

# IMPROVED ESTIMATION METHOD OF REGION OF STABILITY FOR NONLINEAR AUTONOMOUS SYSTEMS

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**Abstract:** An iterative method for estimationg the stability region by constructing a sequence of Lyapunov functions is analyzed and improved in this paper. Vanelli and Vidyasagar proved that there exists a sequence of special kind of Lyapunov functions  $V_m$  that can be used to estimate the domain of attraction (DOA) for an asymptotically stable equilibria. Based on this idea a corrected proof is given and the iterative method has been implemented in a Mathematica-package to find the appropriate approximating functions  $V_m$ . The use and the properties of the method is illustrated on three simple examples.

**Keywords:** region of attraction, maximal Lyapunov functions

## 1. INTRODUCTION

The stability and the stability region of (controlled) industrial systems are important properties to be determined. The reason for that is that it is highly desirable to have a system which is globally stable; that is why stability theory is continuously in the focus of both theoretical researchers and industrial partitioners.

A substantial part of the different methods described in the literature is based on the classical results of Lefschetz and La Salle using a suitably chosen Lyapunov function, see for example (Chesi, 2005), (Camilli *et al.*, 2000), (Kaslik *et al.*, n.d.), (Vanelli and Vidyasagar, 1985) and (Yoshizawa, 1966). The subject of this paper is to use the idea of Vanelli and Vidyasagar (Vanelli and Vidyasagar, 1985) to develop a practically useful algorithm to estimate the stability region (or domain of attraction, DOA) of autonomous nonlinear systems by improving the original method.

In the following we will consider the nonlinear autonomous system centered such that the origin is its asymptotically stable equilibrium point:

$$\dot{x}(t) = f(x(t)) \tag{1}$$

By the region or domain of attraction  $S(M)$  of the set  $M$  (which need not to be an attractor) we mean the union of all trajectories with the property that their limit sets are non-empty and

contained by  $M$  itself. Based on this composition we can define the domain of attraction of the origin as a set having only one element. The *domain of attraction of the origin* is the set

$$S(0) = \{x_0 : x(t, x_0) \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

where  $x(t, x_0)$  denotes the solution of the system in (1) corresponding to the initial condition  $x(0) = x_0$ .

In this paper it is shown by following the ideas of (Vanelli and Vidyasagar, 1985) that there exists a sequence of special kind of Lyapunov functions  $V_m$  that can be used to estimate the set  $S(0)$  through estimating a Lyapunov function of special kind. An iterative method will be given to find these appropriate functions  $V_m$ . The given algorithm is able to find unbounded domains of attraction, too. Usually, the first few number of iterations can show if the domain is bounded or not. Throughout the paper it is assumed that the function  $f$  is smooth enough that (1) has unique solution corresponding to each initial condition  $x(0) = x_0$ .

## 2. MAXIMAL LYAPUNOV FUNCTIONS

The proofs of the statements in this section can be found in (Vanelli and Vidyasagar, 1985).

**Theorem 2.1.** *Suppose we can find a set  $A \subseteq \mathbb{R}^n$  containing the origin in its interior, a continuous function  $V : A \rightarrow \mathbb{R}_+$  and a positive definite function  $\phi$  such that*

1.  $V(0) = 0, V(x) > 0 \forall x \in A \setminus \{0\}$
2. *The function  $\dot{V}(x_0) = \lim_{t \rightarrow 0^+} \frac{V(x(t, x_0)) - V(x_0)}{t}$  is well defined at all  $x \in A$  and satisfies  $\dot{V}(x) = -\phi(x), \forall x \in A$ .*
3.  $V(x) \rightarrow \infty$  as  $x \rightarrow \partial A$  and/or  $\|x\| \rightarrow \infty$ .

Then  $A = S$ .

Suppose  $V$  is a continuous function on some ball  $B_\delta$  such that  $V(0) = 0$  and  $\dot{V}$  is negative definite. Then one could prove that  $V$  is positive definite. This fact shows that if we can find a function  $V$  and a positive definite function  $\phi$  such that  $V(0) = 0$  and  $\nabla V(x)' f(x) = -\phi(x)$  then  $V$  is guaranteed to be positive definite.

**Definition 2.2.** *A function  $V_m : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$  is called maximal Lyapunov function for the system described in (1) if*

1.  $V_m(0) = 0, V_m(x) > 0, x \in S \setminus \{0\}$
2.  $V_m(x) < \infty \Leftrightarrow x \in S$
3.  $V_m(x) \rightarrow \infty$  as  $x \rightarrow \partial S$  and/or  $\|x\| \rightarrow \infty$
4.  $\dot{V}_m$  is well-defined and negative definite over  $S$ .

### 3. COMPUTATION OF THE DOMAIN OF ATTRACTION (DOA)

We need a function  $V$  and a positive definite function  $\phi$  satisfying  $V(0) = 0$  and

$$\dot{V}(x) = -\phi(x) \quad (2)$$

over some neighborhood of the origin. Then the boundary of the domain of attraction is defined by the limit  $V(x) \rightarrow \infty$ .

A systematic procedure will be described and discussed here to solve (2) supposing that the Taylor series expansion exists for  $f$  around the origin. Express  $f$  as

$$f(x) = \sum_{i=1}^{\infty} F_i(x)$$

where the functions  $F_i, i \geq 1$  are homogeneous functions of degree  $i$ . For  $i = 1$  we have

$$F_1(x) = Ax, \quad A \in \mathbb{R}^{n \times n}.$$

For the sake of brevity let  $F_i(x) = 0, i \leq 0$ .

Our candidate Lyapunov function should excess any limit as  $x$  gets closer to the boundary of set  $S$  or as  $\|x\| \rightarrow \infty$ . For this reason we put a function  $D(x)$  to the denominator, i.e.

$$V(x) = \frac{N(x)}{D(x)}$$

where  $N(x)$  and  $D(x)$  are polynomials in  $x$ . Thus,  $V(x) \rightarrow \infty$  as  $x \rightarrow \partial S$  and this suggests that  $x \in \partial S$  when  $D(x) = 0$ . According to our results so far, the boundary of  $S$  is defined by solving  $D(x) = 0$  for  $x$ . We obtain a recursive technique to find this boundary by defining

$$V(x) = \frac{\sum_{i=2}^{\infty} R_i(x)}{1 + \sum_{i=1}^{\infty} Q_i(x)}$$

where  $R_i$  and  $Q_i$  are homogeneous functions of degree  $i$ . The most straightforward idea for  $\phi$  is  $x' \Omega x$  where  $\Omega > 0$ . Substituting this expression into (2) we obtain

$$\dot{V}(x) = \nabla V(x)' f(x) = -\phi(x) = -x' \Omega x.$$

Based on these two equations we get

$$\left( \left( 1 + \sum_{i=1}^{\infty} Q_i \right) \sum_{i=2}^{\infty} \nabla R_i' - \left( \sum_{i=1}^{\infty} \nabla Q_i' \right) \sum_{i=2}^{\infty} R_i \right) \sum_{i=1}^{\infty} F_i = -x' \Omega x \left( 1 + \sum_{i=1}^{\infty} Q_i \right)^2.$$

From this finally we obtain

$$\begin{aligned} \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} \nabla R_i' F_k + \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i \nabla R_j' F_k - \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} Q_i' R_j F_k = \\ -x' \Omega x \left( 1 + 2 \sum_{i=1}^{\infty} Q_i + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} Q_i Q_j \right). \end{aligned}$$

Equating the coefficients of the same degrees of the two sides of this equality we get for degree 2 that

$$\nabla R_2' F_1 = -x' Q x$$

and the general solution when degree  $k$  is greater than or equal to 3 is

$$\sum_{i=2}^k \nabla R_i' F_{k+1-i} + \sum_{i=1}^{k-2} \sum_{j=2}^{k-1} (Q_i \nabla R_j' - \nabla Q_i' R_j) F_{k+1-i-j} = -x' Q x \left( 2Q_{k-2} + \sum_{i=1}^{k-3} Q_i Q_{k-2-i} \right).$$

Thus in each step of the algorithm we get the following *linear under-determined set of equations* as the equivalent form of the previous two equations:

$$A_n \cdot y = b_n \quad (3)$$

where  $A_n$  are matrices of appropriate dimension. Consider the nonlinear system of equations

$$\dot{x} = f(x) = \sum_{i=1}^{\infty} F_i(x). \quad (4)$$

First, select homogeneous functions  $R_n$  and  $Q_{n-2}$ ,  $n \geq 3$ , such that the coefficients of  $R_n$  and  $Q_n$  solve the constrained minimization problem yielded by (3)

$$\begin{aligned} & \min e_n(y) \\ & \text{s.t. } A_n \cdot y = b_n \end{aligned} \quad (5)$$

where  $e_n(y)$  is the square of 2-norm of the coefficients of degree greater than or equal to  $n+1$  in the expression of  $\dot{V}_n$ . Furthermore, according to the theorem of La-Salle about invariant sets one can choose the largest positive value  $C^*$  such that the level set

$$V_n = \frac{\sum_{i=2}^n R_i}{1 + \sum_{i=1}^{n-2} Q_i} = C^*$$

is contained in the region given by

$$\Omega = \left\{ x : \dot{V}_n(x) \leq 0 \right\}.$$

Then the set

$$S_A = \{x : V_n(x) < C^*\} \quad (6)$$

is contained in the region of attraction  $S$ . If this is the case then the iteration should be stopped as soon as the desired accuracy has been reached.

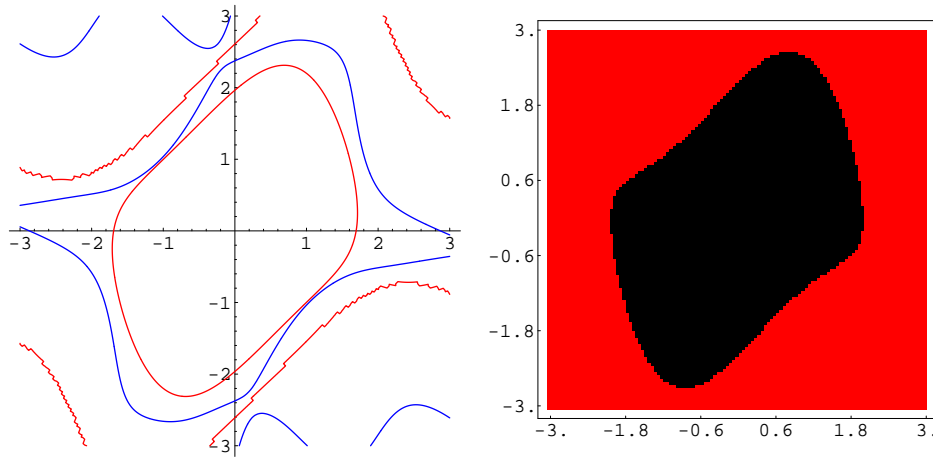
If  $e_n(y^*) = 0$  for some  $y^*$  then the iteration can be stopped and

$$\dot{V}_n = -x' \Omega x$$

where  $\Omega > 0$ . In this case the domain of attraction is defined by  $D(x) = 0$ . This means that the domain of attraction  $S$  is given by the formula

$$S = \left\{ x : \sum_{i=1}^{n-2} Q_i > -1 \right\}. \quad (7)$$

Would any of the afore-mentioned cases happen for the major part of the systems, the iteration usually stops in less than 10 cycles.



(a) DOA by the algorithm of Vanelli and (b) DOA by direct scanning of the domain Vidyasagar

Figure 1: DOA of the Van der Pole system

#### 4. CASE STUDIES

In this section three examples will be shown. In the first and second examples the minimization problem in (5) cannot be solved such that  $e(n)$  becomes small enough that we could apply (7). Instead, (6) is used for the estimation of the domain of attraction. In the last example the minimization problem can be solved and one can use (7) to get a proper region.

In the picture showing the level sets figure 1(a) the red colour curve shows  $V = C^*$  while the blue one shows those points where  $\dot{V} = 0$ .

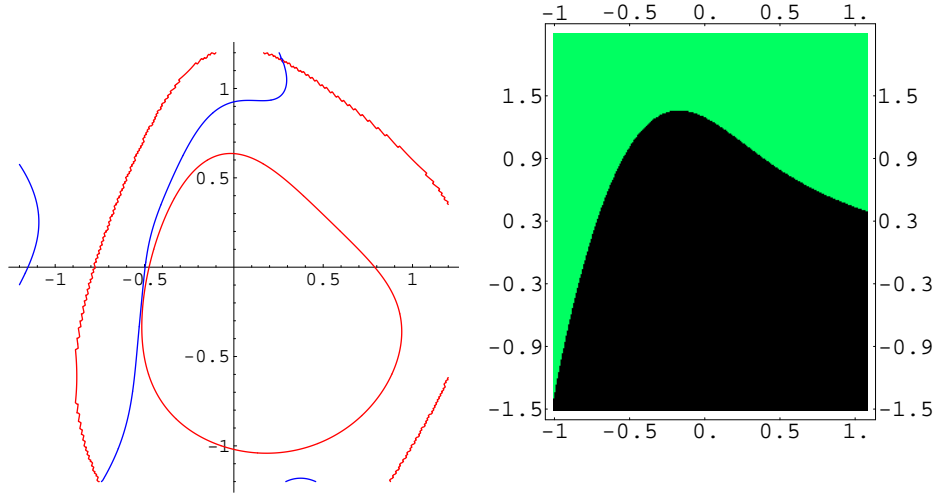
##### 4.1 Van der Pole-equation

The Van der Pole system we took as example is described with the equation-system

$$\begin{aligned}\dot{z}_1 &= -z_2 \\ \dot{z}_2 &= z_1 - z_2 + z_1^2 z_2\end{aligned}$$

The origin is an asymptotically stable equilibrium point of the system thus the method based on maximal Lyapunov functions can be used in this case. Applying the iteration steps 9 times we get  $C^* = 6.6$  and the minimum value of  $e(n)$  was found to be 0.0168945. The stability region is the innermost area bounded by the red colored curve in figure 1(a). We stopped the iteration at step 9 because further steps could not increase the minimum significantly.

To verify the given region we used a direct method by scanning the points over the region  $[-3, -3] \times [-3, -3]$  and examining if the system remains stable or not. The resulted set can be seen in figure 1(b). We can ascertain that the regions found by the two different methods are pretty the same.



(a) DOA by the algorithm of Vanelli and Vidyasagar (b) DOA by direct scanning of the domain

Figure 2: DOA of the Lotka-Volterra system

#### 4.2 Lotka-Volterra

In this example we take a Lotka-Volterra system described by the equations

$$\begin{aligned}\dot{z}_1 &= -z_1 \left( (z_1 - 1)(z_1 - 3) + \frac{1}{2}z_2 \right) \\ \dot{z}_2 &= z_2 (-2.1 + z_1)\end{aligned}$$

Both the origin and point  $(2.1, 1.98)$  both are equilibria of the system. By shifting the second equilibrium to the origin we get the following centralized system

$$\begin{aligned}\dot{x}_1 &= -0.42x_1 - 2.3x_1^2 - x_1^3 - 1.05x_2 - \frac{1}{2}x_1x_2 \\ \dot{x}_2 &= 1.98x_1 + x_1x_2\end{aligned}$$

Similarly to the previous case after 6 steps we get  $C^* = 1.65$  and the minimum of  $e(n)$  is 2915.040375. See the region in figure 2(a) bounded by the inner red curve.

By scanning the points over the region  $[-1, 1.5] \times [-1.5, 1.5]$  we find that the region estimated by the Vanelli-method is a subset of the one we found by direct search, see figure 2(b).

#### 4.3 Fermentation system

In this example we show a simple model of a fermentation system (Szederkényi *et al.*, 2002) described by the equations

$$\begin{aligned}\dot{z}_1 &= -0.802228z_1 + \frac{z_1z_2}{0.03 + z_2 + 0.5z_2^2} \\ \dot{z}_2 &= 0.802228(10 - z_2) - \frac{z_1z_2}{0.03 + z_2 + 0.5z_2^2}\end{aligned}$$

which has one asymptotically stable equilibrium point (4.89067, 0.218662). Shifting the system by this point we get the transformed equation system

$$\begin{aligned}\dot{x}_1 &= -0.802228(4.49067 + x_1) + \frac{2.13881 + 0.437324x_1 + 9.79134x_2 + 2x_1x_2}{0.545137 + 2.43732x_2 + x_2^2} \\ \dot{x}_2 &= 7.84686 - 0.802228x_2 + \frac{-4.27762 - 0.874648x_1 - 19.5627x_2 - 4x_1x_2}{0.545137 + 2.43732x_2 + x_2^2}\end{aligned}$$

After 3 steps of iteration the minimization problem can be solved that  $e(n)$  becomes zero so we can apply (7) and we get the region of stability as it is seen in figure 4.3. Note that the given region is very small and it meets our expectations as it is described in (Szederkényi *et al.*, 2002).

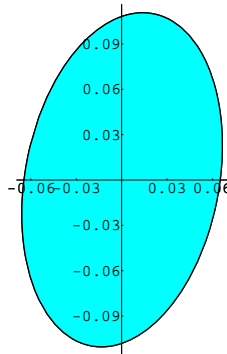


Figure 3: Stability region of the fermentation system

## 5. CONCLUSION AND FUTURE WORK

In this paper an improved algorithm based on constructing maximal Lyapunov functions (Vanelli and Vidyasagar, 1985) is shown to estimate the DOA of nonlinear autonomous systems. The advantage of this algorithm is that one does not have to know the solution of the system starting from different initial values; only a minimization problem (a linear programming problem) needs to be solved in each step of the recursive approximation procedure. Moreover, the applicable system class is wider than that of the majority of available algorithms can handle (they are mainly restricted to polynomial systems). And lastly, this algorithm is faster than Zubov's method as only about maximum ten steps are needed instead of a few times ten. In addition, it is accurate enough that one could use it instead of the cumbersome (however more exact) scanning approach.

A few problems arose during the analysis and implementation of the described algorithm, they can be divided into two main areas. One of them is the deeper investigation of the method to find broader classes of systems which can be analyzed, for example, the class of periodic non-autonomous systems or the so-called partial stable systems (for definitions see (Rouche *et al.*, 1977)) are promising. Another important problem is to seek for restrictive conditions under which the number of iterations could be estimated.

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## REFERENCES

- Camilli, F., L. Grüne and F. Wirth (2000). A regularization of Zubov's equation for robust domains of attraction. Technical Report 00-06. Universität Bremen. submitted.
- Chesi, Graziano (2005). Domain of attraction: estimates for non-polynomial systems via lmis. In: *Proc. of 16th IFAC World Congress on Automatic Control*.
- Kaslik, E., A. M. Balint and St. Balint (n.d.). Methods for determination and approximation of the domain of attraction. *Submitted to Elsevier Science*.
- Rouche, Nicolas, P. Habets and M. Laloy (1977). *Stability theory by Liapunov's direct method*. Springer-Verlag. New York.
- Szederkényi, G., N. R. Kristensen, K. M. Hangos and S. B. Jorgensen (2002). Nonlinear analysis and control of a continuous fermentation process. *Computers and Chemical Engineering* **26**, 659–670.
- Vanelli, A. and M. Vidyasagar (1985). Maximal lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica* **21**, 69–80.
- Yoshizawa, T. (1966). Stability theory by lyapunov's second method. *The Mathematical Society of Japan* **9**, 223.