AN ASPECT OF OPTIMAL ACTIVE DETECTION

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Abstract: The paper deals with an active change detection problem in a stochastic discrete-time linear multiple model framework. An active detector decides on changes in an observed system and also generates an auxiliary input signal that should improve change detection. Design of the optimal active detector is formulated as minimization of an appropriate criterion. The general solution is obtained using Bellman's principle of optimality. The main contribution of the paper is an analysis of active change detection in case of two scalar models and detection horizon of two steps. It is shown that the auxiliary input signal can improve change detection only if two models differ in certain parameters.

Keywords: Active change detection, input signal design, Optimization

1. INTRODUCTION

A change detection problem arises in many practical applications, ranging from the time series analysis to the fault detection in industrial processes. Its importance is still growing and the availability of faster computers allows to use advanced approaches to detector design.

In general, the aim is to design a detector that provides information about changes in the observed system. There are many design approaches, but stress will be laid on the model-based approach, where detector design is based on a given model of the observed system. Detectors can be divided on *passive* and *active* detectors. Passive detectors use measurements to decide on changes without influencing the observed system. More advanced detectors are called active because they generate an auxiliary input that improves change detection.

The observed system is usually described by a multiple model in known approaches to the active change detection problem. Stochastic discrete-time linear Gaussian models and the sequential probability ratio test (SPRT) are considered in (Zhang, 1989). Firstly, only two models are considered and the SPRT is used to detect a change. The auxiliary input signal is designed to optimize a selected property of the SPRT, namely the average sampling number (ASN) and the probability of a wrong decision. In the case of more than two models the SPRT must be modified (extended SPRT) to accommodate such situation. The extended SPRT is not optimal and design of the auxiliary input signal is based on minimization of the weighted sum of the criteria, where the weights are given by belief that the observed system will behave according to particular model after the change. This leads to the auxiliary input signal that can increase the probability of a wrong decision.

Another approach uses a deterministic multiple model and disturbances are modelled as signals bounded in their amplitude or energy (Nikoukhah, 1998). It is considered that system behavior does not change during a test period and the membership approach is used to determine the valid model. The aim is to find the auxiliary input signal that allows to surely determine the valid model at time of the test period. If such auxiliary input signal exist then the detector provides exact information about current behavior of the observed system. Unfortunately, it is not possible to guarantee that system behavior does not change during the test period.

In (Kerestecioğlu, 1993) an attempt to provide more general active change detection formulation was presented. The active detector was designed by minimization of a criterion that penalizes a terminal decision and the cost of all measurements needed to take this decision. The similar idea was also presented in (Šimandl and Herejt, 2003). A solution for the case, where the auxiliary input signal depends on the decision was elaborated deeply using three information processing strategies (IPS's), namely open loop, open loop feedback and closed loop strategy. The solutions obtained using these three IPS were related to solutions obtained using approaches that are well-known in change detection field. Further extensions of the idea are summarized in (Šimandl and Punčochář, 2006).

This paper presents an analysis of the active change detection problem. The goal is to discuss the influence of the auxiliary input signal on change detection in dependence on the difference between models. Only two scalar Gaussian models are considered in this analysis because a general analysis would be too difficult to perform. Firstly, the change detection problem for this special case is formulated and the solution for the detection horizon $F = 1$ is presented. A state estimation problem is briefly mentioned and then the analysis is provided. The improvement in change detection depends on the amplitude of the auxiliary input signal and the differences between corresponding parameters of two models. It is shown that some parameter differences are important and if these differences are zeros than the auxiliary input signal can not lead to better decisions.

The paper is organized as follows. Section 2 is devoted to the formulation of the active change detection problem. The observed system, the general form of the active detector and an additive criterion are described. The general solution based on the dynamic programming and a brief description of the state estimation problem are presented in Section 3. Section 4 is focused on the analysis of the active change detection. The last Section 5 provides a conclusion.

2. PROBLEM FORMULATION

The change detection problem is considered on the finite detection horizon F and the observed system is described at each time step $k \in \mathcal{T} = \{0, 1, \ldots, F\}$ by the linear Gaussian model in the state space form

$$\begin{aligned} \mathbf{x}_{k+1} &= A(\mu_k)\,\mathbf{x}_k + B(\mu_k)\,\mathbf{u}_k + G(\mu_k)\,\mathbf{w}_k, \\ \mathbf{y}_k &= C(\mu_k)\,\mathbf{x}_k + H(\mu_k)\,\mathbf{v}_k, \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathcal{R}^{n_x}$ is the immeasurable state of the system, $\mathbf{u}_k \in \mathcal{U}_k \subseteq \mathcal{R}^{n_u}$ and $\mathbf{y}_k \in \mathcal{R}^{n_y}$ denote input and output of the system, respectively. The noises $\mathbf{w}_k \in \mathcal{R}^{n_w}$ and $\mathbf{v}_k \in \mathcal{R}^{n_v}$ are mutually independent white Gaussian sequences with zero-mean and identity covariance matrices. The initial condition of the state $\mathbf{x}_0$ is independent of the noises and it has also Gaussian probability density function (pdf) with mean value $\mathbf{x}'_0$ and covariance matrix $P'_{x,0}$. The scalar discrete immeasurable variable $\mu_k \in \mathcal{M} = \{1, \ldots, N\}$ denotes the model, which is valid at time step

k. The set $\mathcal{M}$ is given in advance and the known matrices $A(\mu_k)$, $B(\mu_k)$, $C(\mu_k)$, $G(\mu_k)$ and $H(\mu_k)$ have appropriate dimensions. The switching between models is described by transition probabilities $P_{i,j} = P(\mu_{k+1} = j|\mu_k = i)$ and initial condition of the model is given by the known probability $P(\mu_0)$.

The aim of active detection is to determine the true model and the auxiliary input signal at each time step. The active detector should be a causal system. It means that the active detector can use only information obtained up to the current time step. The general form of the active detector for each time step $k \in \mathcal{T}$ is the following

$$\begin{bmatrix} d_k \\ \mathbf{u}_k \end{bmatrix} = \boldsymbol{\rho}_k\left(\mathbf{I}_0^k\right), \tag{2}$$

where $\boldsymbol{\rho}_k$ is unknown vector function describing the active detector. The decision $d_k \in \mathcal{M}$ is a point estimate of the variable μ_k and all available information received up to the time step k is denoted $\mathbf{I}_0^k = \left[\mathbf{y}_0^{k^T}, \mathbf{u}_0^{k-1^T}, d_0^{k-1^T}\right]^T$. Note that the time sequence of a variable from time step i to time step j, where $i, j \in \mathcal{T}, i < j$ is denoted as $\mathbf{y}_i^j = \left[\mathbf{y}_i^T, \ldots, \mathbf{y}_j^T\right]^T$.

The active detector should make as less as possible wrong decisions. This request is expressed by the additive criterion over whole detection horizon F

$$J(\boldsymbol{\rho}_0^F) = \mathrm{E}\left\{\sum_{k=0}^{F} L(d_k, \mu_k)\right\} \rightarrow \min, \tag{3}$$

where $\mathrm{E}\{\cdot\}$ is expectation operator and $L(d_k, \mu_k)$ is a cost function satisfying

$$\forall i, j \in \mathcal{M} : L(i,i) < L(i,j). \tag{4}$$

The cost function is chosen by the designer and it should respect real costs connected with wrong decisions. If these costs are known exactly, then proposed design provides the active detector that minimizes average costs over whole detection horizon. It should be noted that this formulation of the active change detection problem is very similar to the formulation of the optimal stochastic control.

3. ACTIVE DETECTOR DESIGN

The goal is to find the active detector (2) minimizing the additive criterion (3) given the constraints (1). The solution can be obtained using Bellman's principle of optimality, which leads to the well-known dynamic programming. The general backward recursive equation for time steps $k = F, F-1, \ldots, 0$ is

$$V_k^*\left(\mathbf{I}_0^k\right) = \min_{\substack{d_k \in \mathcal{M} \\ \mathbf{u}_k \in \mathcal{U}_k}} \mathrm{E}\left\{L(d_k, \mu_k) + V_{k+1}^*\left(\mathbf{I}_0^{k+1}\right)|\mathbf{I}_0^k, \mathbf{u}_k, d_k\right\}, \tag{5}$$

where $\mathrm{E}\{\cdot|\cdot\}$ is the conditional expectation operator and the Bellman function $V_k^*\left(\mathbf{I}_0^k\right)$ represents the minimum of the conditional mean value of the current and future costs. The initial condition for backward recursive equation is $V_{F+1}^* = 0$ and the value of the criterion can be expressed as $J^* = J\left(\boldsymbol{\rho}_0^{F^*}\right) = \mathrm{E}\left\{V_0^*(\mathbf{I}_0)\right\}$.

It is obvious that the conditional probability $P\left(\mu_k|\mathbf{I}_0^k, \mathbf{u}_k, d_k\right)$ and the pdf $p\left(\mathbf{y}_{k+1}|\mathbf{I}_0^k, \mathbf{u}_k, d_k\right)$ are needed to evaluate the recursive equation (5) at each time step k. Using properties of the system (1) it is possible to write the following identities

$$\begin{aligned} P\left(\mu_k|\mathbf{I}_0^k,\mathbf{u}_k,d_k\right) &= P\left(\mu_k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right), \\ p\left(\mathbf{y}_{k+1}|\mathbf{I}_0^k,\mathbf{u}_k,d_k\right) &= p\left(\mathbf{y}_{k+1}|\mathbf{y}_0^k,\mathbf{u}_0^k\right). \end{aligned} \tag{6}$$

Then the recursive equation (5) can be modified in the following way. Firstly, using identities (6) it can be easily shown that the Bellman function satisfies $V_k^*\left(\mathbf{I}_0^k\right) = V_k^*\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right), \forall k \in \mathcal{T}$. Further, the conditional mean value of the cost function $L(d_k,\mu_k)$ is independent of the input signal $\mathbf{u}_k$ and the conditional mean value of the Bellman function $V_{k+1}(\mathbf{y}_0^{k+1},\mathbf{u}_0^k)$ is independent of the decision d_k. Now, it is obvious that the backward recursive equation (5) can be rewritten as

$$\begin{aligned} V_k^*\left(\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right) &= \min_{d_k\in\mathcal{M}} \mathrm{E}\left\{L\left(d_k,\mu_k\right)|\mathbf{y}_0^k,\mathbf{u}_0^{k-1},d_k\right\} + \\ &\quad \min_{\mathbf{u}_k\in\mathcal{U}_k} \mathrm{E}\left\{V_{k+1}^*\left(\mathbf{y}_0^{k+1},\mathbf{u}_0^k\right)|\mathbf{y}_0^k,\mathbf{u}_0^k\right\}. \end{aligned} \tag{7}$$

The optimal decision d_k^* and the optimal auxiliary input signal $\mathbf{u}_k^*$ are

$$d_k^* = \arg\min_{d_k\in\mathcal{M}} \mathrm{E}\left\{L\left(d_k,\mu_k\right)|\mathbf{y}_0^k,\mathbf{u}_0^{k-1},d_k\right\}, \tag{8}$$

$$\mathbf{u}_k^* = \arg\min_{\mathbf{u}_k\in\mathcal{U}_k} \mathrm{E}\left\{V_{k+1}^*\left(\mathbf{y}_0^{k+1},\mathbf{u}_0^k\right)|\mathbf{y}_0^k,\mathbf{u}_0^k\right\}. \tag{9}$$

Hence, the active detector consists of two independent parts, the optimal detector and the optimal input signal generator.

The state estimation and output prediction problem will be briefly discussed. The conditional probability $P\left(\mu_k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right)$ can be expressed as

$$P\left(\mu_k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right) = \sum_{\mu_0}\cdots\sum_{\mu_{k-1}} P\left(\mu_0^k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right) = \sum_{\mu_0^{k-1}} P\left(\mu_0^k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right), \tag{10}$$

where μ_0^k is the model sequence and its conditional probability can be recursively computed using the equation

$$P\left(\mu_0^k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right) = \frac{p\left(\mathbf{y}_k|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-1},\mu_0^k\right)P\left(\mu_k|\mu_{k-1}\right)P\left(\mu_0^{k-1}|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-2}\right)}{p\left(\mathbf{y}_k|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-1}\right)}. \tag{11}$$

The pdf $p\left(\mathbf{y}_k|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-1}\right)$ is only normalization constant independent of the model sequence and the probability $P\left(\mu_0^{k-1}|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-2}\right)$ is known from previous time step. The predictive pdf $p\left(\mathbf{y}_0^{k+1}|\mathbf{y}_0^k,\mathbf{u}_0^k\right)$ can be written as

$$p\left(\mathbf{y}_0^{k+1}|\mathbf{y}_0^k,\mathbf{u}_0^k\right) = \sum_{\mu_0^{k+1}} p\left(\mathbf{y}_{k+1}|\mathbf{y}_0^k,\mathbf{u}_0^k,\mu_0^{k+1}\right)P\left(\mu_{k+1}|\mu_k\right)P\left(\mu_0^k|\mathbf{y}_0^k,\mathbf{u}_0^{k-1}\right). \tag{12}$$

In general, the predictive pdf's $p\left(\mathbf{y}_k|\mathbf{y}_0^{k-1},\mathbf{u}_0^{k-1},\mu_0^k\right)$ have to be known to evaluate the equations (11) and (12). These predictive pdf's can be computed easily, because given the model sequence μ_0^{k+1} it is possible to use standard Kalman filter to find the predictive pdf of the output $\mathbf{y}_{k+1}$ and the pdf of the state $\mathbf{x}_k$ at each time step k. The number of needed Kalman filters grows exponentially with time according to N^{k+1}, but this issue will not be discussed in this article. There are some approaches dealing with this problem, see e.g. (Boers and Driessen, 2005) for more information and references.

4. ANALYSIS OF ACTIVE DETECTION FOR TWO SCALAR MODELS

The general analysis of the active change detection problem would be too complex. Therefore, the case of two scalar models and the detection horizon $F = 1$ is analyzed. At each time step $k \in \mathcal{T} = \{0, 1\}$ the system is described by one of the following model

$$\begin{aligned} \mu_k = 1 : x_{k+1} &= a(1)x_k + b(1)u_k + g(1)w_k \\ y_k &= c(1)x_k + h(1)v_k \end{aligned} \tag{13}$$

$$\begin{aligned} \mu_k = 2 : x_{k+1} &= a(2)x_k + b(2)u_k + g(2)w_k \\ y_k &= c(2)x_k + h(2)v_k \end{aligned} \tag{14}$$

The cost function is chosen in the following form

$$\begin{aligned} L(d_k, \mu_k) = 0 &\iff \mu_k = d_k, \\ L(d_k, \mu_k) = 1 &\iff \mu_k \neq d_k. \end{aligned} \tag{15}$$

This cost function leads to the decision d_k that is the point estimate of the variables μ_k in the maximum a posteriori probability sense.

Firstly, the backward recursive equation will be presented. Considering the initial condition $V_2^* = 0$ the Bellman function at time step $k = 1$ is

$$\begin{aligned} V_1^*\left(y_0^1, u_0\right) &= \min_{d_1 \in \mathcal{M}} \sum_{\mu_1} L(d_1, \mu_1) P\left(\mu_1 | y_0^1, u_0\right) = \\ &\min\left[P\left(\mu_1 = 1 | y_0^1, u_0\right), P\left(\mu_1 = 2 | y_0^1, u_0\right)\right], \end{aligned} \tag{16}$$

where the function $\min[\cdot,\cdot]$ returns the minimum value of two input arguments. The optimal decision d_1^* is exactly given by the minimization, but the optimal input signal u_1^* is not determined. This is caused by the fact that the input signal u_1 can not influence the output of the system on the considered detection horizon. Thus, the input u_1 can take an arbitrary value from the subset $\mathcal{U}_1$. The Bellman function at time step $k = 0$ is

$$\begin{aligned} V_0^*(y_0) &= \min_{d_0 \in \mathcal{M}} \sum_{\mu_0} L(d_0, \mu_0) P(\mu_0 | y_0) + \\ &\min_{u_0 \in \mathcal{U}_0} \int V_1^*\left(y_0^1, u_0\right) p(y_1 | y_0, u_0)\, dy_1. \end{aligned} \tag{17}$$

In this case, both the optimal decision d_0^* and optimal input signal u_0^* are determined as arguments of corresponding minimizations.

The question is how the input signal u_0 influences the value of the criterion J or equivalently the conditional mean value of the Bellman function $V_1^*(y_0^1, u_0)$. By subsequent substitution (16), (10) and (11) into (17) and after cancellation out common terms the second term of the sum in (17) can be written as

$$\begin{aligned} \min_{u_0 \in \mathcal{U}_0} \int \min\Bigg[&\sum_{\mu_0 \in \mathcal{M}} p(y_1 | y_0, u_0, \mu_0, \mu_1 = 1) P(\mu_1 = 1 | \mu_0) P(\mu_0 | y_0), \\ &\sum_{\mu_0 \in \mathcal{M}} p(y_1 | y_0, u_0, \mu_0, \mu_1 = 2) P(\mu_1 = 2 | \mu_0) P(\mu_0 | y_0) \Bigg] dy_1. \end{aligned} \tag{18}$$

Given the fixed input u_0 the both arguments of inner minimization in the above expression are two terms weighted Gaussian sums with respect to the variable y_1. Considering standard Kalman filter it is obvious that the input signal u_0 influences only mean values of the pdf's $p(y_1|y_0, u_0, \mu_0^1) = \mathcal{N}\{m_{\mu_0,\mu_1}, C_{\mu_0,\mu_1}\}, \mu_0^1 \in \mathcal{M} \times \mathcal{M}$. The relations for computing these mean values are given in Table 1, where the columns corresponds to the Gaussian sums and $\hat{x}_0(1)$, $\hat{x}_0(2)$ are the state estimates obtained from Kalman filter for situations $\mu_0 = 1$ and $\mu_0 = 2$, respectively. Considering the expression (18) it is clear that the optimal input signal

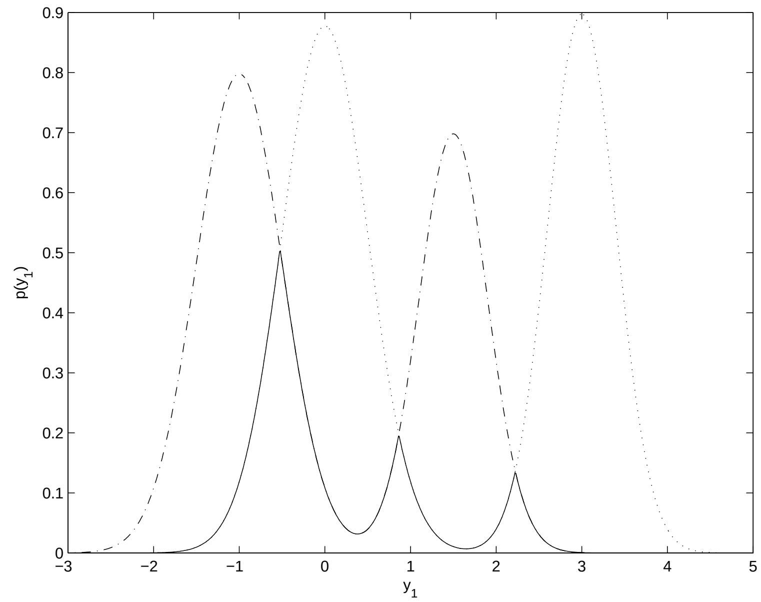
Table 1: Mean values of the predictive pdf's $p\left(y_1|y_0, u_0, \mu_0^1\right)$

	$\mu_1 = 1$	$\mu_1 = 2$
$\mu_0 = 1$	$m_{1,1} = c(1)a(1)\hat{x}_0(1) + c(1)b(1)u_0$	$m_{1,2} = c(2)a(1)\hat{x}_0(1) + c(2)b(1)u_0$
$\mu_0 = 2$	$m_{2,1} = c(1)a(2)\hat{x}_0(2) + c(1)b(2)u_0$	$m_{2,2} = c(2)a(2)\hat{x}_0(2) + c(2)b(2)u_0$

u_0^* minimizes an area under function that is given as minimum of the Gaussian sums, see Figure 1. This area is determined by overlap between the Gaussian sums. It leads to the general conclusion that the optimal input signal u_0^* moves the Gaussian sums apart as much as possible. The important question is how the differences between parameters of the models influence the active change detection. In the following four cases only the relevant parameters will be considered and conclusions are based directly on expressions in Table 1.

- The parameters of the models satisfy $a(1) = a(2)$, $b(1) = b(2)$, $c(1) = c(2)$. All differences between mean values $m_{1,1}$, $m_{1,2}$, $m_{2,1}$ and $m_{2,2}$ are independent of input signal u_0. Thus, the input signal can not improve change detection.
- The parameters of the models satisfy $a(1) \neq a(2)$, $b(1) = b(2)$, $c(1) = c(2)$. Again, the input signal can not improve change detection because the differences between mean values are not functions of the input signal. If a longer detection horizon is considered then input signal can improve change detection.
- The parameters of the models satisfy $a(1) = a(2)$, $b(1) \neq b(2)$, $c(1) = c(2)$. The differences $m_{1,1} - m_{1,2}$ and $m_{2,1} - m_{2,2}$ are zero. It means that mean values of the first Gaussian sum are the same as mean values of the second Gaussian sum. There are two subcases.
  - It holds that $h(1) = h(2)$ and$P(\mu_1 = i|\mu_0) = P(\mu_1 = j|\mu_0)\, \forall i, j, \mu_0 \in \mathcal{M} : i \neq j$. In this case the both Gaussian sums are exactly the same functions of the output y_1 and the input signal u_0. Hence the overlap is whole Gaussian sum and this overlap does not change in dependence on the input signal. Hence input signal can not improve change detection.
  - In other cases the input signal can improve change detection because of differences between Gaussian sums. If the input signal tends to infinity then the distance between mean values of the first and second Gaussian sum tends to infinity. However, the overlap of Gaussian sums does not tends to the zero but to a value that depends on the parameters of both models. Thus, even infinity input signal can not guarantee that the optimal decision d_1^* will be almost surely right.
- The parameters of the models satisfy $a(1) = a(2)$, $b(1) = b(2)$, $c(1) \neq c(2)$. The shape of both Gaussian sums is independent of the input signal but their relative position depends on the input signal. Hence, the input signal can improve change detection. If the infinity input signal is used then the overlap between Gaussian sums tends to zero. It means that the optimal decision d_1^* will be almost surely right.

Fig. 1: Gaussian sums: (..) Gaussian sum for $\mu_1 = 1$, (.-) Gaussian sum for $\mu_1 = 2$, (–) minimum of the Gaussian sums



5. CONCLUSION

The paper deals with the analysis of the active detector behavior in the case of two scalar linear Gaussian models and detection horizon of two steps. It was shown that active detection is advantageous only if the models differ in the parameter b or c. For longer detection horizon the active change detection can be also advantageous when the models differ only in the parameter a. Further, it is obvious that the statistical properties of the noises are not important from this point of view. An extension of this analysis to multidimensional system is not straightforward. The following two aspects have to be taken into account: infinite number of equivalent state space representations and a delay longer than one time step. Therefore, this case and the case with more then two models should be addressed in the future.

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