\in X$. Notice that $f(v_0) \neq r_1(v)$, because otherwise $v = j(v_0) \in L$. Using the continuity of both f and r_1, find a neighborhood V of v in $[-\infty, +\infty]^P$ such that $f(x \restriction P_0) \neq r_1(x)$ whenever $x \in V$ and $x \restriction P_0 \in X$. We may further assume that $x \restriction P_0 \neq 0$ whenever $x \in V$. Then $V \cap K = \emptyset$ and $V \cap j[X] = \emptyset$. □

2.2 Trees

For detailed information concerning theory and applications of trees we refer the readers to the excellent survey [13]. Here we give necessary definitions and facts only.

A *tree* is a partially ordered set $\langle T, < \rangle$ with a minimal element denoted by 0, such that for every $t \in T$ the set $[0, t) = \{x \in T \colon x < t\}$ is well ordered and for every $s, t \in T$ there exists the greatest lower bound $s \wedge t$. The last condition ensures that its *interval topology* induced by open sets of the form $(s, t] = \{x \in T \colon s < x \leqslant t\}$ is Hausdorff. It is clear that this topology is locally compact, namely every interval of the form $[s, t] = \{x \in T \colon s \leqslant x \leqslant t\}$ is compact. We denote by αT the Alexandrov one-point compactification of the tree T. That is, $\alpha T = T \cup \{\infty\}$ and a basic neighborhood of ∞ is of the form

$$\alpha T \setminus \bigcup_{i<n} [0, t_i],$$

where $t_0, \dots, t_{n-1} \in T$, $n \in \omega$. It will be convenient to extend the partial order of T onto αT by setting $t < \infty$ for every $t \in T$. By this way, $\infty = \sup \mathcal{C}$ whenever $\mathcal{C}$ is an unbounded chain in T.

Given a tree T, the order type of $[0, t)$ is called the *height* of t in T and denoted by $\operatorname{ht}_T(t)$ or just $\operatorname{ht}(t)$. The *height* of T is the supremum of all numbers $\operatorname{ht}_T(t)$, where $t \in T$. The set $\{t \in T \colon \operatorname{ht}_T(t) = \alpha\}$ is called the *α-th level* of T. A *branch through* T is a maximal linearly ordered subset of T. Given $t \in T$ we denote by t^+ the set of all *immediate successors* of t in T, that is, $t^+ = \{s \in T \colon t < s \wedge [t, s] = \{t, s\}\}$. We say that T is *finitely/countably branching* if t^+ is finite/countable for each $t \in T$.

Let $\langle T, < \rangle$ be a tree and let $\langle X, < \rangle$ be a linearly ordered set. We say that T is *X-embeddable* if there exists a $<$-preserving function from T into X. Note that such a function may not be one-to-one. We shall be particularly interested in $\mathbb{R}$-embeddable trees, where $\mathbb{R}$ denotes the real line. $\mathbb{Q}$-embeddable trees are often called *special*. A tree is $\mathbb{Q}$-embeddable if and only if it is covered by countably many antichains.

A subset A of a partially ordered set $\langle P, < \rangle$ is an *initial segment* if $(\leftarrow, y] \subseteq A$ whenever $x \in A$, where $(\leftarrow, x] = \{y \in P \colon y \leqslant x\}$. If P is a tree, then we say that A is an *initial subtree*.

We shall need the following criterion for continuity of maps defined on trees.

Lemma 2.2. *Let $\langle T, < \rangle$ be a tree and let $f \colon \alpha T \to X$ be a continuous map into a topological space X. Then f is continuous if and only if it satisfies the following two conditions.*

(1) $\lim_{\alpha<\lambda} f(t_\alpha) = f(t)$, whenever λ is a regular infinite cardinal and $\{t_\alpha\}_{\alpha<\lambda}$ is a strictly increasing sequence in T with $\sup_{\alpha<\lambda} t_\alpha = t$, where $t = \infty$ if $\{t_\alpha\}_{\alpha<\lambda}$ is unbounded.

(2) $\lim_{n\to\infty} f(t_n) = f(\infty)$, whenever $\{t_n\}_{n\in\omega}$ is an antichain in T.

Proof. It is obvious that the above conditions are necessary. Assume $f \colon \alpha T \to X$ satisfies (1) and (2). It is clear that f is continuous at each point $t \in T$. Indeed, if for some neighborhood U of $f(t)$, $f^{-1}[U]$ were not a neighborhood of t, then there would exist an increasing sequence $\{s_\alpha\}_{\alpha<\lambda} \subseteq [0, t] \setminus f^{-1}[U]$ with $t = \sup_{\alpha<\lambda} t_\alpha$, where λ is the cofinality of $[0, t)$. This would contradict (1).

Fix a neighborhood U of $f(\infty)$ and let $B = \alpha T \setminus f^{-1}[U]$. If B contains a sequence $\{t_\alpha\}_{\alpha<\lambda}$ which has no upper bound in T then we have that $\infty = \lim_{\alpha<\lambda} t_\alpha$ and U witnesses that (1) fails. Thus, every chain in B is bounded in T. Suppose B contains an infinite antichain $\{b_n\}_{n\in\omega}$. Then $\infty = \lim_{n\to\infty}\{b_n\}_{n\in\omega}$ while, on the other hand, U witnesses the failure of (2).

It follows that all antichains in B are finite. It is well known and not hard to prove that a tree with this property can have only finitely many branches, say $S_0, \dots, S_{k-1}$. For each $i < k$ let a_i be an upper bound of S_i in T. Then $V = \alpha T \setminus \bigcup_{i<k} [0, a_i]$ is a neighborhood of ∞ such that $f[V] \subseteq U$. $\square$

We shall consider trees with countable branches only. In this case, condition (1) can be replaced by

(1') $\lim_{n\to\infty} f(t_n) = f(t)$ whenever $\{t_n\}_{n\in\omega}$ is a strictly increasing sequence in T with $t = \sup_{n\in\omega} t_n$.

2.3 Trees of sets

A *tree of sets* indexed by a fixed tree T is a family of sets $\{A_t \colon t \in T\}$ satisfying the following conditions:

(3) $A_t \supseteq A_s$ whenever $t \leqslant s$ in T.

(4) $A_t \cap A_s = \emptyset$ whenever $t, s \in T$ are incomparable.

(5) $\bigcap_{s\in C} A_s = A_t$ whenever $C \subseteq T$ is a chain with $t = \sup C$.

(6) $\bigcap_{t \in C} A_t = \emptyset$ whenever $C \subseteq T$ is an unbounded chain.

Assume further that X is a fixed set such that $A_s \subseteq X$ for every $s \in T$. Then $\{A_t : t \in T\}$ can be regarded as a topological subspace of the Cantor cube 2^X, identified with the powerset of X (a set corresponds to its characteristic function).

Proposition 2.3. *Let $\{A_t : t \in T\}$ be a tree of sets. Then the map $f : \alpha T \to \mathcal{P}(X)$, defined by $f(t) = A_t$ for $t \in T$ and $f(\infty) = \emptyset$, is continuous.*

Proof. It is clear that condition (2) of Lemma 2.2 is satisfied, since any sequence of pairwise disjoint sets converges to the empty set. Condition (1) is also satisfied, because $\bigcap_{\alpha<\lambda} A_{t_\alpha} = \lim_{\alpha<\lambda} A_{t_\alpha}$ whenever $\{t_\alpha\}_{\alpha<\lambda}$ is increasing. Thus, the continuity of f follows from Lemma 2.2. □

A tree of sets $\{A_t : t \in T\}$ will be called *proper* if $A_t \neq \emptyset$ for every $t \in T$ and $A_t \neq A_s$ whenever $t \neq s$.

Corollary 2.4. *Let $\{A_t\}_{t \in T}$ be a proper tree of sets. Then*

$$\{A_t : t \in T\} \cup \{\emptyset\} \subseteq \mathcal{P}(A_0)$$

is homeomorphic to αT.

Note that every tree is isomorphic to a proper tree of sets. Namely, given a tree T, the family $\{V_t : t \in T\}$, where $V_t = \{s \in T : t \leqslant s\}$, is a tree of pairwise different nonempty subsets of T. The above corollary implies that αT is homeomorphic to $\{V_t : t \in T\} \cup \{\emptyset\}$, where $\emptyset$ corresponds to ∞.

2.4 Expanding trees

We describe a well known operation on a tree that replaces each element by a copy of another fixed tree.

Fix a tree T and another, possibly much smaller, tree S. For instance, let $S = 2^{<2}$ or $S = 2^{<\omega}$. We would like to insert a copy of S at each node of T. For this aim, for each $t \in T$ choose a tree $\{D_s(t)\}_{s \in S}$ of subsets of t^+ such that $D_0(t) = t^+$. Now let $T' = T \cup (T \times (S \setminus \{0\}))$, where we declare that

(7) $t < \langle t, s \rangle$ for every $s \in S \setminus \{0\}$,

(8) $\langle t, s \rangle < \langle t, s' \rangle$ whenever $s < s'$ in S and

(9) $\langle t, s \rangle < r$ whenever $r \in D_s(t)$.

It is clear that this defines a tree order on T', extending the order of T. We shall call it an *S-expansion* of T. This construction of course depends on the choice of $D_s(t)$ for $s \in S$, $t \in T$.
In case where the tree T is already a tree of nonempty sets (so, in particular, the order is reversed inclusion) and $|D_s(t)| > 1$ for every $\langle t, s\rangle \in T \times S$, one can represent the S-expansion with respect to D as another tree of sets:

$$T' = T \cup \{A_s(t) \colon s \in S,\ t \in T\}, \quad \text{where } A_s(t) = \bigcup D_s(t).$$

Recall that $D_s(t)$ is a family of pairwise disjoint nonempty sets.
The above construction can of course be generalized in such a way that for each $t \in T$ one adds a different tree S_t above t. We shall not need this type of expansions here.
Below we present a sample application of expansions of trees.

Proposition 2.5. *Let T be an $\mathbb{R}$-embeddable tree of cardinality $\leqslant 2^{\aleph_0}$ and let $f\colon T \to \mathbb{R}$ be a $<$-preserving function. Then there exist a countably branching tree T' containing T as a subtree and a $<$-preserving function $f'\colon T' \to \mathbb{R}$ such that $f' \restriction T = f$.*

Proof. Given $t \in T$ let $W_n(t) = \{x \in t{+}\colon f(x) \geqslant f(t)+1/n\}$ and let $P_n(t) = W_n(t) \setminus W_{n-1}(t)$. Further, using the fact that $|P_n(t)| \leqslant 2^{\aleph_0}$, choose a Cantor tree of sets $\{P_s(t,n)\}_{s\in 2^{<\omega}}$ such that $P_\emptyset(t,n) = P_n(t)$ and $|\bigcap_{s\in C} P_s(t,n)| \leqslant 1$ whenever $C \subseteq 2^{<\omega}$ is an infinite chain.
Now let $S = \{\emptyset\} \cup \omega \cup (\omega \times 2^{<\omega})$ regarded as a tree with the order imposed by conditions

- $(^*)$ $\emptyset < n < \langle n, \emptyset\rangle$ for $n \in \omega$;
- $(^*_*)$ $\langle n, s\rangle \leqslant \langle m, r\rangle$ iff $n = m$ and $s \subseteq r$.

That is, S is obtained by "joining" countably many copies of the Cantor tree $2^{<\omega}$.
Given $t \in T$, define $D_\emptyset(t) = t^+$, $D_n(t) = P_n(t)$ and $D_{\langle n,s\rangle}(t) = P_s(t,n)$.
Let T' be the S-expansion of T with respect to the partitioning D. The tree T' is obviously countably branching. It remains to extend f onto $T \times (S \setminus \{\emptyset\})$.
Fix $t \in T$. Define $f'(t) = f(t)$ and $f'(t,n) = f(t)+1/(2n)$. By assumption, $f(x) \geqslant f(t)+1/n$ whenever $x \in D_n(t)$, therefore so far defined f' preserves the order. Given $n \in \omega$, choose a $<$-preserving function $\varphi\colon 2^{<\omega} \to \mathbb{R}$ so that $f(t) + 1/(2n) < \varphi(r) < f(t) + 1/n$. Finally, given $s = \langle n, r\rangle \in S$, define $f'(t,s) = \varphi(r)$. It is clear that f' is $<$-preserving. □

2.5 Sierpiński tree and Rosenthal compacta

The *Sierpiński tree* $\sigma\mathbb{Q}$ is defined to be the set of all bounded well ordered subsets of the rationals $\mathbb{Q}$, with the "being initial segment" order. We write "$s \sqsubseteq t$" for "s is an initial segment of t". Actually, the relation $\sqsubseteq$ is defined on all subsets of $\mathbb{Q}$: $x \sqsubseteq y$ iff $x \subseteq y$ and $\inf(y \setminus x) \geqslant \sup(x)$, agreeing that $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = +\infty$. The tree $\sigma\mathbb{Q}$ is $\mathbb{R}$-embeddable, which is witnessed by the function $\varphi(t) = \sum_{q_n\in t} 2^{-n}$, where $\{q_n\}_{n\in\omega}$ is a one-to-one enumeration of $\mathbb{Q}$. Note that the function $t \mapsto \sup(t)$ is not $<$-preserving.

In some situations it is more convenient to consider the tree $w\mathbb{Q}$ which consists of all (not necessarily bounded) well ordered subsets of the rationals. The tree $w\mathbb{Q}$ is complete in the sense that all chains are bounded from above; in particular every branch of $w\mathbb{Q}$ has the maximal element—an unbounded well ordered subset of $\mathbb{Q}$.

A well known and useful fact (see e.g. [11, Thm. 4]) is that the tree $\sigma\mathbb{Q}$ is universal for countably branching $\mathbb{R}$-embeddable trees. For completeness we include a short proof.

Proposition 2.6. *Let T be a countably branching tree with a $<$-preserving function $f\colon T \to \mathbb{R}$. Then there exists a tree embedding $\psi\colon T \to \sigma\mathbb{Q}$ such that $\psi[T]$ is an initial segment of $\sigma\mathbb{Q}$ and $f(t) \geqslant \sup \psi(t)$ for every $t \in T$.*

Proof. Denote by $\mathcal{H}$ the family of all partial tree embeddings $h\colon S \to \sigma\mathbb{Q}$ satisfying the above condition, i.e. $f(s) \geqslant \sup h(s)$ for $s \in S$, such that S is an initial segment of T and given $t \in S$, either $t^+ \subseteq S$ or t is maximal in S. It is rather clear that the union of a chain of maps in $\mathcal{H}$ belongs to $\mathcal{H}$. Thus, by the Kuratowski-Zorn Lemma, there exists a maximal element $\psi \in \mathcal{H}$. We claim that $\operatorname{dom}(\psi) = T$. Suppose this is not the case and fix a minimal $t \in T \setminus S$, where $S = \operatorname{dom}(\psi)$. Suppose that t is in the closure of S. Fix $s_0 < s_1 < \dots$ in S with $t = \sup_{n\in\omega} s_n$ and let $x = \bigcup_{n\in\omega} \psi(s_n)$. Clearly, $x \in \sigma\mathbb{Q}$ because it is well ordered and $\sup(x) = \lim_{n\to\infty} \sup \psi(s_n) \leqslant \sup_{n\in\omega} f(s_n) \leqslant f(t)$. Thus $\psi \cup \{\langle t, x\rangle\} \in \mathcal{H}$, a contradiction. We conclude that t is not in the closure of S. Let $s \in S$ be the immediate predecessor of t. Note that s must be a maximal element of S and hence $s^+ \cap S = \emptyset$. We shall extend ψ to $S \cup s^+$. By assumption, s^+ is countable and hence we may find a one-to-one map $j\colon s^+ \to \mathbb{Q}$ so that $\sup(s) < j(r) \leqslant f(r)$ for $r \in s^+$. Finally, define $\psi'\colon S \cup s^+ \to \sigma\mathbb{Q}$ by $\psi' \restriction S = \psi$ and $\psi'(r) = \psi(s) \cup \{j(r)\}$ for $r \in s^+$. Clearly $\psi' \in \mathcal{H}$, a contradiction. $\square$

The above result says in particular that every countably branching $\mathbb{R}$-embeddable tree is isomorphic to a subtree of $\sigma\mathbb{Q}$. In fact, because of Proposition 2.5, every $\mathbb{R}$-embeddable tree of cardinality $\leqslant 2^{\aleph_0}$ can be embedded into $\sigma\mathbb{Q}$ (see also [11, Proof of Thm. 4]).

A theorem of Kurepa [5] says that the tree $\sigma\mathbb{Q}$ is not $\mathbb{Q}$-embeddable. Below we sketch an elegant proof of this result, due to Todorčević.

Suppose $\sigma\mathbb{Q}$ is $\mathbb{Q}$-embeddable and write $\sigma\mathbb{Q} = \bigcup_{n\in\omega} A_n$, where each A_n is an antichain. Construct by induction a sequence $t_0 \sqsubseteq t_1 \sqsubseteq \dots$ of elements of $\sigma\mathbb{Q}$ and a sequence of reals $r_0 \geqslant r_1 \geqslant \dots$, so that $r_n > \sup t_n$ and for each $n \in \omega$ the following condition is satisfied.

(★) If there exists $s \in A_n$ such that $t_{n-1} \sqsubseteq s$ and $\sup s < r_{n-1}$ then t_n has this property, $t_n \in A_n$ and $r_n < r_{n-1}$; otherwise $t_n = t_{n-1}$ and $r_n = r_{n-1}$.

Actually, condition (★) gives a "recipe" for constructing both sequences. It is easy to see that the sequences $\{t_n\}_{n\in\omega}$ and $\{r_n\}_{n\in\omega}$ cannot be eventually constant. Let $t_\infty = \bigcup_{n\in\omega} t_n$. Then t_∞ is a well ordered subset of $\mathbb{Q}$ bounded from above by r_0, i.e. $t_\infty \in \sigma\mathbb{Q}$. Find $k \in \omega$ such that $t_\infty \in A_k$. Then $t_{k-1} \sqsubseteq t_\infty$ and $\sup t_\infty < r_{k-1}$, therefore by (★), $t_k \in A_k$ has the same property. But this means that A_k contains two different comparable elements t_k and t_∞, a contradiction.

To finish this section, we present a short proof of the result of Todorčević [11] saying that the one-point compactification of the tree $w\mathbb{Q}$ is Rosenthal compact.

Let $P = \mathcal{P}(\mathbb{Q})$ be endowed with the Cantor set topology. Given $t \in w\mathbb{Q}$, define

$$A_t = \{x \in P \colon t \sqsubseteq x\}.$$

It is straight to check that A_t is closed. Note that $A_t \supseteq A_s$ whenever $t \sqsubseteq s$ and $A_t \cap A_s = \emptyset$ whenever s and t are incomparable. Given an increasing sequence $\{t_n\}_{n\in\omega} \subseteq w\mathbb{Q}$ with $t = \bigcup_{n\in\omega} t_n$, we have that $A_t = \bigcap_{n\in\omega} A_{t_n} = \lim_{n\to\infty} A_{t_n}$. In other words, $\{A_t\}_{t\in w\mathbb{Q}}$ is a proper tree of nonempty sets. By Corollary 2.4, $\alpha(w\mathbb{Q})$ is homeomorphic to $\{A_t \colon t \in w\mathbb{Q}\} \cup \{\emptyset\}$. Finally, notice that the characteristic function of each A_t is of the first Baire class [1]. This shows that $\alpha(w\mathbb{Q})$ is Rosenthal compact.

3 Main results

In this section we collect our main results concerning trees, finitely determined compacta, Rosenthal compacta and renorming properties of their spaces of continuous functions.

A result of Todorčević [12] says that a hereditarily separable Rosenthal compact is 2-fibered. It is not known whether every Rosenthal compact is a continuous image of a metrizably fibered compact. A partial positive result is proved in [4, Prop. 2.14], namely a Rosenthal compact is metrizably determined if it can be represented as a set of Baire class one functions each having countably many points of discontinuity. Note that for each natural number n the space $X_n = (\alpha\omega_1)^n$ is a Rosenthal compact that is $(n+1)$-determined and not n-determined [4, Thm. 4.5]. That X_n is Rosenthal follows from two facts: $\alpha\omega_1$ is easily seen to be Rosenthal and finite products of Rosenthal compacta are Rosenthal.

3.1 Trees, 2- and 3-determined compacta

In this subsection we address the question which trees induce 2-determined or 3-determined compacta.

Theorem 3.1. *Let T be a tree. Then αT is 2-determined if and only if T is $\mathbb{R}$-embeddable and $|T| \leqslant 2^{\aleph_0}$.*

Proof. Assume T is $\mathbb{R}$-embeddable and $|T| \leqslant 2^{\aleph_0}$. We are going to describe a countable closed family $\mathcal{N}$ such that for every $t \in T$, $\{t, \infty\} = \bigcap \mathcal{N}_t$ for some $\mathcal{N}_t \subseteq \mathcal{N}$. By compactness, this will show that $\alpha T = T \cup \{\infty\}$ is 2-determined.

Fix a $<$-increasing function $h \colon T \to \mathbb{R}$. We may assume that h is continuous (define $h'(t) = h(t)$ if t is a successor element and $h'(t) = \sup_{s<t} h(s)$ if $t \in T$ is a limit; then $h' \colon T \to \mathbb{R}$ is

[1] The characteristic function of a set is of the first Baire class if and only if this set is at the same time F_σ and G_δ.

continuous and $<$-preserving). Let $\mathcal{I}$ denote the collection of all closed rational intervals of $\mathbb{R}$. Given $A \subseteq T$ denote by $\min A$ the set of all minimal elements of A. For each $I \in \mathcal{I}$ fix a countable collection $\mathcal{F}_I$ of subsets of $\min h^{-1}[I]$ which separates the points of $\min h^{-1}[I]$ (here we use the fact that $|T| \leqslant 2^{\aleph_0}$). Given $F \in \mathcal{F}_I$ define

$$F' = \{\infty\} \cup \{t \in h^{-1}[I] \colon (\exists\, s \in F)\ s \leqslant t\}.$$

Observe that F' is closed. Indeed the set $h^{-1}[I] \cup \{\infty\}$ is closed and if $t \in h^{-1}[I] \setminus F'$ then $(0,t] \cap F' = \emptyset$, by the definition of F'.

Define $\mathcal{N} = \{F' \colon F \in \mathcal{F}_I,\ I \in \mathcal{I}\}$. Then $\mathcal{N}$ is a countable family consisting of closed subsets of αT. Fix $t \in T$ and fix $s \neq t$. We are going to find $N \in \mathcal{N}$ with $t \in N$ and $s \notin N$.

Assume first that $h(s) \neq h(t)$. Find $I \in \mathcal{I}$ such that $h(t) \in I$ and $h(s) \notin I$. Next, find $F \in \mathcal{F}_I$ such that the unique element $s \in \min h^{-1}[I] \cap (0,t]$ belongs to F. Then $t \in F' \in \mathcal{N}$ and $s \notin F'$. Assume now that $h(s) = h(t)$. Let r be the maximal element below s, t. Find $I \in \mathcal{I}$ such that $s, t \in h^{-1}[I]$ and $h(r) \notin I$. Let $s_0, t_0 \in \min h^{-1}[I]$ be such that $s_0 \leqslant s$ and $t_0 \leqslant t$. Find $F \in \mathcal{F}_I$ such that $t_0 \in F$ and $s_0 \notin F$. Then $t \in F' \in \mathcal{N}$ and $s \notin F'$. This completes the proof of sufficiency.

Now suppose that αT is 2-determined. Clearly, $|T| \leqslant 2^{\aleph_0}$. Further, there exist a cover $\mathcal{C}$ of αT consisting of at most 2-element sets and a countable family $\mathcal{N}$ of closed sets which is a network for $\mathcal{C}$. Fix the families $\mathcal{C}$ and $\mathcal{N}$. We may assume that $\mathcal{N}$ is closed under finite intersections.

Define

$$T_0 = \{t \in T \colon \{t, \infty\} \in \mathcal{C}\}.$$

We claim that $T \setminus T_0$ has a countable height. For suppose otherwise and choose $t_\alpha \in T \setminus T_0$ so that $\operatorname{ht}(t_\alpha) \geqslant \alpha$, $\alpha < \omega_1$. There is $s_\alpha \in T$ such that $C_\alpha = \{t_\alpha, s_\alpha\} \in \mathcal{C}$. Now $U_\alpha = [0, t_\alpha] \cup [0, s_\alpha]$ is a neighborhood of C_α. Find $N_\alpha \in \mathcal{N}$ with $C_\alpha \subseteq N_\alpha \subseteq U_\alpha$. Notice that

$$\alpha \leqslant \operatorname{ht}(t_\alpha) \leqslant \sup_{x \in N_\alpha} \operatorname{ht}(x) = \max\{\operatorname{ht}(t_\alpha), \operatorname{ht}(s_\alpha)\} < \omega_1.$$

On the other hand, since $\mathcal{N}$ is countable, there should exist N so that $N = N_\alpha$ for uncountably many $\alpha < \omega_1$. This is a contradiction.

Thus we may assume that $T = T_0$, since $T \setminus T_0$ is clearly $\mathbb{Q}$-embeddable. Write $\mathcal{N} = \{N_n\}_{n \in \omega}$. Given $t \in T$, define

$$\varphi(t) = \{n \in \omega \colon [0,t] \cap N_n = \emptyset\}.$$

Clearly, $\varphi(t) \subseteq \varphi(s)$ whenever $s \leqslant t$. Assume $s < t$. Since $\{t, \infty\} \in \mathcal{C}$ and $[0,s]$ is closed and disjoint from $\{t, \infty\}$, by compactness there exists $N_i \in \mathcal{N}$ such that $\{t, \infty\} \subseteq N_i$ and $[0,s] \cap N_i = \emptyset$ (recall that $\mathcal{N}$ is closed under finite intersections). This shows that $\varphi(t) \neq \varphi(s)$ whenever $s < t$. Finally, setting $h(t) = -\sum_{n \in \varphi(t)} 2^{-n}$, we see that T is $\mathbb{R}$-embeddable. $\square$

Theorem 3.2. *Let T be an $(\mathbb{R} \cdot \mathbb{R})$-embeddable tree of cardinality $\leqslant 2^{\aleph_0}$. Then αT is 3-determined.*

Proof. First, replace $\mathbb{R} \cdot \mathbb{R}$ by a Dedekind complete line $X := \mathbb{R} \cdot [0,1]$ and let $p\colon X \to \mathbb{R}$ be the projection onto the first coordinate. Note that p is continuous with respect to the interval topology on X. Let $f\colon T \to X$ be $<$-preserving. As in the proof of Theorem 3.1, modify f by setting $f'(t) = \sup_{s<t} f(s)$ when $\mathrm{ht}_T(t)$ is a limit ordinal and $f'(t) = f(t)$ otherwise. Here we have used the fact that X is complete. Thus, we may assume that h is continuous. Let $f\colon T \to \mathbb{R}$ be the composition of h and the projection p. Then f is continuous, $\leqslant$-preserving and for each $\lambda \in \mathbb{R}$ the tree

$$T_\lambda = \{0_T\} \cup f^{-1}(\lambda)$$

is $\mathbb{R}$-embeddable (recall our requirement that every tree must have the minimal element — that is why we have added 0_T into T_λ). Note that $Y_\lambda = f^{-1}(\lambda) \cup \{\infty\}$ is closed in αT. By (the proof of) Theorem 3.1, for each $\lambda \in \mathbb{R}$ there is a family $\{F_n(\lambda)\}_{n\in\omega}$ consisting of closed subsets of Y_λ, whose all maximal intersections are of the form $\{t,\infty\}$, where $t \in f^{-1}(\lambda)$. Let F_n be the closure in αT of the union $\bigcup_{\lambda\in\mathbb{R}} F_n(\lambda)$. Let $\mathcal{N}$ be a countable family of closed sets defined in the proof of Theorem 3.1. Finally, let

$$\mathcal{M} = \mathcal{N} \cup \{F_n\colon n \in \omega\}.$$

We claim that all maximal nonempty intersections of elements of $\mathcal{M}$ are of the form $\{s,t,\infty\}$, where $s \leqslant t$, $f(s) = f(t)$ and $f(s') < f(s)$ whenever $s' < s$.

It is clear that $\{\infty\} = \bigcap \mathcal{M}$. Fix $t \in T$ and let $C = \bigcap\{M \in \mathcal{M}\colon t \in M\}$. Fix $s \in C$, $s \neq t$. By the proof of Theorem 3.1, we know that $f(s) = f(t) = f(r)$, where $r = s \wedge t$ (otherwise we would be able to separate s from t by an element of $\mathcal{N}$). Suppose s is a minimal element of $f^{-1}(\lambda)$. Then necessarily $s \leqslant t$, since otherwise $r < s$ and we would have $f(r) < f(s)$. It remains to show that the case $s \notin \min f^{-1}(\lambda)$ is impossible.

For suppose $s \notin \min f^{-1}(\lambda)$ and find $n \in \omega$ such that $t \in F_n(\lambda)$ and $s \notin F_n(\lambda)$. Find $u < t$ in $f^{-1}(\lambda)$ such that $U = (u,s]$ is disjoint from $F_n(\lambda)$. Now observe that U is a neighborhood of s not only in Y_λ but also in αT. Consequently, $U \cap F_n(\delta) = \emptyset$ for every $\delta \in \mathbb{R}$. In particular, $U \cap F_n = \emptyset$ and hence $s \notin F_n$, $t \in F_n$. This shows that $s \notin C$, a contradiction. □

One could go further this direction and try to characterize trees whose one-point compactifications are k-fibered for $k > 2$. We stop here, because our examples will give us only 2- or 3-determined compacta.

Theorem 3.3. *Let T be a tree. The following properties are equivalent.*

(a) T is $\mathbb{R}$-embeddable and $|T| \leqslant 2^{\aleph_0}$.

(b) T admits a one-to-one continuous map onto a separable metric space.

(c) αT is a continuous image of a 2-fibered compact.

(d) αT is 2-determined.

Proof. (a) $\Longrightarrow$ (b) In view of Proposition 2.6 and the remarks after it, there exists a tree embedding $j\colon T \to w\mathbb{Q}$. That is, f is a one-to-one map satisfying $f(t) < f(t') \iff t < t'$. We may modify f so that $f[T]$ becomes closed in $w\mathbb{Q}$. Indeed, define inductively $f'\colon T \to w\mathbb{Q}$ by setting $f'(t) = f(t)$ if the T-level of t is a successor ordinal and $f'(t) = \sup_{s<t} f'(s)$ if the T-level of t is a limit ordinal.

Thus, we may assume without loss of generality that T is a closed subtree of $w\mathbb{Q}$. Let $h\colon T \to \mathcal{P}(\mathbb{Q})$ be the inclusion map. Clearly, h is one-to-one and since T is closed in $w\mathbb{Q}$, h is continuous.

(b) $\Longrightarrow$ (c) Let $h\colon T \to K$ be as in (b). Enlarging K if necessary, we may assume that K is a compact metric space. Then $\alpha_h T$ is a 2-fibered compact that maps onto αT.

(c) $\Longrightarrow$ (d) This is trivial.

(d) $\Longrightarrow$ (a) Clearly, (d) implies that $|T| \leqslant 2^{\aleph_0}$. The fact that T is $\mathbb{R}$-embeddable is included in Theorem 3.1. $\square$

Corollary 3.4. *There exists a* 2*-fibered Rosenthal compactification* K_0 *of* $w\mathbb{Q}$ *whose remainder is homeomorphic to the Cantor set.*

Proof. By the implication (a) $\Longrightarrow$ (b) of the above theorem, there is a one-to-one continuous map $f\colon w\mathbb{Q} \to K$ such that K is a compact metric space—in fact, after learning its proof we know that $K = \mathcal{P}(\mathbb{Q})$ and f is the inclusion map. Let $K_0 = \alpha_f(w\mathbb{Q})$. By the result of Todorčević [11], $\alpha(w\mathbb{Q})$ is Rosenthal compact, therefore by Lemma 2.1, K_0 is Rosenthal compact too. Finally, K_0 is 2-fibered, because the canonical retraction of K_0 onto $\mathcal{P}(\mathbb{Q})$ is 2-to-1. $\square$

3.2 Two examples

We shall construct two trees T_1 and T_2, whose one-point compactifications K_1 and K_2 are 3-determined Rosenthal compacta and their spaces of continuous functions have certain renorming properties.

Given a tree T, denote by $\mathcal{C}_0(T)$ the Banach space of all continuous functions $f\colon T \to \mathbb{R}$ vanishing at infinity, i.e. such that for every $\varepsilon > 0$ there is a compact set $F \subseteq T$ satisfying $|f(t)| < \varepsilon$ for $t \in T \setminus F$. The space $\mathcal{C}_0(T)$ is naturally identified with the subspace of $\mathcal{C}(\alpha T)$ consisting of all $f \in \mathcal{C}(\alpha T)$ such that $f(\infty) = 0$.

An important work of Haydon [2] contains several results on renorming properties of spaces of the form $\mathcal{C}_0(T)$, where T is a tree. In particular, it turns out that the existence of a Kadec renorming of $\mathcal{C}_0(T)$ (i.e. a renorming such that the weak and the norm topologies coincide on the unit sphere) is equivalent to σ-fragmentability and it is also equivalent to the existence of a $\leqslant$-preserving function $f\colon T \to \mathbb{R}$ which has no bad points, where a point $t \in T$ is called *good* for f if there are a finite set $F \subseteq t^+$ and $\varepsilon > 0$ such that $f(s) > f(t) + \varepsilon$ for every $s \in t^+ \setminus S$. As one can easily guess, a point is *bad* for f if it is not good.

Proposition 3.5. *Assume T is an $\mathbb{R}$-embeddable tree and $\mathcal{C}_0(T)$ is σ-fragmentable. Then T is $\mathbb{Q}$-embeddable.*

Proof. Let $f\colon T \to \mathbb{R}$ be a $<$-preserving map and let $g\colon T \to \mathbb{R}$ be a $\leqslant$-increasing function with no bad points (which exists by Haydon's result [2, Thm. 6.1]). Then $h = f + g$ is a $<$-preserving map with the property that for every $t \in T$ there is $\varepsilon(t) > 0$ such that

$$h(t) + \varepsilon(t) \leqslant h(s) \qquad \text{whenever} \quad s > t.$$

Let T_n be the set of all $t \in T$ such that $\varepsilon(t) \geqslant 1/n$. We claim that $\langle T_n, < \rangle$ is a tree of height $\leqslant \omega$. Indeed, if $\{t_\alpha\colon \alpha \leqslant \omega\}$ were strictly increasing in T_n then $h(t_k) \geqslant h(t_0) + k/n$ for every $k < \omega$ and therefore $h(t_\omega)$ would not be a real number. It follows that each T_n is a countable union of antichains, which shows that T is $\mathbb{Q}$-embeddable. □

Let us note that $\mathcal{C}_0(T)$ has a Kadec renorming whenever T is a special tree. Indeed, given a $<$-preserving function $f\colon T \to \mathbb{Q}$, one can take an order preserving embedding $h\colon \mathbb{Q} \to C$, where $C \subseteq \mathbb{R}$ is a Cantor set, such that $h[\mathbb{Q}]$ consists of all points of C which are isolated from the right. Then hf is $<$-increasing with no bad points, therefore by a theorem of Haydon [2], $\mathcal{C}_0(T)$ has a Kadec renorming.

From Corollary 3.4 we obtain

Corollary 3.6. *There exists a 2-fibered Rosenthal compact whose space of continuous functions is not σ-fragmentable and, in particular, does not have any equivalent Kadec norm.*

We are now going to construct a tree, which results a 3-determined Rosenthal compact, not a continuous image of any first countable Rosenthal compact. The tree will be a certain S_2-expansion of $\sigma\mathbb{Q}$, where $S_2 = \{\emptyset, 0, 1\}$ is the unique 3-element binary tree, i.e. $\emptyset < 0$, $\emptyset < 1$ and $0, 1$ are incomparable. Given $t \in \sigma\mathbb{Q}$, choose $r \in \mathbb{Q}$ so that $\sup t < r$ and define intervals $I_0(t) = [\sup t, r)$, $I_1(t) = [r, +\infty)$. Additionally, let $I_\emptyset = [\sup t, +\infty)$. Recall that every $u \in t^+$ is of the form $u = t \cup \{x\}$, where x is a rational number from the interval $I_\emptyset(t)$. Actually, $\{I_s(t)\}_{s \in S_2}$ is a tree of real intervals indexed by S_2. Now define

$$D_s(t) = \{t \cup \{x\} \in t^+ \colon x \in I_s(t)\}.$$

Finally, let $T_1 = w\mathbb{Q} \cup T'$, where T' is the S_2-expansion of $\sigma\mathbb{Q}$ with respect to D.

Theorem 3.7. *There exists a scattered Rosenthal compact K_1 satisfying the following conditions.*

(I) K_1 is 3-determined, not 2-determined.

(II) $\mathcal{C}(K_1)$ is not σ-fragmentable.

Proof. Let $K_1 = \alpha T_1$, where T_1 is the above tree. We first show that K_1 is Rosenthal compact. Recall that T_1 is represented as $w\mathbb{Q} \cup (\sigma\mathbb{Q} \times \{0,1\})$, where $t < \langle t, i \rangle < u$ whenever $t \in \sigma\mathbb{Q}$, $i \in \{0,1\}$ and $u \in t^+$. Define, as in Subsection 2.5, $A_t = \{x \in \mathcal{P}(\mathbb{Q}) \colon t \sqsubseteq x\}$. Then $\{A_t\}_{t \in w\mathbb{Q}}$ is a tree of closed subsets of the Cantor set $\mathcal{P}(\mathbb{Q})$.

Given $t \in \sigma\mathbb{Q}$, $i \in \{0,1\}$, define

$$A_{\langle t,i \rangle} = \{x \in A_t \colon \inf(x \setminus t) \in I_i(t)\}.$$

Observe that $A_{\langle t,1 \rangle}$ is a closed set and $A_{\langle t,0 \rangle} = A_t \setminus A_{\langle t,1 \rangle}$ is relatively open in A_t. It follows that both sets are at the same time F_σ and G_δ in $\mathcal{P}(\mathbb{Q})$. In particular, the characteristic functions of the sets A_t, $t \in T_1$ are of the first Baire class and $A_t \neq A_r$ if $t \neq r$. In order to show that K_1 is Rosenthal compact, by Proposition 2.3, it suffices to check that $\{A_t\}_{t \in T_1}$ is indeed a (proper) tree of sets. It is clear that $A_t \cap A_r = \emptyset$ whenever $t \neq r$ and $A_t \supseteq A_r$ whenever $t \leqslant r$. Given a strictly increasing sequence $\{u_n\}_{n \in \omega} \subseteq T_1 \setminus w\mathbb{Q}$, one can find $\{t_n\}_{n \in \omega} \subseteq \sigma\mathbb{Q}$ such that $u_n < t_n < u_{n+1}$ for every $n \in \omega$. Thus, conditions (5) and (6) in the definition of the tree of sets follow from the fact that $\{A_t\}_{t \in w\mathbb{Q}}$ is already a tree of sets.

We now show (I). Let $f \colon w\mathbb{Q} \to \mathbb{R}$ be strictly $\sqsubseteq$-preserving and define $g(t) = \langle f(t), 0 \rangle$ for $t \in w\mathbb{Q}$, $g(t,i) = \langle f(t), 1 \rangle$ for $\langle t, i \rangle \in \sigma\mathbb{Q} \times \{0,1\}$. Then $g \colon T_1 \to \mathbb{R} \cdot \{0,1\}$ is $<$-preserving, showing that T_1 is $(\mathbb{R} \cdot \{0,1\})$-embeddable. By Theorem 3.2, K_1 is 3-determined. In order to show that K_1 is not 2-determined, by Theorem 3.1, it suffices to show that T_1 is not $\mathbb{R}$-embeddable. Suppose $f \colon T_1 \to \mathbb{R}$ is $<$-preserving and given $t \in \sigma\mathbb{Q}$ let

$$\varepsilon(t) = \min_{i=0,1} |f(t,i) - f(t)|.$$

Then $\varepsilon(t) > 0$, so choose $r(t) \in \mathbb{Q}$ satisfying $f(t) < r(t) < f(t) + \varepsilon(t)$. Then $r \colon \sigma\mathbb{Q} \to \mathbb{Q}$ is $<$-preserving. On the other hand, $\sigma\mathbb{Q}$ is not $\mathbb{Q}$-embeddable, according to Kurepa's theorem [5]. This shows (I).

Finally, suppose that $\mathcal{C}(K_1)$ is σ-fragmentable. Since $\mathcal{C}_0(T_1) \cong \mathcal{C}(K_1)$, by Haydon's result [2, Thm. 6.1], there exists a $\leqslant$-preserving function $f_1 \colon T_1 \to \mathbb{R}$ with no bad points. Since S_2 is a finite tree, the restriction $f \restriction \sigma\mathbb{Q}$ has no bad points either. But, $\sigma\mathbb{Q}$ does not possess such a function, by the proof of Proposition 3.5. This contradiction shows (II). $\square$

A modification of the above tree expansion gives another example of a Rosenthal compact, showing that a positive renorming property does not imply 2-determination.

Theorem 3.8. *There exists a scattered Rosenthal compact K_2 satisfying the following conditions.*

(I) K_2 is 3-determined, not 2-determined.

(II) $\mathcal{C}(K_2)$ has an equivalent Kadec norm.

Proof. Let $S_\omega = 2^{<\omega}$ be the Cantor tree. We shall construct a tree $T_2 = w\mathbb{Q} \cup T'$, where T' is an S_ω-expansion of $\sigma\mathbb{Q}$ with respect to D defined below. As before, let $D_\emptyset(t) = t^+$. Let $\{I_s(t)\}_{s \in S_\omega}$ be a fixed Cantor tree of subsets of the interval $[\sup t, +\infty)$, such that each $I_s(t)$ is of the form $[a, b)$ and

$$I_s(t) = I_{s^\frown 0}(t) \cup I_{s^\frown 1}(t)$$

for every $s \in S_\omega$, $i = 0, 1$, where $^\frown$ denotes the usual concatenation of sequences. Additionally, assume that $I_s(t)$ is below $I_r(t)$ whenever s is lexicographically below r in the tree S_ω. Finally, define

$$D_s(t) = \{t \cup \{x\} \in t^+ \colon x \in I_s(t)\}.$$

This finishes the definition of the tree T_2.

Let $K_2 = \alpha T_2$. Let $\{A_t\}_{t \in w\mathbb{Q}}$ be as before and define

$$A_{\langle t,s \rangle} = \{x \in A_t \colon \inf(x \setminus t) \in I_s(t)\}.$$

Notice that $A_{\langle t,s^\frown 1\rangle}$ is closed in $A_{\langle t,s\rangle}$ and $A_{\langle t,s^\frown 0\rangle} = A_{\langle t,s\rangle} \setminus A_{\langle t,s^\frown 1\rangle}$. This implies that all the sets $A_{\langle t,s\rangle}$ are simultaneously $F\sigma$ and G_δ. A similar argument as in the proof of Theorem 3.7 shows that $\{A_t\}_{t \in T_2}$ is a proper tree of sets, which shows that K_2 is representable as a space of the first Baire class functions on the Cantor set $\mathcal{P}(\mathbb{Q})$ (the characteristic functions of A_t, $t \in T_2$ plus the constant zero function).

Properties (I) is proved like in Theorem 3.7. The only difference is that the tree T_2 is $(\mathbb{R} \cdot \mathbb{N})$-embeddable, not $(\mathbb{R} \cdot \{0, 1\})$-embeddable.

In order to show (II), notice that the tree T_2 is binary, therefore the constant zero function $f \colon T_2 \to \mathbb{R}$ has no bad points. By Haydon's theorem [2, Thm. 6.1], we conclude that $\mathcal{C}(K_2) \cong \mathcal{C}_0(T_2)$ has a Kadec renorming. □

Let us note in closing that our examples of trees give spaces of continuous functions which have rotund renormings. This is because of another result of Haydon [2]. On the other hand, the paper [2] contains a very similar to T_1 construction of a tree Υ such that $\mathcal{C}_0(\Upsilon)$ fails to have a rotund renorming. To be more precise, Υ is an S_2-expansion of the tree

$$\Gamma = \{t \in \omega^{<\omega_1} \colon t \text{ is one-to-one and } |\omega \setminus \operatorname{rng}(t)| = \aleph_0\}.$$

The order is inclusion or, in other words, extension of functions. The tree Γ is easily seen to be $\mathbb{R}$-embeddable: the function $h(t) = \sum_{n \in \operatorname{rng}(t)} 2^{-n}$ is strictly order preserving. Moreover, it is not hard to prove that Γ contains an isomorphic copy of $\sigma\mathbb{Q}$. On the other hand, the spaces $\alpha\Gamma$ and $\alpha\sigma\mathbb{Q}$ are not homeomorphic, because the existence of a rotund renorming distinguishes their spaces of continuous functions.

Note that $\alpha\Gamma$ is a 2-determined Rosenthal compact. We do not know whether $\alpha\Upsilon$ is Rosenthal compact.

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