Design of Square-Root Derivative-Free Smoothers

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The problem of recursive state estimation of discrete-time stochastic dynamic systems from noisy or incomplete measurement data has been a subject of considerable research interest for the last several decades.

The general solution of the estimation problem, based on Bayesian approach, is given by the Functional Recursive Relations (FRR's) for computation of probability density functions (pdf's) of the state conditioned by the measurements. These pdf's provide a full description of the immeasurable state. The FRR's are known for all three parts of the estimation problem which can be distinguished, according to relation between time instant of the estimated state and time instant of the last measurement, to prediction, filtering, and smoothing.

The closed form solution of the FRR's is available only for a few special cases [1, 2], e.g. for linear Gaussian system, where the solution of the filtering problem is given by the well-known Kalman Filter. As a solution of smoothing problem, the Rauch-Tung-Striebel Smoother [1, 3] can be used. The alternative approach for smoothing is based on the doubling of the state dimension and on the utilisation of common filtering techniques [4, 5]. The multi-step prediction can be imagined as a multiply application of the one-step prediction known from the filtering algorithm [3, 5]. In other cases it is necessary to apply some approximative methods.

The local methods are often based on approximation of the nonlinear functions in the state or measurement equation so that the Kalman technique can be used for the FRR's solution. This approach causes that all conditional pdf's of the state estimate are given by the first two moments, i.e. mean value and covariance matrix. This rough approximation of the a posteriori estimates induces local validity of the state estimates and consequently impossibility to ensure the convergence of the local filter estimates. Moreover, resulting estimates of the local filters are suitable mainly for point estimates. On the other hand, the advantage of the local methods can be found in the relative simplicity of the FRR's solution.

The standard local nonlinear filtering methods are based on the approximation of nonlinear functions in the state or the measurement equation with the Taylor expansion. The FRR's solution based on the Taylor expansion first order approximation leads to the Extended Kalman Filter or to the Iterated Kalman Filter [1]. Generally, the more exact Second Order Filter [6, 7] utilises the Taylor expansion second order approximation. The Taylor expansion first order can be used to design of the extended Rauch-Tung-Striebel Smoother and the multi-step predictor as well [3].

In the last decade the novel approaches to the local filter design, based on the polynomial interpolation [8–11] or on the unscented transformation [9–13], have been published. The approximation of the nonlinear functions by means of the Stirling's polynomial interpolation first or second order leads to the Divide Difference Filters 1st order or to the Divided Difference Filter 2nd order, respectively, which are usually called as the Divided Difference Filters [8]. Instead of direct substitution of the nonlinear functions in the system description an approximation of the "already approximated"pdf's representing state estimate by a set of deterministically chosen weighted points (so called σ-points) can be utilised as a base for the local filters. This transformation is often called as the unscented transformation. The Unscented Kalman Filter [10, 12, 14] or the Gauss-Hermite Filter [9] exemplify this approach. The smoothing local methods utilising the Stirling's interpolation and unscented transformation was very briefly outlined in [15] and properly derived in [5]. Similarly to the standard local approaches the multi-step prediction is realised by the multiply application of the one-step prediction known from the filtering algorithm [5]. It is very important to mention that the estimators based on the unscented transformation and the Stirling's interpolation have common features although the basic idea of these estimators comes out from quite different assumptions [8, 11, 16]. Therefore, these local filters can be called together as the sigma point Kalman estimators or the derivative-free Kalman estimators.

The numerical properties of the derivative-free local filters have been discussed in the several papers. For example the Divided Difference Filters have been directly designed in the square-root form [8] and although the Unscented Kalman Filter was originally derived in the "nonsquare-root" form, its square-root versions have been subsequently derived in [10, 14]. However, the poor attention has been paid to the numerical properties of the novel derivative-free smoothers [5].

Therefore, in the paper the novel square-root smoothing algorithms, which are based on the unscented transformation and the Stirling's interpolation, are introduced. This modification improves not only numerical properties of the smoothing algorithms, but it slightly reduces their computational demands as well. Finally, the theoretical results are illustrated in a numerical example.

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