Examples of State and Parameter Estimation for Linear Model with Uniform Innovations

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State estimation is an important subtask of a range decision making problems. Kalman filtering (KF) [1] is the first-option method for its addressing. However, still there is no well-established methodology of selecting innovation covariances. Also, it is difficult to combine KF with hard restrictions on state ranges. Both these drawbacks can be avoided by assuming that the model innovations are uniform.

In this contribution, state-space model with uniformly distributed innovations is introduced and the Bayesian state estimation proposed, [2]. This extends parameter estimation of the controlled autoregressive model treated in [3]. Similarly as in the latter case, the off-line evaluation of the maximum a posteriori probability (MAP) estimate of unknowns in the linear state-space model with uniform innovations reduces to linear programming (LP). The solution provides either estimates of the noise boundary and parameters or of the noise boundary and states.

The on-line estimation is obtained by applying LP on the sliding window, i.e., by considering only the fixed amount, say $0 < \partial$, of the newest last data and states items.

By swapping between state and parameter estimations, joint parameter and state estimation is obtained. The use of Taylor expansion for approximation of products of unknowns solves also the joint parameter and state estimation. Simulation studies help to get an insight on the potential and restrictions of these heuristic method. This contribution shares the experimentally gained experience with both these solutions of the joint state and parameter estimation.

Problem formalization We consider the standard linear state-space model

$$x_t = Ax_{t-1} + Bu_t + {}^{x}e_t, \quad y_t = Cx_t + Du_t + {}^{y}e_t, \tag{1}$$

known in connection with from Kalman filtering theory. In it, x, u, y are unobserved state, known input and observed output of the system, respectively. They are real column vectors. The subscript $t \in \{0, 1, 2, \ldots\}$ labels discrete time. The involved time-invariant matrices A, B, C, D have appropriate dimensions. Unlike in the KF case, the distributions of vector innovations ${}^{x}e_t$ and ${}^{y}e_t$ are assumed to be uniform

$$f\left({}^{x}e_t\right) = \mathcal{U}\left(0, {}^{x}r\right), \quad f\left({}^{y}e_t\right) = \mathcal{U}\left(0, {}^{y}r\right). \tag{2}$$

$\mathcal{U}\left(\mu, {}^{x}r\right)$ denotes uniform probability density function (pdf) on the box with the center μ and half-width of the support interval ${}^{x}r$. The model parameters A, B, C, D are collected into parameters Θ. Equations (1) together with the assumptions (2) define the linear uniform state-space model (LU).

We assume that the generator of the inputs $u^{1:t} \equiv [u_t', \ldots, u_1']'$ meets natural conditions of control [2]. They formalize assumption that information about unknown quantities for generating u_t can only be extracted from the observed data $d^{1:t-1}$, where $d_t = (y_t, u_t)$. Then, for a given initial state x_0, half-widths ${}^{x}r$, ${}^{y}r$ and parameters Θ, the joint pdf of data and the state trajectory $x^{1:t}$ of the LU model is

$$f\left(d^{1:t}, x^{1:t} \middle| x_0, {}^{x}r, {}^{y}r, \Theta\right) \propto \prod_{i=1}^{n} {}^{x}r_i^{-t} \prod_{j=1}^{m} {}^{y}r_j^{-t} \chi(\mathcal{S}). \tag{3}$$

$\chi(\mathcal{S})$ is the indicator of the support $\mathcal{S}$; $\propto$ denotes equality up to a constant factor. The convex set $\mathcal{S}$ is given by inequalities, $\tau = 1, 2, \ldots, t$,

$$\begin{aligned} -{}^{x}r &\leq x_\tau - Ax_{\tau-1} - Bu_\tau \leq {}^{x}r \\ -{}^{y}r &\leq y_\tau - Cx_\tau - Du_\tau \leq {}^{y}r. \end{aligned} \tag{4}$$

The adopted Bayesian estimation needs to complement conditional pdf (3) by a *prior pdf* $f\left(x_0, {}^{x}r, {}^{y}r \middle| \Theta\right)$ defined as uniform pdf on support given by the initial condition.

If the inequalities (4) are linear in the unknowns and ${}^{x}r < 1$, ${}^{y}r < 1$, then the MAP estimation is equivalent to the problem of LP. We can run either noise boundary and state estimation (parameters Θ are supposed to be known) or noise boundary and parameters estimation (known states $x^{1:t}$ are supposed).

In the case that both parameter and states are unknown, the joint parameters and state estimation is to be performed. Two approaches for the joint parameter and state estimation are considered.

The first one is based on the idea of swapping between the parameter and state estimation. It means that the state $x^{1:t}$ is estimated with parameters Θ fixed at their last point estimates. The resulting estimates of states, $\hat{x}^{t-\partial:t}$ are subsequently used to obtain new estimates of the parameters Θ. Initial values of the estimates are prepared in off-line mode.

The second one linearizes non-linear expressions at the newest point estimates. This common idea is used in various extensions of KF, see e.g. [1]. When the expressions occurring in inequalities (4) are approximated by the first order Taylor expansion linear inequalities are obtained. Thus it suffices to transform them into the standard form of LP and joint parameter and state estimation is gained.

Concluding remarks The proposed approach opens a way for on-line parameter and state estimation for a class of non-uniform distributions with restricted support as well as for Bayesian filtering of non-linear systems.

The main current contributions include feasible care about hard bounds of estimated quantities; joint estimation of parameters, state, and noise bounds; parameter tracking via windowing the joint estimation.

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