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Bernard Ducomet, Šárka Nečasová

Free boundary problem for the equations of spherically symmetric motion of compressible gas with density-dependent viscosity

SUMMARY

We consider a free boundary problem for the equations of spherically symmetric motion of a isentropic gas with a density-dependent viscosity $\mu(\eta) \geqslant \underline{\mu}\eta^{-\lambda}$, where $\underline{\mu}$ and λ are positive constants. We prove that the problem admits a weak solution provided that $0 < \lambda < 1/4$.

Keywords: compressible heat conduction fluids, spherically symmetry, density dependent viscosity

AMS subject classification: 35Q30, 76N10

1 Introduction

We consider the following model of compressible Navier-Stokes system for a spherical symmetric flow in lagrangian (mass) coordinates

$$\begin{cases} \eta_t = (r^2 v)_x, \\ v_t = r^2 \left(-p + \dfrac{\mu}{\eta}(r^2 v)_x\right)_x + f(r,x), \\ e_t = \pi_x + \left(-p + \dfrac{\mu}{\eta}(r^2 v)_x\right)(r^2 v)_x, \\ r_t = v, \end{cases} \tag{1}$$

in the domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$, where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radius r depend on the lagrangian mass coordinates (x,t), with

$$r(x,t) := r_0(x) + \int_0^t v(x,s)\ ds, \tag{2}$$

where

$$r_0(x) := \left[R_0^3 + 3\int_0^x \eta^0(y)\ dy \right]^{1/3}, \quad \text{for } x \in \Omega.$$

The stress σ is given by

$$\sigma(\eta, \theta) := -p(\eta, \theta) + \frac{\mu(\eta)}{\eta}(r^2 v)_x.$$

In order to simplify the exposition, we assume in all the following that the bulk viscosity coefficient ν is zero.

We take the perfect gas law $p := \frac{R\theta}{\eta}$ for the pressure and $e := c_V \theta$ for the internal energy and π is the heat flux $\pi(\eta, \theta) := \frac{\kappa(\eta,\theta) r^4}{\eta} \theta_x$.

We consider a free boundary problem for (1): the motion is supposed to take place in a domain Ω surrounding a fixed ball (modelling an inert hard core) with radius $R_0 := r_0(0)$, and the external surface of Ω is free.

So we consider the boundary conditions

$$\begin{cases} v|_{x=0} = 0, \quad \pi|_{x=0} = 0, \\ -p(\eta,\theta) + \dfrac{\mu(\eta)}{\eta}(r^2 v)_x \Big|_{x=M} = 0, \quad \pi|_{x=M} = 0, \end{cases} \tag{3}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad r|_{t=0} = r^0(x), \quad \theta|_{t=0} = \theta^0(x) \qquad \text{on } \Omega. \tag{4}$$

The mass force f has the form

$$f(r, x) = -G\frac{M_0 + j_0 x}{r^2},$$

with $G > 0$, $M_0 \geqslant 0$, and $j_0 = 0$ or 1. The case $j_0 = 1$ corresponds to a selfgravitating fluid, the simpler case $j_0 = 0$ supposes that selfgravitation is neglected and only the newtonian attraction by an effective central mass M_0 is taken into account.

The viscosity coefficient $\mu(\eta)$ supposed to be continuous on $\mathbf{R}^+$, is such that $\mu' \in L^\infty_{loc}(\mathbf{R}^+)$ and satisfy the classical thermodynamic requirement

$$0 \leqslant \mu(s) \ \text{ for } s > 0, \tag{5}$$

together with the conditions

$$\frac{d}{d\xi}\mu(\xi) \leqslant 0, \quad \mu(\xi)\xi^\lambda \geqslant \underline{\mu} \ \text{ and } \lim_{\epsilon \to 0} \int_\epsilon^a \frac{\mu(\xi)}{\xi}\, d\xi = \infty \ \text{ for any } a > 0, \tag{6}$$

where the positive constant λ satifies

$$0 < \lambda < 1/4.$$

The thermal conductivity satisfies the inequalities

$$\underline{\kappa}(1+\theta^q) \leqslant \kappa(\eta,\theta) \leqslant \overline{\kappa}(1+\theta^q) \quad \text{for } q \geqslant 1. \tag{7}$$

We study weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \quad \eta_t \in L^\infty([0,T],L^2(\Omega)), \quad \sqrt{\rho}\,(r^2v)_x \in L^\infty([0,T],L^2(\Omega)), \\ v \in L^\infty([0,T],L^4(\Omega)), \quad v_t \in L^\infty([0,T],L^2(\Omega)),\ \sigma_x \in L^\infty([0,T],L^2(\Omega)), \\ \qquad \theta \in L^\infty([0,T],L^2(\Omega)), \quad \sqrt{\rho}\,\theta_x \in L^\infty([0,T],L^2(\Omega)). \end{cases} \tag{8}$$

and

$$r \in C(Q) \ \text{ and for all } t \in [0,T], x \to r(x,t) \text{ is strictly increasing on } \Omega, \tag{9}$$

where $Q_T := \Omega \times (0,T)$.

We also assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \text{ on } \Omega,\ \eta^0 \in L^1(\Omega), \\ v_0 \in L^2(\Omega), \quad \sqrt{\rho^0}\, v_x^0 \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \quad \inf_\Omega \theta^0 \geqslant 0. \end{cases} \tag{10}$$

In the last decades, significant progress on the compressible Navier-Stokes system or Navier-Stokes-Fourier system with positive constant viscosity coefficients has been achieved by many authors. In one dimension, it is well known that global solutions exist for large initial data and are time-asymptotically stable. In dimension greater than one, Matsumura and Nishida proved the existence of global smooth solutions and obtained the decay rates of solutions for sufficiently small initial data (see [21, 22, 23]). For large initial data the global existence and large-time behavior of solutions to the Navier-Stokes-Fourier system have also been obtained in the spherically symmetric case (see [11, 10, 9]). Concerning the global existence for general large initial data in general domains the first fundamental work in the case of isentropic fluids was done by P.L. Lions [24] and then extended by Feireisl [6]. Completely new theory for the full Navier-Stokes-Fourier system was studied by Feireisl [7]. Taking into account some physical considerations, Liu, Xin and Yang [15] introduced modified compressible Navier-Stokes equations with density dependent viscosity coefficients for isentropic fluids. For one-dimensional compressible Navier-Stokes equations, or in the spherically symmetric case there exists a very abundant literature [13, 14, 15, 18, 25, 26, 27]. For multi-dimensional case we can finally mention works by Bresch and Desjardin [2, 3], the article of Mellet and Vasseur [16] and the work by Feireisl [8].

1.1 Formulation of the problem and main result

Definition 1.1. *We call* (η, v, θ) *a weak solution of (1) if it satisfies*

$$\eta(x,t) = \eta^0(x) + \int_0^t \left(r^2 v_x + \frac{2\eta v}{r}\right)(x,s)\ ds, \tag{11}$$

for a.e. $x \in \Omega$ *and any* $t > 0$*, and if, for any test function* $\phi \in L^2([0,T], H^1(\Omega))$ *with* $\phi_t \in L^1([0,T], L^2(\Omega))$ *such that* $\phi(\cdot, T) = 0$*, one has*

$$\int_Q \left[\phi_t v + \left(r^2\phi_x + \frac{2\eta\phi}{r}\right) p - \frac{\mu\phi_x r^4}{\eta} v_x - 2\mu\frac{\phi\eta v}{r^2} + \phi f\right]\ dx\ dt$$
$$= \int_\Omega \phi(0,x)\ v^0(x)\ dx, \tag{12}$$

and

$$\int_Q \left[\phi_t e + \frac{\kappa r^4\theta_x}{\eta}\phi_x - r^2 v\sigma\phi_x - r^2 v\sigma_x\phi\right]\ dx\ dt$$
$$= \int_\Omega \phi(0,x)\ \theta^0(x)\ dx. \tag{13}$$

Then our main result is the following

Theorem 1. *Suppose that the initial data satisfy (10) and that* T *is an arbitrary positive number.*

Then the problem (1)(3)(4) possesses a global weak solution satisfying (8) and (9) together with properties (11), (12) and (13).

Moreover one has the uniqueness result

Theorem 2. *Suppose that the initial data satisfy (10) and that* T *is an arbitrary positive number.*

Then the problem (1)(3)(4) possesses a global unique weak solution satisfying (8) and (9) together with properties (11), (12) and (13).

2 Proof of the existence

In the spirit of [9], we first suppose that the solution is classical in the following sense

$$\left\{\begin{array}{c} \eta \in C^1(Q_T), \quad \rho > 0, \\ \\ v, \theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)). \end{array}\right. \tag{14}$$

and

$$r > 0 \ \text{ for all } t \in [0,T], \quad r(M,t) < \infty. \tag{15}$$

Let us introduce the primitive function

$$F(r,x) := -G\left(\frac{1}{r_0} - \frac{1}{r}\right)(M_0 + j_0 x),$$

and the auxiliary function $\Phi(s) = s - \log s - 1$ for $s > 0$.

Let N be an arbitrary positive number, and let $K = K(N)$, $K_i = K_i(N)$, $i = 0, 1, ...$, be positive non-decreasing functions of N which may possibly depend on the physical constants of the problem G, M_0, M etc.

Lemma 1. *Under the following condition on the data*

$$\|v^0\|_{L^2(\Omega)} + \|\eta^0\|_{L^1(\Omega)} + \|\theta^0\|_{L^1(\Omega)} \leqslant N, \tag{16}$$

1. *The following energy equality holds*

$$\int_\Omega \left[\frac{1}{2}v^2 + e - F(r,x)\right] \, dx = \int_\Omega \left[\frac{1}{2}(v^0)^2 + e^0 - F(r^0,x)\right] \, dx. \tag{17}$$

2. *The following entropy inequality holds*

$$\int_0^T \int_\Omega \left(\frac{\kappa(\eta,\theta)r^4}{\eta\theta^2}\ \theta_x^2 + \frac{\mu(\eta)}{\eta\theta}\ [(r^2v)_x]^2\right) dx\, dt \leqslant K(N) + R\int_\Omega \eta\, dx. \tag{18}$$

3. *The following estimate holds*

$$\|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^1(\Omega))} \leqslant K(N). \tag{19}$$

Proof. 1. Multiplying the second equation (1) by v, adding the result to the third equation (1), integrating by part using (3), (4) and the relations $\frac{dF}{dt} = f$ and $r_t = v$, one gets the energy identity (17).

2. Computing the time-derivative s_t of the entropy $s := c_V \log\theta + R\log\eta$, and using (17), we get

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}v^2 + e - F(r,x) - s\right) dx = \int_\Omega \left(\frac{\kappa(\eta,\theta)r^4}{\eta\theta^2}\ \theta_x^2 + \frac{\mu(\eta)}{\eta\theta}\ [(r^2v)_x]^2\right) dx,$$

which implies (18).

3. The estimate (19) follows from (17) $\square$

Lemma 2. *Under the previous condition on the data, there exists a positive constant $\underline{\eta}$ depending on T and N such that*

$$\underline{\eta} \leqslant \eta(x,t) \quad \textit{for } (t,x) \in Q_T. \tag{20}$$

Proof. From the second relation (1)

$$(\mathcal{M})_t = p(\eta,\theta) - \frac{d}{dt}\int_x^M \frac{v}{r^2}\ dy - \int_x^M \frac{2v^2}{r^3}\ dy + \int_x^M \frac{f(r,y)}{r^2}\ dy,$$

with $\mathcal{M}(\xi) := \int_{\eta_0}^{\xi} \frac{\mu(s)}{s}\ ds$, where $\eta_0 = \inf_\Omega \eta^0$.

Integrating between 0 and $t > 0$ we get

$$\mathcal{M}(\eta) - \mathcal{M}(\eta^0) = \int_0^t p\ ds - \int_x^M \frac{v}{r^2}\ dy - \int_0^t \int_x^M \frac{2v^2}{r^3}\ dy\ ds$$

$$+ \int_0^t \int_x^M \frac{f(r,y)}{r^2}\ dy\ ds + \int_x^M \frac{v^0}{r^{0^2}}\ dy.$$

Using Cauchy-Schwarz inequality and (17)

$$\mathcal{M}(\eta) > -\int_0^M \frac{|v|}{r^2}\ dy - \int_0^t \int_0^M \frac{2v^2}{r^3}\ dy\ ds - G\int_0^t \int_x^M \frac{M_0 + M}{r^4}\ dy\ ds$$

$$-\int_x^M \frac{|v^0|}{r^{0^2}}\ dy > -K(N,T).$$

As $\xi \to \mathcal{M}(\xi)$ is increasing and maps $(0,\eta_0)$ on $(-\infty,0)$, (20) is proved □

Lemma 3. *Under the previous condition on the data, there exists a positive constant K depending on T and N such that*

$$\int_\Omega \eta^{1-\lambda}(x,t)\ dx \leqslant K(N,T)\ \textit{for}\ t \in (0,T). \tag{21}$$

Proof. From the second relation (1)

$$-\sigma = \int_x^M \frac{v_t - f}{r^2}\ dy,$$

then

$$\frac{d}{dt}\int_{\underline{\eta}}^{\eta} \mu(\xi)\ d\xi = p(\eta,\theta)\eta - \eta\int_x^M \frac{v_t - f}{r^2}\ dy$$

$$= R\theta + \eta\int_x^M \left(\frac{2v^2}{r^3} + \frac{f}{r^2}\right)\ dy - \eta\frac{d}{dt}\int_x^M \frac{v}{r^2}\ dy,$$

or, integrating on Q_T

$$\int_{Q_T} \frac{d}{dt}\int_{\underline{\eta}}^{\eta} \mu(\xi)\ d\xi\ dx\ ds$$

$$= R\int_{Q_T} \theta\ dx\ ds + \int_{Q_T} \eta\int_x^M \left(\frac{2v^2}{r^3} + \frac{f}{r^2}\right)\ dy\ dx\ ds - \int_{Q_T} \eta\frac{d}{ds}\int_x^M \frac{v}{r^2}\ dy\ dx\ ds. \tag{22}$$

Integrating by parts in the last term gives

$$\int_0^t \int_\Omega \eta \frac{d}{ds} \int_x^M \frac{v}{r^2}\, dy\, dx\, ds = \int_\Omega \eta \int_x^M \frac{v}{r^2}\, dy\, dx - \int_\Omega \eta^0 \int_x^M \frac{v^0}{{r^0}^2}\, dy\, dx$$

$$- \int_0^t \int_\Omega (r^2 v)_x \int_x^M \frac{v}{r^2}\, dy\, dx\, ds = \int_\Omega \eta \int_x^M \frac{v}{r^2}\, dy\, dx - \int_\Omega \eta^0 \int_x^M \frac{v^0}{{r^0}^2}\, dy\, dx$$

$$- \int_0^t \int_\Omega v^2\, dx\, ds.$$

Plugging into (22), we get

$$\int_{Q_T} \frac{d}{dt} \int_{\underline{\eta}}^{\eta} \mu(\xi)\, d\xi\, dx\, ds = R \int_{Q_T} \theta\, dx\, ds + \int_{Q_T} \eta \int_x^M \left(\frac{2v^2}{r^3} - \frac{v}{r^2} + \frac{f}{r^2} \right) dy\, dx\, ds$$

$$+ \int_\Omega \eta^0 \int_x^M \frac{v^0}{{r^0}^2}\, dy\, dx + \int_0^t \int_\Omega v^2\, dx\, ds.$$

As the formula $r_x = \frac{\eta}{r^2}$ rewrites $\eta = \left(\frac{r^3}{3}\right)_x$, we can integrate by parts the second contribution in the right-hand side

$$\int_{Q_T} \eta \int_x^M \left(\frac{2v^2}{r^3} - \frac{v}{r^2} + \frac{f}{r^2} \right) dy\, dx\, ds = \frac{1}{3} \int_0^t \int_\Omega (r^3 - R_0^3) \left(\frac{2v^2}{r^3} - \frac{v}{r^2} + \frac{f}{r^2} \right) dy\, dx\, ds.$$

Then

$$\int_{Q_T} \frac{d}{dt} \int_{\underline{\eta}}^{\eta} \mu(\xi)\, d\xi\, dx\, ds = R \int_{Q_T} \theta\, dx\, ds + \frac{1}{3} \int_0^t \int_\Omega (r^3 - R_0^3) \left(\frac{2v^2}{r^3} - \frac{v}{r^2} + \frac{f}{r^2} \right) dy\, dx\, ds$$

$$+ \int_\Omega \eta^0 \int_x^M \frac{v^0}{{r^0}^2}\, dy\, dx + \int_0^t \int_\Omega v^2\, dx\, ds. \tag{23}$$

We bound the right-hand side by

$$R \int_{Q_T} \theta\, dx\, ds + \frac{1}{3} \int_0^t \int_\Omega \left(2v^2 - rv + rf\right)\, dx\, ds + \frac{1}{3} \int_0^t \int_\Omega \left(2v^2 + R_0|v| + R_0|f|\right)\, dx\, ds$$

$$+ \int_\Omega \eta^0 \int_x^M \frac{v^0}{{r^0}^2}\, dy\, dx + \int_0^t \int_\Omega v^2\, dx\, ds.$$

Using (17), this quantity is bounded from above by

$$K_1(N,T) - \frac{1}{3} \int_0^t \int_\Omega (rv - rf)\, dx\, ds$$

$$= K_1(N,T) - \frac{1}{3} \int_0^t \int_\Omega \left(\frac{r^2}{2}\right)_t dx\, ds - \frac{1}{3} \int_0^t \int_\Omega G \frac{M_0 + j_0 x}{r}\, dx\, ds$$

$$\leqslant K_1(N,T) + M\frac{R_0^2}{6} := K_2(N,T). \tag{24}$$

Now from (6) we get the lower bound

$$\int_{\underline{\eta}}^{\eta} \mu(\xi)\ d\xi \geqslant \mu_0 \int_{\underline{\eta}}^{\eta} \xi^{-\lambda}\ d\xi \geqslant \frac{\mu_0}{1-\lambda}\eta^{1-\lambda} - K_3(N,T). \tag{25}$$

Finally, plugging (24) and (25) into (23), we obtain (21) $\square$

Lemma 4. *Under the previous condition on the data, there exists a positive constant $\overline{\theta}$ depending on T and N such that*

$$\underline{\theta} \leqslant \theta(x,t) \quad \textit{for } (t,x) \in Q_T. \tag{26}$$

Proof. Multiplying the third relation (1) by θ^{-2} one gets

$$c_V\left(\frac{1}{\theta}\right)_t = \left(r^4\frac{\kappa}{\eta}\left(\frac{1}{\theta}\right)_x\right)_x - \frac{\mu}{\eta}\left(\frac{1}{\theta}(r^2v)_x - \frac{R}{\mu}\right)^2 + \frac{R^2}{\eta\mu}.$$

Then, after lemma 2 and property (6)

$$c_V\left(\frac{1}{\theta}\right)_t \leqslant \left(r^4\frac{\kappa}{\eta}\left(\frac{1}{\theta}\right)_x\right)_x + \frac{R^2}{\underline{\eta}^{1-\lambda}\underline{\mu}}.$$

Using the maximum principle, this implies that

$$\max_{Q_T}\frac{1}{\theta} \leqslant K(N,T) := \max_{\Omega}\frac{1}{\theta^0} + \frac{R^2}{c_V\underline{\eta}^{1-\lambda}\underline{\mu}}T,$$

which gives (26) $\square$

Now defining the positive function

$$M_\theta(t) := \max_{\Omega}\theta(x,t) \quad \text{for any } 0 \leqslant t < T,$$

one get also an integrated (density-dependent) upper bound for θ.

Lemma 5. *Under the previous condition on the data, there exists a positive constant K depending on T and N such that*

$$M_\theta(t) \leqslant K(N,T)\left(1 + \max_{\Omega}\eta^\lambda(x,t)\right) \quad \textit{for } t < T. \tag{27}$$

Proof. As the mean value of $\psi(x,t) := \theta(x,t) - \frac{1}{M}\int_\Omega \theta(y,t)\ dy$ on Ω is zero there exists a $y(t)$ in Ω such that $\psi(y) := \psi(y(t),t) = 0$.

Then we have

$$\psi(x) \leqslant K + \int_{y(t)}^{x} |\psi_z| dz.$$

$$\leqslant \frac{1}{2}\int_\Omega \frac{\kappa r^4}{\theta^2\eta}\ \theta_x^2\ dx + \frac{1}{2}\int_\Omega \frac{\theta^2\eta}{\kappa r^4}\ dx \leqslant \frac{1}{2}\int_\Omega \frac{\kappa r^4}{\theta^2\eta}\ dx + K\int_\Omega \theta\eta\ dx.$$

After lemma 1 and 3, we get (27) $\square$

Lemma 6. *Under the previous condition on the data, there exists a positive constant* $\overline{\eta}$ *depending on* T *and* N *such that*

$$\overline{\eta} \geqslant \eta(x,t) \quad for\ (t,x) \in Q_T. \tag{28}$$

Proof. From the second relation (1) one gets

$$\frac{v_t}{r^2} - \frac{f}{r^2} + p_x = \mathcal{M}_{tx}.$$

Then

$$\left(\mathcal{M}_x - \frac{v}{r^2}\right)_t = p_x + 2\frac{v^2}{r^3} - \frac{f}{r^2}.$$

Multiplying by $\mathcal{M}_x - \frac{v}{r^2}$ and integrating on Q_T, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx \leqslant \int_\Omega p_x \left(\mathcal{M}_x - \frac{v}{r^2}\right) dx + \int_\Omega \left(2\frac{v^2}{r^3} - \frac{f}{r^2}\right)\left(\mathcal{M}_x - \frac{v}{r^2}\right) dx$$

$$:= \mathcal{A}_1 + \mathcal{A}_2.$$

Let us estimate all of the contributions in the right-hand size

$$\mathcal{A}_1 = \int_\Omega \left(-R\frac{\theta}{\eta^2}\eta_x + \frac{R\theta_x}{\eta}\right)\left(\mathcal{M}_x - \frac{v}{r^2}\right) dx$$

$$= -R\int_\Omega \frac{\theta}{\eta\mu(\eta)}\,\mathcal{M}_x\left(\mathcal{M}_x - \frac{v}{r^2}\right) dx + R\int_\Omega \frac{\theta_x}{\eta}\left(\mathcal{M}_x - \frac{v}{r^2}\right) dx := \mathcal{A}_{11} + \mathcal{A}_{12}.$$

One gets

$$\mathcal{A}_{11} = -R\int_\Omega \frac{\theta}{\eta\mu(\eta)}\,\mathcal{M}_x^2\, dx + R\int_\Omega \frac{\theta}{r^2\eta\mu(\eta)}\,\mathcal{M}_x\, v\, dx$$

$$\leqslant -\frac{R}{2}\int_\Omega \frac{\theta}{\eta\mu(\eta)}\,\mathcal{M}_x^2\, dx + K\max_\Omega \frac{\theta}{\eta\mu(\eta)},$$

and

$$\mathcal{A}_{12} = R\int_\Omega \frac{\theta_x\mathcal{M}_x}{\eta}\, dx + R\int_\Omega \frac{\theta_x v}{r^2\eta}\, dx$$

$$\leqslant \frac{R}{2}\int_\Omega \frac{\theta}{\eta\mu(\eta)}\,\mathcal{M}_x^2\, dx + \frac{R}{2R_0^2}\int_\Omega \frac{\kappa r^2}{\eta\theta^2}\theta_x^2\,\frac{\theta}{\kappa}\, dx + \frac{R}{2}\int_\Omega \frac{\kappa r^2}{\eta\theta^2}\theta_x^2\,\frac{\theta}{\kappa}\, dx + \frac{R}{2}\int_\Omega \frac{\theta^2 v^2}{\kappa\eta r^6}\, dx.$$

Using the bounds (7) assumed for κ and lemma 2 we get

$$\mathcal{A}_{12} \leqslant \frac{R}{4}\int_\Omega \frac{\theta}{\eta\mu(\eta)}\,\mathcal{M}_x^2\, dx + K\max_\Omega \eta^\lambda + K\max_\Omega \theta(x,t).$$

In the same manner, we have

$$\mathcal{A}_2 \leqslant 2\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx + 2\int_\Omega \left(2\frac{v^2}{r^3} - \frac{f}{r^2}\right)^2 dx.$$

In order to bound the last term, we observe that

$$(r^2v)^2 \leqslant \int_0^x 2r^2|v\ (r^2v)_x|\ dy \leqslant \frac{1}{2}\int_0^x \frac{\mu r^4}{\eta} v^2[(r^2v)_x]^2 dy + \frac{1}{2}\int_0^x \frac{\eta^{1+\lambda}}{r^4} dy.$$

So

$$\frac{v^2}{r^6} \leqslant K\frac{1}{2}\int_\Omega \frac{\mu r^4}{\eta} v^2[(r^2v)_x]^2 dy + K\max_\Omega \eta^\lambda\ \frac{1}{r^3}\int_0^x \eta\ dy,$$

and then

$$\begin{aligned}\int_\Omega \frac{v^4}{r^6}\ dx &\leqslant \int_\Omega v^2 \left(K\int_\Omega \frac{\mu r^4}{\eta} v^2[(r^2v)_x]^2 dy + K\max_\Omega \eta^\lambda\right)\ dx \\ &\leqslant K\int_\Omega \frac{\mu r^4}{\eta} v^2[(r^2v)_x]^2 dy + K\max_\Omega \eta^\lambda.\end{aligned}$$

As clearly $\int_\Omega \frac{f^2}{r^4}\ dx \leqslant K$, we obtain by using Cauchy-Schwarz inequality and lemma 2 that

$$\mathcal{A}_2 \leqslant 2\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx + K(1+\max_\Omega \eta^\lambda).$$

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx \leqslant 2\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx + K(1+\max_\Omega \eta^\lambda) + K\max_\Omega \theta.$$

After lemma 5 one obtains the differential inequality

$$\frac{dY}{dt} \leqslant Y + K(1+\max_\Omega \eta^\lambda),$$

with $Y(t) := \frac{1}{2}\int_\Omega \left(\mathcal{M}_x - \frac{v}{r^2}\right)^2 dx$.

Integrating on $(0,t)$, we obtain finally the estimate

$$\int_\Omega \mathcal{M}_x^2 dx \leqslant K(1+\max_\Omega \eta^\lambda) \quad \text{for any } t \in [0,T].$$

Now we use the argument of [13]

$$\begin{aligned}\eta^{1-\lambda}(x,t) &\leqslant K + K\left(\int_\Omega \mathcal{M}_x^2 dx\right)^{1/2}\left(\int_\Omega \frac{\eta^{2(1-\lambda)}}{\mu^2(\eta)} dx\right)^{1/2} \\ &\leqslant K + K\left(\int_\Omega \eta^2 dx\right)^{1/2} \max_\Omega \eta^{\lambda/2} \leqslant K + K\max_\Omega \eta^{1/2+\lambda},\end{aligned}$$

and we end with the inequality

$$\eta^{1-\lambda}(x,t) \leqslant K + K\max_\Omega \eta^{1/2+\lambda}. \tag{29}$$

As, for any $0 < \lambda < 1/4$, we have $1-\lambda < 1/2+\lambda$, inequality (29) implies (28) $\square$

Following now the strategy of the paper [9], we first prove the existence of a strong solution. Then we will prove the existence of a weak solution.

Let us multiply the equation $(1)_2$ by $r^2\left(-p+\frac{\mu}{\eta}(r^2v)_x\right)_x$ and integrate on Ω, rewriting the term r^2v_t as

$$r^2v_t = ((r^{1/2}v)_t)r^{3/2} - \frac{1}{2}rv^2.$$

We get then

$$\int_0^1\left\{(r^{1/2}v)_t r^{3/2}\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)_x\right\}dx = -\int_0^1(r^2v)_{xt}\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)dx$$
$$+\tfrac{3}{2}\int_0^1(rv^2)_x\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)dx.$$

It implies that

$$-\int_0^1(r^2v)_{xt}\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)dx = -\tfrac{1}{2}\tfrac{d}{dt}\int_0^1\left(\partial_x(r^2v)\tfrac{\mu}{\eta}-p\right)^2dx$$
$$+\int_0^1\left\{\left((r^2v)_x\tfrac{\mu}{\eta}-p\right)\left(\left(\tfrac{\mu}{\eta}\right)_t(r^2v)_x-p_t\right)\right\}dx = -\tfrac{1}{2}\tfrac{d}{dt}\int_0^1\left((r^2v)_x\tfrac{\mu}{\eta}-p\right)^2dx$$
$$+\int_0^1\left\{\left((r^2v)_x\tfrac{\mu}{\eta}-p\right)\left(\left(\tfrac{\mu'}{\eta}+\tfrac{\mu}{\eta^2}\right)((r^2v)_x)^2-\tfrac{R\theta_t}{\eta}-\tfrac{R\theta(r^2v)_x}{\eta^2}\right)\right\}dx. \tag{30}$$

Then

$$\tfrac{1}{2}\tfrac{d}{dt}\int_0^1\left((r^2v)_x\tfrac{\mu}{\eta}-p\right)^2dx+\int_0^1\left[r^2\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)_x\right]^2dx$$
$$= -\int_0^1\left\{f(r,x)r^2\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)_x\right\}dx+\tfrac{3}{2}\int_0^1\left\{(rv^2)_x\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)\right\}dx$$
$$-\int_0^1\left\{\left((r^2v)_x\tfrac{\mu}{\eta}-p\right)\left[((r^2v)_x)^2\left(\tfrac{\mu'}{\eta}+\tfrac{\mu}{\eta^2}\right)-\tfrac{R\theta_t}{\eta}-\tfrac{R\theta(r^2v)_x}{\eta^2}\right]\right\}dx$$
$$-\int_0^1\left\{r^1v^2\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)\right\}dx. \tag{31}$$

Let us estimate the terms of the right hand side. One gets first

$$\int_0^1\left\{f(r,x)r^2\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)_x\right\}dx$$
$$\leqslant\left(\int_0^1 f(r,x)^2dx\right)^{1/2}\left(\int_0^1\left[r^2\left(-p+\tfrac{\mu}{\eta}(r^2v)_x\right)_x\right]^2dx\right)^{1/2}.$$

Now the second contribution reads

$$\begin{aligned}&\Big| \int_0^1 \tfrac{3}{2} \int_0^1 (rv^2)_x \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right) dx\Big| \leqslant \\ &\leqslant \left(\int_0^1 \left|(rv^2)_x\right|^2 dx\right)^{1/2} (\int_0^1 |\tfrac{\mu}{\eta}(r^2v)_x - p|^2 dx)^{1/2}.\end{aligned} \tag{32}$$

The third one leads to

$$\begin{aligned}&\Big| -\tfrac{1}{2}\int_0^1 rv^2 \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)_x dx\Big| = \Big| \int_0^1 \left[(rv^2)_x \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)\right] dx\Big| \\ &\leqslant \left(\int_0^1 \left|(rv^2)_x\right|^2 dx\right)^{1/2} \left(\int_0^1 \left|-p + \tfrac{\mu}{\eta}(r^2v)_x\right|^2 dx\right)^{1/2}.\end{aligned} \tag{33}$$

Then

$$\begin{aligned}&\Big| \int_0^1 \left\{\left((r^2v)_x \tfrac{\mu}{\eta} - p\right)\left([(r^2v)_x]^2 \left(\tfrac{\mu'}{\eta} + \tfrac{\mu}{\eta^2}\right) + \tfrac{R\theta(r^2v)_x}{\eta^2}\right)\right\} dx\Big| \\ &\leqslant c(\underline{\mu}, \underline{\eta}, N) \sup_{0\leqslant x\leqslant 1}(r^2v)_x \left[\int_0^1 \left|(r^2v)_x \tfrac{\mu}{\eta} - p\right|^2 dx\right]^{1/2} \left[\int_0^1 \left|(r^2v)_x\right|^2 dx\right]^{1/2} \\ &\leqslant c(\underline{\mu}, \underline{\eta}, N) \left(\int_0^1 [(r^2v)_{xx}]^2 dx\right)^{1/2} \left(\int_0^1 \left|(r^2v)_x \tfrac{\mu}{\eta} - p\right|^2 dx\right)^{1/2} \left(\int_0^1 [(r^2v)_x]^2 dx\right)^{1/2}.\end{aligned} \tag{34}$$

Finally

$$\begin{aligned}&\Big|\int_0^1 \tfrac{R\theta_t}{\eta} \left((r^2v)_x \tfrac{\mu}{\eta} - p\right) dx\Big| = \Big| \int_0^1 \tfrac{Rc_v}{\eta} \left[\pi_x + \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)(r^2v)_x\right] \left((r^2v)_x \tfrac{\mu}{\eta} - p\right) \Big| \\ &= \Big| \int_0^1 \left[\tfrac{Rc_v}{\eta} \left(\left(\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x\right)_x + \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)(r^2v)_x\right)\right] \left((r^2v)_x \tfrac{\mu}{\eta} - p\right) dx\Big| \\ &= \Big| \int_0^1 \left\{\left[\tfrac{Rc_v}{\eta} \left((r^2v)_x \tfrac{\mu}{\eta} - p\right)\right]_x (\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)\right\} dx + \int_0^1 \left\{\tfrac{Rc_v}{\eta} \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)^2 (r^2v)_x\right\} dx\Big| \\ &\leqslant \left(\int_0^1 \left|\tfrac{Rc_v}{\eta} \left((r^2v)_x \tfrac{\mu}{\eta} - p\right)_x\right|^2 dx\right)^{1/2} \left(\int_0^1 (\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)^2 dx\right)^{1/2} + \\ &+ | \int_0^1 \left\{[\tfrac{Rc_v}{\eta}]_x \left((r^2v)_x \tfrac{\mu}{\eta} - p\right) (\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)\right| + \\ &\left(\int_0^1 [(r^2v)_{xx}]^2 dx\right)^{1/2} \int_0^1 \tfrac{Rc_v}{\eta} \left(-p + \tfrac{\mu}{\eta}(r^2v)_x\right)^2 \; dx.\end{aligned} \tag{35}$$

$$\Big|\int_0^1 \Big\{[\tfrac{Rc_v}{\eta}]_x \Big((r^2v)_x\tfrac{\mu}{\eta}-p\Big)(\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)\Big|\Big\} = \Big|\int_0^1 \tfrac{Rc_v\eta_x}{\eta^2}\Big((r^2v)_x\tfrac{\mu}{\eta}-p\Big)(\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)\Big| =$$

$$|\int_0^1 \tfrac{\mathcal{M}_x}{\mu(\eta)\eta}\Big((r^2v)_x\tfrac{\mu}{\eta}-p\Big)(\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)\Big| \leqslant$$

$$\leqslant c(\bar\mu,\bar\eta,N)\sup_{0\le qx\leqslant 1}\Big((r^2v)_x\tfrac{\mu}{\eta}-p\Big)\{\int_0^1|\mathcal{M}_x\tfrac{1}{\mu(\eta)\eta}|^2dx\}^{1/2}\{\int_0^1|(\tfrac{\kappa(\eta,\theta)r^4}{\eta}\theta_x)|^2dx\}^{1/2}. \tag{36}$$

Then after some computation we find the following inequality

$$\frac{1}{2}\frac{d}{dt}\int_0^1 r^4\left(\frac{\mu}{\eta}(r^2v)_x-p\right)^2 \leqslant K. \tag{37}$$

Secondly, multiplying $(1)_2$ by v and adding with $(1)_3$ we get

$$(e+\tfrac{1}{2}v^2)_t = r^2(-p+\tfrac{\mu}{\eta}(r^2v)_x)_xv + f(r,x)v + (-p+\tfrac{\mu}{\eta}(r^2v)_x)(r^2v)_x. \tag{38}$$

Multiplying by $(e+\frac{1}{2}v^2)$ and integrating with respect to x we get the following

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\int_0^1(e+\tfrac{1}{2}v^2)^2dx = \int_0^1 r^2(-p+\tfrac{\mu}{\eta}(r^2v)_x)_x(e+\tfrac{1}{2}v^2)dx\\&+\int_0^1 f(r,x)v(e+\tfrac{1}{2}v^2)dx+\int_0^1(-p+\tfrac{\mu}{\eta}(r^2v)_x)(r^2v)_x(e+\tfrac{1}{2}v^2)dx.\end{aligned} \tag{39}$$

Now, adding together inequality (37) and estimating (39) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\mu}{\eta}(r^2v)_x-p\right)^2dx+\frac{1}{2}\frac{d}{dt}\int_0^1(e+\frac{1}{2}v^2)^2dx \leqslant K(\underline{\mu},\bar\eta,N). \tag{40}$$

It implies that

$$\int_0^{\bar t}\int_0^1 v_t^2dx \leqslant K, \tag{41}$$

and

$$\sup_{0\leqslant t\leqslant\bar t}\int_0^1 r^4(\frac{\mu}{\eta})^2[(r^2v)_x]^2dx \leqslant K. \tag{42}$$

3 Proof of the existence of a strong solution

Theorem 3. *Let the conditions on the data*

$$v_0,\theta_0\in C^{2+\nu}([0,1]),\ \eta_0\in C^{1+\nu}([0,1]), f\in C^2([r_\gamma,\infty[) \text{ with } 0<\nu<1,$$

$$\inf_{0\leqslant x\leqslant 1}\eta_0(x)>0,\ \inf_{0\leqslant x\leqslant 1}\theta_0(x)>0,$$

and the following extra condition of compatibility

$$v_0|_{x=0}=0,\ \big[-R\frac{\theta_0}{\eta_0}+\frac{\mu}{\eta_0}\eta_0(r^2v_0)_x\big]=0,$$

be satisfied.

The system of equations (1) together with conditions (3)-(7), where r is defined in (2) then for $\bar{t} \in (0, \infty)$, has a solution v, η, θ such that

$$v, \theta \in C^{2+\nu, 1+\frac{\nu}{2}}((0,1) \times (0, \bar{t})), \ \rho \in C^{1+\nu, 1+\frac{\nu}{2}}((0,1) \times (0, \bar{t})).$$

Proof. Since

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left(\mathcal{M}_x - \frac{v}{r^2} \right)^2 dx \leqslant 2 \int_{\Omega} \left(\mathcal{M}_x - \frac{v}{r^2} \right)^2 dx + K(1 + \max_{\Omega} \eta^{\lambda}) + K \max_{\Omega} \theta$$

and together with (39)-(41) it follows that

$$(r^2 v)_{xx} \in L^2((0,1) \times (0, \bar{t})).$$

Now, multiplying $(1)_3$ by $\left(\frac{r^4 \kappa}{\eta} \theta_x \right)_x$ and integrating with respect to x, we get

$$\theta_x \in L^{\infty}(0, \bar{t}), L^2(0,1)), \ \theta_{xx} \in L^2((0,1), \times (0, \bar{t})), \ \theta_t \in L^2((0,1) \times (0, \bar{t})).$$

Then differentiating $(1)_2$ with respect to x we obtain from the previous information that

$$\eta_{xt} \in L^2((0,1) \times (0, \bar{t})).$$

It follows that

$$\eta \in C^{1/2}([0,1] \times [0, \bar{t}]).$$

We differentiate $(1)_2$ with respect to t and multiply by v_t we get

$$\frac{1}{2} \frac{d}{dt} \int_0^1 (v_t)^2 dx = \int_0^1 \left\{ \left[r^2 \left(-p + \frac{\mu}{\eta} (r^2 v)_x \right)_x \right]_t v_t + (f(r,x))_t v_t \right\} dx.$$

From this we will get

$$v_t \in L^{\infty}(0, \bar{t}, L^2((0,1)), \text{ and } (r^2 v_t)_x \in L^2(Q).$$

Then it implies

$$v_{xt} \in L^2(]0,1[\times]0, \bar{t}]), \text{ and } u \in C^{1/2}([0,1] \times [0, \bar{t}]).$$

Again differentiating with respect to t $(1)_3$ and multiplying by θ_t we get

$$\tfrac{1}{2} \tfrac{d}{dt} \int_0^1 (\theta_t)^2 dx = \int_0^1 \left[\pi_x + \left(-p + \tfrac{\mu}{\eta} (r^2 v)_x \right) (r^2 v)_x \right]_t \theta_t dx,$$

which implies

$$\theta_t \in L^{\infty}(0, \bar{t}; L^2(0,1)), \text{ and } \ r^2 \theta_{xt} \in L^2(]0,1[\times]0, \bar{t}[),$$

and

$$\theta \in C^{1/2}([0,1] \times [0, \bar{t}]).$$

Then we obtain

$$\eta_x \in C^{1/2}(0,1)\times(0,\bar{t})),$$

and it implies that

$$r \in C^{1/2}(0,1)\times(0,\bar{t})).$$

If we consider the equation $(1)_3$ as a linear equation in θ, it follows that

$$\|\theta\|_{C^{2+2\nu^*,1+\nu^*}} \leqslant c + c\|v_x\|_{C^0} + \|v_x\|_{C^{2\nu^*,\nu^*}},$$

where $\eta^* = (1/2)(\min\,(\nu,1/2))$ and assuming that the equation $(1)_2$ is linear in v, it follows finally that

$$\|v\|_{C^{2+2\nu^*,1+\nu^*}} \leqslant c + c\|v_x\|_{C^0} + \|v_x\|_{C^{2\nu^*,\nu^*}},$$

which implies that

$$v,\theta \in C^{2+2\nu^*,1+\nu^*}((0,1)\times(0,\bar{t})).$$

If $\nu \leqslant 1/2$ then $2\nu^* = \nu$ and we get the required regularity. In the case $\nu > 1/2$, it is necessary to iterate all the previous steps again (see [1])).

4 Proof of Theorem 1

From the previous arguments and a priori estimates, we know that there exists subsequences $(u_k, \eta_k, \theta_k, r_k)$ such that

- $v_k \to v$ in $L^p(0,\bar{t},C^0(0,1))$ strongly and in $L^p(0,\bar{t},H^1(0,1))$, weakly for any $1 < p < \infty$,
- $v_k \to v$ a.e. in $(0,1)\times(0,\bar{t})$ and in $L^\infty(0,\bar{t},L^4(0,1))$ * weakly,
- $(v_k)_t \to v_t$ in $L^2(0,\bar{t},L^2(0,1))$ weakly,
- $\theta_k \to \theta$ in $L^2(0,\bar{t},C^0(0,1))$ strongly and in $L^2(0,\bar{t},H^1(0,1))$ weakly,
- $\theta_k \to \theta$ a.e. in $(0,1)\times(0,\bar{t})$ and in $L^\infty(0,\bar{t};L^2(0,1))$,
- $r_k \to r$ in $C^0((0,1)\times(0,\bar{t}))$,
- $r^2(\frac{\mu}{\eta_k}(r^2 v_k)_x - \frac{\theta_k}{\eta_k})$ converge to A_1 in $L^2(0,\bar{t},H^1(0,1))$ weakly,
- $\frac{\kappa(\eta,\theta)r^4}{\eta}(\theta_k)_x \to A_2$ in $L^2(0,\bar{t},L^2(0,1))$ weakly,
- $\frac{\mu}{\bar{\eta}}\partial_x(r^2 u_k) \to A_3$ in $L^\infty(0,\bar{t},L^2(0,1))$ weakly *.

After the definition of $r(x,t)$, one has

$$r(x,t) = r_0(x) + \int_0^t v(x,t')dt' \text{ a. e. } (0,1)\times(0,\bar{t}),$$

then

$$r_k(x,t)-r_k(y,t)=(\int_y^x \eta_k(s,t)ds)^{1/3}$$

$$\geqslant \epsilon(x-y) \ \bigvee(x,y,t)\in(0,1)\times(0,x)\times(0,\bar{t}).$$

Then from the previous computations we get

$$r(x,t)-r(y,t)\geqslant \epsilon(x-y) \ \bigvee(x,y,t)\in(0,1)\times(0,x)\times(0,\bar{t}),$$

and finally

$$f_k r_k \to f\ r \text{ in } C^0((0,1)\times(0,\bar{t})).$$

Moreover, it implies that

- $\eta_k\to\eta$ a.e. in $(0,1)\times(0,\bar{t})$ and$L^s((0,1)\times(0,\bar{t}))$ strongly for all $s\in(1,\infty)$,
- $A_1=(\frac{\mu}{\eta}(r^2v)_x-p)$ in $L^2(0,\bar{t};H^1(0,1))$,
- $A_2=\frac{\kappa(\eta,\theta)r^4}{\eta}\theta_x$ in $L^2(0,\bar{t},L^2(0,1))$,
- $A_3=\frac{\mu}{\bar{\eta}}(r^2v)_x$ in $L^\infty(0,\bar{t},L^2(0,1))$.

So we can pass to the limit in the weak formulation of $(1)_2$ and $(1)_3$, and we get a weak solution of (1).

5 Proof of Theorem 2

Let η_i, v_i,θ_i, $i=1,2$ be two solutions of (1), and let us consider the differences: $\eta=\eta_1-\eta_2$, $\theta=\theta_1-\theta_2$ and $v=v_1-v_2$.

The following auxiliary result holds

Proposition 1.

$$|r_2^m-r_1^m|\leqslant c\int_0^1(\eta_2-\eta_1)dx.$$

Proof. from the definition of $r(x,t)$, we see that

$$\begin{aligned} &r_2^m-r_1^m=(r_2^4)^{m/2}-(r_1^3)^{m/3}\\ &=\tfrac{m}{3}r_*^{m-3}(r_2^3-r_1^3)=\tfrac{m}{3}r_*^{m-3}3\textstyle\int_0^x(\eta_2-\eta_1)ds\leqslant c\int_0^1(\eta_2-\eta_1)dx,\end{aligned}$$

where

$$1\leqslant r_k\leqslant c, r_*=r_1+\epsilon(r_2-r_1)\quad\square$$

Now, we subtract $(1)_2$ for η_2,v_2,θ_2 from $(1)_2$ for η_1,v_1,θ_1 in order to get

$$\begin{aligned}&\textstyle\int_0^1(v_2-v_1)_t\phi\ dx=-\Big\{\int_0^1\{\frac{Rr_2^2}{\eta_2}(\theta_2-\theta_1)+Rr_2^2\big(\frac{\eta_1-\eta_2}{\eta_1\eta_2}\big)+\frac{R\theta_1}{\eta_1}(r_2^2-r_1^2)\}\phi_x dx\\ &+\textstyle\int_0^1\Big\{\frac{r_2^2\mu_2}{\eta_2}\big((r_2^2v_2)_x-(r_1^2v_1)_x\big)+(r_1^2v_1)_x\{\frac{\mu_2}{\eta_2}(r_2^2-r_1^2)+r_1^2\Big(\frac{\mu_2(\eta_1-\eta_2)+\eta_1(\mu_2-\mu_1)}{\eta_1\eta_2}\Big)\}\phi_x\Big\}dx\Big\}\\ &+\textstyle\int_0^1(f_2-f_1)\phi\ dx.\end{aligned}\tag{43}$$

Now we set $\phi = v_2 - v_1$ and we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v_2-v_1)^2 dx + \int_0^1 \frac{r_2^4\mu_2}{\eta_2}\Big((v_2-v_1)_x\Big)^2 dx$$

$$= -\{\int_0^1 \frac{Rr_2^2}{\eta_2}(\theta_2-\theta_1)(v_2-v_1)_x dx + \int_0^1 Rr_2^2\big(\frac{\eta_1-\eta_2}{\eta_1\eta_2}\big)(v_2-v_1)_x dx$$

$$+\int_0^1 (r_1v_1)_x\left\{\frac{\mu_2}{\eta_2}(r_2^2-r_1^2)(v_2-v_1)_x + r_1^2\{\frac{\mu_2(\eta_1-\eta_2)+\eta_1(\mu_2-\mu_1)}{\eta_1\eta_2}\}\right\}(v_2-v_1)_x dx$$

$$+\int_0^1\left\{\frac{r_2^2\mu_2}{\eta_2}(r_2^2-r_1^2)v_{1,x} + ((r_2^2)_x v_2 - (r_1^2)_x v_1)\right\}(v_2-v_1)_x dx$$

$$+\int_0^1 (f_2-f_1)(v_2-v_1)_x dx. \tag{44}$$

Let us consider the various contributions

- $I_1 = \int_0^1 \frac{Rr_2^2}{\eta_2}(\theta_2-\theta_1)(v_2-v_1)_x dx,$
- $I_2 = \int_0^1 Rr_2^2\frac{\eta_1-\eta_2}{\eta_1\eta_2}(v_2-v_1)_x dx,$
- $I_3 = \int_0^1 (r_1v_1)_x\{\frac{\mu_2}{\eta_2}(r_2^2-r_1^2)(v_2-v_1)_x$
  $+r_1^2\{\frac{\mu_2(\eta_1-\eta_2)+\eta_1(\mu_2-\mu_1)}{\eta_1\eta_2}\}\}(v_2-v_1)_x dx,$
- $I_4 = \int_0^1\{\frac{r_2^2\mu_2}{\eta_2}(r_2^2-r_1^2)v_{1,x} + ((r_2^2)_x v_2 - (r_1^2)_x v_1)\}(v_2-v_1)_x dx,$
- $I_5 = \int_0^1 (f_2-f_1)(v_2-v_1)_x dx.$

$$I_1 \leqslant \frac{Rc}{\bar\eta}\|\theta_2-\theta_1\|_2\|v_2-v_1\|_2$$

$$I_2 \leqslant \frac{Rc}{\bar\eta^2}\|\eta_2-\eta_1\|_2\|v_2-v_1\|_2$$

$$I_3+I_4 \leqslant c\int_0^1(\eta_2-\eta_1)dx\|(v_2-v_1)_x\|_2 + c\|\eta_2-\eta_1\|_2\|(v_2-v_1)_x\|_2 \tag{45}$$

$$+c\|v_2-v_1\|_2\|v_2-v_2)_x\|_2+$$

$$\int_0^x(\eta_2-\eta_1)ds\ \|(v_2-v_1)_x\|_2.$$

Now subtracting $(1)_3$ for η_2, v_2, θ_2 from $(1)_3$ for η_1, v_1, θ_1, we get

$$\int_0^1 c_v(\theta_2-\theta_1)_t\psi dx = -\Big\{\int_0^1\{\frac{k_2r_2^4}{\eta_2}(\theta_2-\theta_1)_x\psi_x\}dx$$

$$+\int_0^1\theta_{1,x}\Big\{\frac{K_2}{\eta_2}(r_2^4-r_1^4) + \frac{r_1^4}{\eta_1\eta_2}\big(K_2(\eta_2-\eta_2)+\eta_1(K_2-K_1)\big)\Big\}\psi_x dx\Big\}$$

$$+\int_0^1\frac{R\theta_2}{\eta_2}((r_2^2v_2)_x - x(r_1^2v_1)_x)\psi + R(r_1^2v_1)_x(\frac{\theta_2}{\eta_2}-\frac{\theta_1}{\eta_1})\psi dx \tag{46}$$

$$+\int_0^1\frac{\mu_2}{\eta_2}\Big(((r_2^2v_2)_x)^2 - ((r_1^2v_1)_x)^2\Big)\psi dx + \int_0^1((r_1^2v_1)_x)^2\Big(\frac{\mu_2}{\eta_2}-\frac{\mu_1}{\eta_1}\Big)\psi\ dx.$$

Setting $\psi = \theta_2 - \theta_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}c_v\int_0^1(\theta_2-\theta_1)^2dx + \int_0^1\frac{k_2}{\bar{\eta}}r_2^4(\theta_2-\theta_1)_x^2dx$$

$$= -\Big\{\int_0^1\theta_{1,x}\Big\{\frac{K_2}{\eta_2}(r_2^4-r_1^4)+\frac{r_1^4}{\eta_1\eta_2}\big(K_2(\eta_2-\eta_2)+\eta_1(K_2-K_1)\big)\Big\}(\theta_2-\theta_1)_xdx\Big\}$$

$$+\int_0^1\frac{R\theta_2}{\eta_2}((r_2^2v_2)_x(r_1^2v_1)_x)(\theta_2-\theta_1)+R(r_1^2v_1)_x(\frac{\theta_2}{\eta_2}-\frac{\theta_1}{\eta_1})(\theta_2-\theta_1)dx$$

$$+\int_0^1\frac{\mu_2}{\eta_2}\Big(((r_2^2v_2)_x)^2-((r_1^2v_1)_x)^2\Big)(\theta_2-\theta_1)dx+\int_0^1((r_1^2v_1)_x)^2\Big(\frac{\mu_2}{\eta_2}-\frac{\mu_1}{\eta_1}\Big)(\theta_2-\theta_1)dx. \tag{47}$$

Considering the integrals

- $J_1=\int_0^1\theta_{1,x}\Big\{\frac{K_2}{\eta_2}(r_2^4-r_1^4)+\frac{r_1^4}{\eta_1\eta_2}\big(K_2(\eta_2-\eta_2)+\eta_1(K_2-K_1)\big)\Big\}(\theta_2-\theta_1)_xdx,$
- $J_2=\int_0^1\frac{R\theta_2}{\eta_2}((r_2^2v_2)_x-(r_1^2v_1)_x)(\theta_2-\theta_1)+R(r_1^2v_1)_x(\frac{\theta_2}{\eta_2}-\frac{\theta_1}{\eta_1})(\theta_2-\theta_1)\ dx,$
- $J_3=\int_0^1\frac{\mu_2}{\eta_2}\Big(((r_2^2v_2)_x)^2-((r_1^2v_1)_x)^2\Big)(\theta_2-\theta_1)\ dx,$
- $J_4=\int_0^1((r_1^2v_1)_x)^2\Big(\frac{\mu_2}{\eta_2}-\frac{\mu_1}{\eta_1}\Big)(\theta_2-\theta_1)\ dx,$

we estimate them as follows

$$J_1\leqslant c\Big|\int_0^x(\eta_2-\eta_1)ds\Big|\ \|(\theta_2-\theta_1)_x\|_2+c_1\|\eta_2-\eta_1\|_2\|(\theta_2-\theta_1)_x\|_2.$$

$$\begin{aligned}
J_2 &= |\int_0^1\{\frac{R\theta_2r_2}{\eta_2}(v_2-v_1)_x(\theta_2-\theta_1)+\frac{R\theta_2}{\eta_2}(r_2-r_1)v_{2,x}(\theta_2-\theta_1)\\
&+ \frac{R\theta_2v_2}{r_2}(\eta_2-\eta_1)(\theta_2-\theta_1)+\frac{2R\theta_2\eta_1}{r_2\eta_2}(v_2-v_1)(\theta_2-\theta_1)+\\
&+ \frac{2R\theta_2\eta_1v_1}{\eta_2}\big(\frac{1}{r_2}-\frac{1}{r_1}\big)(\theta_2-\theta_1)+\\
&+ R(r_1^2v_1)_x\frac{1}{\eta_2}(\theta_2-\theta_1)(\theta_2-\theta_1)+R(r_1^2v_1)_x\theta_1\Big(\frac{1}{\eta_2}-\frac{1}{\eta_1}\Big)(\theta_2-\theta_1)\}dx|\\
&\leqslant \|\theta_2-\theta_1\|_2\{C_1\|(v_2-v_1)_x\|_2+C_2\int_0^x(\eta_2-\eta_1)^2dy+C_3\|\eta_2-\eta_1\|_2\\
&+ C_4\|v_2-v_1\|_2+C_5\int_0^x|\eta_2-\eta_1|\ dy+C_5\|\eta_2-\eta_1\|_2\}+\|\theta_2-\theta_1\|_2^2.
\end{aligned}$$

$$\begin{aligned}
J_4 &= \int_0^1\{(r_1^2v_1)_x^2\mu_2(\frac{1}{\eta_2}-\frac{1}{\eta_1})(\theta_2-\theta_1)+(r_1^2v_1)_x^2\frac{1}{\eta_1}(\mu_2-\mu_1)(\theta_2-\theta_1)\}\ dx\\
&\leqslant C_6\{\|\eta_2-\eta_1\|_2+C_7\|\mu_2-\mu_1\|_2\}\|\theta_2-\theta_1\|_2.
\end{aligned}$$

From equation $(1)_1$ it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1(\eta_2-\eta_1)^2dx &= \int_0^1\{r_2^2(v_2-v_1)_x(\eta_2-\eta_1)\\ &+ v_{1,x}(r_2^2-r_1^2)(\eta_2-\eta_1)+\frac{2\eta_2}{r_2}(v_2-v_1)(\eta_2-\eta_1)\\ &- \frac{2v_1}{r_2}(\eta_2-\eta_1)^2\}dx,\end{aligned}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int_0^1(\eta_2-\eta_1)^2dx \leqslant c(\int_0^1(\eta_2-\eta_1)^2dx)^2+$$
$$\|\eta_2-\eta_1\|_2\{\|(v_2-v_1)_x\|_2+\|v_2-v_1\|_2+\int_0^1(\eta_2-\eta_1)^2\ dx\}.$$

Combining the previous estimates, only remains to be controlled the term J_3 in order to get uniqueness.

In fact, rewriting it as

$$\left(\int_0^1(((r_2^2v_2)_x-(r_1^2v_1)_x)((r_2^2v_2)_x+(r_1^2v_1)_x))^2dx\right)^{1/2}\|\theta_2-\theta_1\|_2^2,$$

we get

$$J_3\leqslant\|\theta_2-\theta_1\|_2\left\{C_{10}\|\eta_2-\eta_1\|_2+C_{11}\|(v_2-v_1)_x\|_2+\int_0^1|\eta_2-\eta_1|\ dx+C_{12}\|v_2-v_1\|_2\right\}.$$

All the previous estimates imply the uniqueness, which ends the proof of Theorem 2 □

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Bernard Ducomet
Département de Physique Théorique et Appliquée, CEA/DAM Ile de France
BP 12, F-91680 Bruyères-le-Châtel, France
E-mail: bernard.ducomet@cea.fr

Šárka Nečasová
Mathematical Institute AS ČR
Žitna 25, 115 67 Praha 1, Czech Republic
E-mail: matus@math.cas.cz