COMPUTATIONAL ASPECTS OF CONTROLLER DESIGN AND QUALITY EVALUATION

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Abstract: An important part of controller design is the controller tuning. Tuning is done by searching such values of controller tuning parameters in order to achieve aims on the closed loop given by user. The fulfillment of these aims is measured by quality function. To evaluate the quality a Monte Carlo simulation is used. To estimate how long simulation is needed, on-line stopping rules are proposed. The Kullback-Leibler divergence is utilized to measure stabilization of the quality.

Keywords: Adaptive Control, LQG Controller, Optimization

1. INTRODUCTION

Computational efficiency is an important factor of controller design using Monte Carlo method. Controller tuning is a process aiming at the correct controller set-up to fulfill given constraints and requirements. A controller depends on certain parameters, called tuning knobs, which have to be set properly to obtain desired control loop behavior. Model-based predictive controllers are, in some sense, optimal. However, the optimality is conditioned by the perfect model fit to the controlled plant. And, the optimality from user's point of view need not match the kind of optimality acceptable for the controller. The reasons for controller tuning are:

1. The assumed controller uses a system model that does not fit to the reality due to incorrectly identified model parameters, or even the structure or type of model is not perfect. The tuning knobs have to be set to suppress control error caused by the model mismatch. In another words the controller setting influences its robustness.

2. The optimality criterion of the controller is not able to express the user's desired kind of optimality. The selected controller requires a different formulation of the task. The tuning process converts the desired optimality into the form acceptable by controller.

The tuning is an optimization task searching the best tuning knob values. The controller behavior is evaluated from predicted closed loop performance. The prediction is calculated using simulation of a model identified from data measured on real plant and user supplied prior information.

However, nested optimization and simulation procedures brings significant computational demands. Thus it is necessary to make computation as short as possible while keeping satisfactory precision. This problem is solved by introducing on-line stopping rules which decide how long the simulation has to be to obtain stabilized results. The decision is made on-line, so instead of making some global estimate, the simulation length fits to the actually simulated data.

2. TUNING AS A BAYESIAN DECISION TASK

This section describes the controller tuning as a Bayesian decision making task and applied for the particular case of the closed loop, user-defined constraints, controller quality and tuning parameters. The particular construction elements are described in terms of experience, action, innovation and decision making. First of all let us present the tuned closed loop.

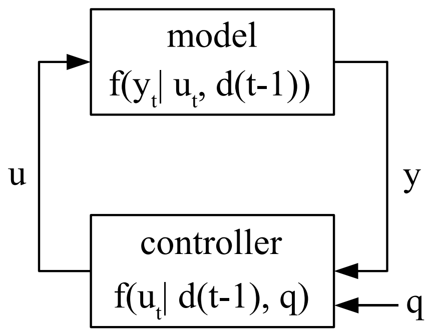

Fig. 1: Closed loop.

The classical interconnection between controlled system and controller, see Figure 1, generates closed loop data $d(T) = (d_1, d_2, \ldots, d_T)$ of length T. The data $d_t = (u_t, y_t)$ collect the input u_t driven by the controller and the output y_t measured on the controlled system. The closed loop forms a stochastic system as the controlled model is considered to be influenced by a random disturbance. Thus the model behavior is described by pdf $f(y_t|u_t, d(t-1))$. The controller is described generally as a random one by pdf $f(u_t|d(t-1), q)$, where q denotes the tuning parameter. The closed loop data $d(T)$ are therefore also a random variable described by pdf obtained by application of the chain rule over the horizon T

$$f(d(T)|q) = \prod_{t=1}^{T} f(y_t|u_t, d(t-1)) f(u_t|d(t-1), q). \tag{1}$$

2.1 Decision Making

For the purpose of controller tuning, the user's requirements imposed on the desired closed loop behavior are represented by a pair of so called controller quality functions Z_c and Z_o defined on the closed loop data $d(T)$. The first function Z_c represents the constraints imposed on the data. It is a mapping

$$Z_c : d(T)^* \mapsto \mathbf{R}^{\mathring{c}}, \tag{2}$$

where $\mathring{c}$ denotes the number of independent constraints. The constraints are considered being met if the expected value of function Z_c is non-positive. The second function Z_o is a mapping

$$Z_o : d(T)^* \mapsto \mathbf{R}. \tag{3}$$

It represents a loss function which is decreasing with increasing controller performance with respect to the output error.

The aim of the tuning as finding such a tuning parameter value that satisfies the constraints while maximizes the performance is stated as the following optimization task

$$\begin{array}{rl} \text{minimize} & E[Z_o|q] \\ \text{subject to} & E[Z_c|q] \leq 0 \\ \text{over the tuning parameters} & q. \end{array} \tag{4}$$

3. CLOSED LOOP PERFORMANCE EVALUATION

In this section, requirements and constraints imposed on the ideal closed loop behavior are defined. Their fulfillment is measured by the controller quality functions Z_o and Z_c. The construction of these functions is described.

3.1 Loss Function

The control objective expresses commonly the aim assigned to the quality of the regulation process, which should be in a certain sense as good as possible subject to the present constraints. The typical wish on the small output error and the control effort of inputs is expressed by the objective function Z_o

$$Z_o = \frac{1}{T} \sum_{\tau=1}^{T} (d_\tau - d_\tau^{\text{ref}})' W (d_\tau - d_\tau^{\text{ref}}), \tag{5}$$

where the desired signal setpoints are described by the reference trajectory $\{d_\tau^{\text{ref}}\}_{\tau=1}^{T}$ and a positive semi-definite matrix W of appropriate dimensions.

The matrix W is usually diagonal with only those elements being non-zero which correspond to signals in the data record d_t with an important prescribed reference trajectory or setpoint in d_t^{ref}. The particular values define the cost of particular signal output error.

The elements of matrix W are user's choice, but they do not substitute the proposed tuning algorithm of parameters q. The function Z_o is of a secondary importance as the primary goal of tuning is to satisfy the specified constraint.

3.2 Constraints

Constraints are often imposed not only on the magnitudes of input and output quantities but also on their dynamic behavior such as limited increments. To cope with these constraints uniformly, a vector variable c_t containing all constrained dynamic expressions of data quantities is introduced.

A vector c_t is extracted from data $d(t)$ by a mapping $\mathcal{C}$

$$\mathcal{C} : d(t)^* \mapsto \mathbf{R}^{\mathring{c}}, \quad \forall t = 1, \dots, T. \tag{6}$$

Using this mapping, the vector c_t can be obtained for the whole time span that is denoted by $c(T) = \{c_t\}_{t=1}^{T}$. The constraints are defined by a set $C \subset \mathbf{R}^{\mathring{c}}$ of allowed values defining the constraint satisfaction in time t by $c_t \in C$.

A common example of independent time invariant constraints is formed by the cartesian product $C = \bigotimes_{i=1}^{\mathring{c}} C_i$ of intervals, where $\mathring{c}$ is dimension of constrained vector c_t. The intervals C_i are defined

$$C_i = \langle c_i^{\min}, c_i^{\max} \rangle \tag{7}$$

In the most practical tasks, vector c_t contains magnitudes and increments of data records. The corresponding function $\mathcal{C}$ is

$$c_t = \mathcal{C}(d(t)) = [d_t, d_t - d_{t-1}].$$

The constraint function Z_c introduced in (2) is now described using the constraint vectors c_t.

$$Z_c : c(T)^* \mapsto \mathbf{R}^{\mathring{c}}, \tag{8}$$

This redefinition does not change the meaning of the function because the constraint variable $c(T)$ is function (6) of the data $d(T)$.

Two variants of function Z_c for servo control Z_{c_M} and noise compensation Z_{c_P} tasks are used as described in the rest of this section.

3.3 Servo Control Task

The constraint function Z_{c_M} collects information about maximal constraint violation during the simulation run

$$Z_{c_M,i} = \max_{t=1,\dots,T} \operatorname{dist}(c_{i,t}, C_i) - \operatorname{dist}(c_{i,t}, \operatorname{comp}(C_i)), \tag{9}$$

where $\operatorname{comp}(C_i)$ is a set complement of C_i, $Z_{c_M,i}$ is i-th element of Z_{c_M}, and $\operatorname{dist}(x, X)$ denotes a distance between point x and set X. This definition of function is suitable mainly for transient processes, where the constrained signals have one or just a few important peaks, such as servo control tasks. The time T is selected big enough to cover all the instants with significant signal changes.

3.4 Noise Compensation Task

The second function Z_{c_P} evaluates proportional amount of time where constraints are violated over the total length of simulation with some allowed tolerance. In the discrete case, it is the relative frequency of constraint satisfaction

$$Z_{c_P,i} = \alpha_{\min} - \frac{1}{T} \sum_{t=1}^{T} \chi_{C_i}(c_{i,t}), \tag{10}$$

where χ_{C_i} is characteristic function of the set C_i, and number $\alpha_{\min} \in \langle 0, 1 \rangle$ relaxes the requirement of constraint satisfaction to a specified level.

This definition is suitable for situations where the constraints can be violated any time during the simulation. This is the case of noise compensation control, where the control loop generates an ergodic process. Then it holds

$$Z_{c_P,i} \overset{T\to\infty}{\longrightarrow} \alpha_{\min} - \mathbf{P}(c_i \in C_i) \quad \text{almost surely},$$

where $\mathbf{P}(\cdot)$ denotes probability and c_i has dropped the time index because of ergodicity of the process.

4. NUMERICAL EVALUATION

In this section, a numerical approach to estimation of expected value from samples is described. The computational complexity is reduced by introducing stopping rules shortening the simulations.

4.1 Expected Value Estimation

The controller tuning, formulated as the optimization task (4), acts on the conditional expectation of the controller quality functions Z_c and Z_o. However their pdf is not known in a closed form, because of the complexity of the dynamic system model and adaptive controller. Thus the expected value has to be estimated by sampling. To unify the notation in the following text, let $Z_\bullet$ denote all the quality functions distinguished by the content of the placeholder "$\bullet$" for "c_M", "c_P" or "o". The expectation $E[Z_\bullet|q]$ is estimated as sample mean

$$Z_\bullet^N(q) = \frac{1}{N}\sum_{s=1}^{N} Z_{\bullet,s}(q) \stackrel{N\to\infty}{\longrightarrow} E[Z_\bullet|q], \tag{11}$$

Sequence $\{Z_{\bullet,s}(q)\}_{s=1}^N$ denotes N samples of $Z_\bullet$ from $f(Z_\bullet|q)$.

4.2 Number and Length of Simulations

The quantity $Z_\bullet^N$ is evaluated using N independent simulation runs. The length of each run is determined by T. Increasing these two numbers N and T increases precision of the expected value approximation $Z_\bullet^N$ of the controller quality functions. On the other hand, it also increases the computational demands of the evaluation, thus the lengths have to be limited. To solve this tradeoff, the on-line stopping rules are employed. First, the properties of the quality functions with respect to number and length of simulations are described.

The variance of the quantity $Z_\bullet^N$ is indirectly proportional to the number of independent simulation runs N, which is clear from its evaluation (11).

The similar situation occurs for length of simulation T, which has to be long enough in order to:

1. Contain all important reference trajectory changes.
2. Allow the uncertain parameters to vary in order to simulate the controller adaptiveness.
3. Decrease the variance of the controller quality functions.

The item 1 is straightforward. It is used for transient processes, where a kind of constraint measure Z_{c_M} is used. Of course, all responses related to reference trajectory changes have to be included, too.

The situation of items 2 and 3 is more complicated. Both of the items contribute to the precision of the expected value estimate. Even more, the item 2 can be substituted by item 3, because if the variance is low, it means that further parameters changes bring no more information on the controller quality functions.

Increasing the simulation length T for the ergodic case, such as the noise compensation, has the same effect as increasing the number N of the simulations. Thus, one long simulation is sufficient.

The proper values of N and T are decided on-line during simulation using the Chebyshev inequality and the Kullback-Leibler divergence. The on-line stopping is advantageous in comparison with the off-line determination of the length and number of simulations, because it considers the contribution of the actual data and thus stopping is optimal for the current simulation unlike for all possible simulation runs as in the case of a priori selected N and T values.

4.3 On-line Stopping Rule for Number of Simulations

The independent simulation runs are connected mainly with non-stationary servo-control tasks. It is hard to find a reasonable distribution of the quality functions $Z_\bullet$ for different variants of reference trajectory. Thus, a simple non-parametric stopping rule is used. It is activated when the following inequality is satisfied

$$\mathbf{P}(|Z_\bullet^N - EZ_\bullet^N| \geq \gamma) \leq \beta, \tag{12}$$

where parameters β and γ determine the sensitivity of the stopping.

The stopping is based on variance of $Z_\bullet^N$ as shown below. The independency of averaged quality functions (11) resulting to $Z_\bullet^N$ used with Chebyshev inequality yields

$$\mathbf{P}(|Z_\bullet^N - EZ_\bullet^N| \geq \gamma) \leq \frac{\mathrm{var}(Z_\bullet^N)}{N\gamma^2}. \tag{13}$$

As covariance $\mathrm{var}(Z_\bullet^N)$ is unknown, its estimate $Z_{\sigma,\bullet}^N$ is used

$$Z_{\sigma,\bullet}^N = \sum_{s=1}^{N} \frac{(Z_{\bullet,s})^2 - (Z_\bullet^N)^2}{N}, \tag{14}$$

where variable $Z_{\bullet,s}$ has the same meaning as in (11). Then the stopping is triggered after certain minimal number of simulations is performed and when the following inequality is satisfied

$$\frac{Z_{\sigma,\bullet}^N}{N\gamma^2} \leq \beta. \tag{15}$$

4.4 On-line Stopping Rule for Simulation Length

A rule for on-line simulation stopping for the noise compensation task is described here. The function $Z_\bullet$ contains a sum (5) or (10), but the summed terms are correlated, so the approach using the Chebyshev inequality from Section 4.3 cannot be applied. Let the summed terms (20) of Z_o and $\chi_{C_i}(c_{i,t})$ of Z_{c_P} forming the controller quality functions be called partial controller quality and be denoted by v_t. For the noise compensation task we assume the closed loop signals be ergodic and thus also the partial losses are ergodic.

To find a reasonable stopping rule, a simple dynamic model of v_t

$$f(v_t|v(t-1), \Xi). \tag{16}$$

is being estimated in Bayesian way. Let the parameters of the model be denoted by Ξ. When the estimated pdf $f(\Xi|v(t))$ of the model parameters Ξ stabilizes, the stopping takes place. The stabilization of pdf $f(\Xi|v(t))$ is measured by the Kullback-Leibler divergence $\mathcal{D}_{\mathrm{KL}}$ of two successive pdf estimates (Kárný *et al.*, 2005). It is defined by

$$\mathcal{D}_{\mathrm{KL}}(f(\Xi|d(T))\|f(\Xi|d(T-1))) = \int f(\Xi|d(T)) \ln \frac{f(\Xi|d(T))}{f(\Xi|d(T-1))} d\Xi. \tag{17}$$

When this divergence, labeled U_T, becomes smaller than some threshold value ε

$$U_T = \mathcal{D}_{\mathrm{KL}}(f(\Xi|d(T))\|f(\Xi|d(T-1))) \leq \varepsilon, \tag{18}$$

the computation is stopped. At this moment T, the pdf $f(\Xi|d(T))$ is considered to reach the steady state. The stationarity means that more data would not bring significantly more information for the estimate. The dynamic model of variable v (16) is used just for determination of the stopping time while the loss function is calculated by its original defining equation (5) or (10). This approach was mentioned in (Kárný *et al.*, 2005).

Yet there is a better opportunity of calculating the loss function value from the estimated dynamic model of the partial loss v by evaluating its stationary pdf. This approach is used in the next paragraph with ARX model. It is shown that $f(Z|d(T))$ is stabilizing as $f(\Xi|d(T))$ is stabilizing. In other words the divergence

$$\mathcal{D}_{\mathrm{KL}}(f(Z|d(T))\|f(Z|d(T-1))) \tag{19}$$

is decreasing as $\mathcal{D}_{\mathrm{KL}}(f(\Xi|d(T))\|f(\Xi|d(T-1)))$ is decreasing.

The definition of the quantity v_t and the construction of the particular models for the functions Z_o and Z_{c_P} is described in the following paragraphs using ARX and Markov chain models. The stopping rule for whole simulation is triggered when the conditions for both loss and constraint function stopping are activated.

Approximation by ARX Model This section describes a suitable model type (16) of the partial quality v_t used for determination of the stopping time when evaluating the loss function Z_o. The quantity v_t for the function Z_o as the summed term in (5) is the weighted distance between the data variable d_t and its referential value d_t^{ref} in time t

$$v_t = (d_t - d_t^{\mathrm{ref}})' W (d_t - d_t^{\mathrm{ref}}). \tag{20}$$

For purpose of stopping quite a rude dynamic approximation of v_t by a simple autonomous ARX model is used.

$$v_t = a v_{t-1} + k + e_t, \quad e_t \sim \mathcal{N}(0, R). \tag{21}$$

The parameters a, k, and R are collected into the variable Ξ, where $[a, k] = \Xi_\theta$, $R = \Xi_R$ and $\Xi = [\Xi_\theta, \Xi_R]$.

The Bayesian identification of the parameters Ξ leads to the self reproducing Gauss-inverse-Wishart prior/posterior pdf

$$\begin{aligned} f(\Xi|v(t)) &= f(\Xi_\theta, \Xi_R | V_t, \nu_t) = \\ &= \alpha_t |R|^{-\frac{\nu_t}{2}} \exp\left\{ -\frac{1}{2} \mathrm{tr}\left(\Xi_R^{-1} \begin{bmatrix} -I \\ \Xi_\theta \end{bmatrix}' V_t \begin{bmatrix} -I \\ \Xi_\theta \end{bmatrix} \right) \right\}, \end{aligned} \tag{22}$$

where α_t is a normalizing constant. Statistics ν_t and V_t and parameter elements Ξ_θ and Ξ_R written only as θ and R without Ξ.

The stationarity measure for the Z_o function denoted by $U_{o;t}$, by means of the Kullback-Leibler divergence of two successive estimated pdfs of Ξ, has the form (Kárný *et al.*, 2005)

$$U_{o;t} = \mathcal{D}_{\mathrm{KL}}(f(\Xi|v(t))\|f(\Xi|v(t-1))) = \frac{F(\nu_t) + G(\zeta_t) + H(\nu_t, \varrho_t, \zeta_t)}{2}, \tag{23}$$

where

$$\begin{aligned}
F(\nu_t) &= 2\ln\left(\Gamma\left(\frac{\nu_t - 1}{2}\right)\right) - 2\ln\left(\Gamma\left(\frac{\nu_t}{2}\right)\right) + \frac{\partial \ln(\Gamma(\frac{\nu_t}{2}))}{\partial \frac{\nu_t}{2}} \\
G(\zeta_t) &= \ln(1+\zeta_t) - \frac{\zeta_t}{1+\zeta_t} \\
\varrho_t &= \frac{\hat{e}_t^2}{D_{y,t-1}(1+\zeta_t)} \\
H(\nu_t, \varrho_t, \zeta_t) &= (\nu_t - 1)\ln(1+\varrho_t) - \frac{\nu_t \varrho_t}{(1+\varrho_t)(1+\zeta_t)}.
\end{aligned}$$

The quantities ζ_t, $\hat{e}_t$, and $D_{y,t-1}$ see (Kárný *et al.*, 2005).

When the divergence $U_{o;T}$ is less than threshold ε in time T then it is assumed that enough information has been collected and the loss function Z_o (5) is precise enough.

Loss Evaluation from Dynamic Model It is possible to evaluate the mean value of loss function Z_o directly from the dynamic stopping model of v_t (16) instead of its original definition (5), were the stabilization property of EZ_o is implied by stabilization of the dynamic model parameters.

This is obtained by transforming the dynamic model (16) into a static one. First, the transformation for deterministic parameters Ξ is given and then the distribution of uncertain ones is transformed.

Suppose now that the parameters a, k, R of dynamic model (16) are known and stable, $|a| < 1$, then the corresponding static model is given by pdf

$$v_t = \mathcal{N}(p, q), \tag{24}$$

where parameters p, q are given by

$$p = \frac{k}{1-a} \tag{25}$$

$$q = \frac{R}{1-a^2} \tag{26}$$

The new parameter p is a suitable estimate of Z_o, as $Z_o = \frac{1}{T}\sum_{t=1}^{T} v_t$. Thus

$$p = Ev_t \doteq EZ_o,$$

where the $\doteq$ sign means approximately equal as the stopping model (21) is just an approximation.

If the model (16) is unstable, $|a| \geq 1$, the loss Z_o is infinity.

Now we drop the assumption of certain parameters. As the model (16) is estimated in Bayesian way, its parameters are uncertain. Thus parameters p and q of the static model (24) are uncertain, too. The estimate of Z_o is therefore selected as expected value of p

$$EZ_o \doteq Ep \tag{27}$$

The pdf of p is obtained by transforming quantities k, a and R according to (25). Unfortunately, the posterior pdf of parameters Ξ is Gauss-inverse-Wishart and it has infinite support for parameter $\Xi_a = a$. Situation when $|a| \geq 1$ and the estimated model (21) is unstable has non-zero probability. This conforms to the reality where a system model with uncertain parameters connected in closed loop can be with some probability unstabilizable.

As this situation is generally unavoidable, we have to accept that the stopping model (21) is unstable with some low probability $\mathbf{P}(|a| \geq 1)$. However this makes the estimate of EZ_o infinite. When evaluating the Z_o directly from simulation by (5) and the closed loop shows to be unstable, the estimated model is rejected by the tuning algorithm. Thus the results are limited to the stabilizable models only. So when we approximate the EZ_o from stable stopping models only $|a| < 1$ we obtain the same result. Therefore we may restrict the transformation (25) to $|a| < 1$.

Stopping properties of transformed quantity The stopping property with respect to the Kullback-Leibler divergence of parameter pdf $f(\Xi)$ (18) implies the same property for transformed quantity p, which is used to estimate EZ_o (27).

$$\mathcal{D}_{\mathrm{KL}}(f(p|d(T))\|f(p|d(T-1))) \leq \mathcal{D}_{\mathrm{KL}}(f(\Xi|d(T))\|f(\Xi|d(T-1))) \leq \varepsilon$$

This can be proven by writing the transformation (25) restricted on $|a| < 1$, lets denote it G, as a composition $G = S \circ H$ of regular transformation $H : p = \frac{k}{1-a},\ k = k$ and projection S selecting only element p from result of S.

The Kullback-Leibler divergence remains unchanged when transforming the quantity by a regular transformation. Let $f(x)$ and $g(x)$ be pdfs on quantity x. Transformed quantity $y = H(x)$ has pdf $\tilde{f}(y) = f(H^{-1}(y))|J_{H^{-1}}(y)|$ and similarly for pdf g. Then it holds

$$\mathcal{D}_{\mathrm{KL}}(\tilde{f}(y)\|\tilde{g}(y)) = \int f(H^{-1}(y))|J_{H^{-1}}(y)| \ln \frac{f(H^{-1}(y))}{g(H^{-1}(y))} dy =$$
$$= \int f(x) \ln \frac{f(x)}{g(x)} dx = \mathcal{D}_{\mathrm{KL}}(f(x)\|g(x)).$$

The projection transformation decreases the value of the Kullback-Leibler divergence

$$\mathcal{D}_{\mathrm{KL}}(f(a)\|g(a)) \leq \mathcal{D}_{\mathrm{KL}}(f(a,b)\|g(a,b)) \tag{28}$$

This is proven by

$$\mathcal{D}_{\mathrm{KL}}(f(a,b)\|g(a,b)) =$$
$$= \int\int f(a,b) \ln \frac{f(a,b)}{g(a,b)} da db =$$

$$
\begin{aligned}
&= \int f(a) \int f(b|a) \ln \frac{f(b|a)f(a)}{g(b|a)g(b)} dbda = \\
&= \int f(a) \ln \frac{f(a)}{g(a)} da + \int f(a) \int f(b|a) \ln \frac{f(b|a)}{g(b|a)} dbda = \\
&= \mathcal{D}_{\mathrm{KL}}(f(a) \| g(a)) + \int f(a) \mathcal{D}_{\mathrm{KL}}(f(b|a) \| g(b|a)) da \geq \mathcal{D}_{\mathrm{KL}}(f(a) \| g(a))
\end{aligned}
$$

Approximation of Transformed Expected Value Applying Taylor series expansion of transformation $p = G(\Xi)$ in point $E\Xi$ we obtain

$$p = G(E\Xi) + (\Xi - E\Xi)\nabla G(E\Xi) + \frac{1}{2}(\Xi - E\Xi)\nabla^2 G(E\Xi)(\Xi - E\Xi)' + \cdots, \tag{29}$$

which is in expected value

$$Ep = EG(\Xi) = G(E\Xi) + \frac{1}{2}\mathrm{tr}(\nabla^2 G(E\Xi)\mathrm{cov}\Xi) + \cdots. \tag{30}$$

Using just the first order approximation we obtain

$$Ep \doteq \frac{Ek}{1 - Ea}$$

The expected value Ea should be evaluated from the distribution with support only on $|a| < 1$, as described above. Nevertheless, the probability $\mathbf{P}(|a| > 1)$ is low. So using the expected value from original Gauss-inverse-Wishart distribution on Ξ is sufficient. This approximation gives good results comparing to calculation directly from definition of Z_o (5) according to experimental testing.

Markov Chain Estimation The calculation of the constraint function Z_{c_P} (10) includes an estimate of constraint satisfaction probability using characteristic function of the allowed set. To determine precision of this estimate, the task is slightly extended.

Given the i-th element of the constraint quantity $c_{i;t}$ from Section 3.2 and the corresponding constraining interval C_i from (7), let $\{v_t\}_{t=1}^T$ be a sequence indicating the relative position of $c_{i;t}$ to C_i

$$v_{i;t} = \begin{cases} 1 & c_{i;t} > C_i \\ 0 & c_{i;t} \in C_i \\ -1 & c_{i;t} < C_i \end{cases}, \tag{31}$$

where the inequality symbol is understood as it holds for all the elements of the interval on its right side.

The dynamic model (16) of the discrete variable $v_{i;t}$ is represented by Markov chain

$$f(v_{i;t}|g_{i;t-1}, \Xi) = \Xi_{v_i|g_i}, \text{ where } \Xi_{v_i|g_i} \geq 0 \text{ and } \sum_{v_i} \Xi_{v_i|g_i} = 1. \tag{32}$$

The notation of $\sum_{v_i}$ denotes a sum over the whole set of possible values v_i^*, the analogous situation holds also for the product in the following text. As the quantity v_t is now discrete, the symbol f represents a probability function now. The quantity $g_{i;t-1}$ contains the past values of $v_{i;t}$

$$g_{i;t-1} = [v_{i;t-1}, v_{i;t-2}, \ldots, v_{i;t-\eta}].$$

The number η denotes the order of the Markov chain. The parameter $\Xi_{v|g}$ has $3^{\eta+1}$ entries. The following derivations are done for single element of v_t only and the element index i is omitted for the sake of simplicity.

Using the Bayes' rule and the conjugated prior on $f(\Xi)$ defined by the statistic $V_{0,v|g}$

$$f(\Xi) \propto \prod_g \prod_v \Xi_{v|g}^{V_{0,v|g}-1},$$

we obtain the posterior pdf of the parameters Ξ

$$f(\Xi|v(t)) = \frac{\prod_g \prod_v \Xi_{v|g}^{V_{v|g;t}-1}}{B(V_t)},$$

where

$$V_{v|g;t} = V_{0,v|g} + \sum_{\tau=1}^{t} \delta(v, v_t)\delta(g, g_t)$$

with $\delta(\cdot,\cdot)$ being the Kronecker delta and the normalizing factor

$$B(V_t) = \prod_g \frac{\prod_v \Gamma(V_{v|g;t})}{\Gamma(\sum_v V_{v|g;t})}.$$

The stopping rule uses the Kullback-Leibler divergence to determine if there is collected enough information about the constraint function Z_{c_P}. The calculation is stopped whenever the divergence of two successive pdfs, denoted by $U_{c;T}$, is less or equal to threshold ε

$$U_{c;T} = \mathcal{D}_{\mathrm{KL}}(f(\Xi|v(T)) \| f(\Xi|v(T-1))) \le \varepsilon. \tag{33}$$

Derivation of this divergence for the Markov chain model is done through converting it to the Dirichlet model, for which the divergence is analyzed in (Kárný *et al.*, 2005).

Parameters $\Xi_{v|g}$ are independent for different past data g. Thus

$$f(\Xi|v(t)) = \prod_g f(\Xi_{\bullet|g}|v(t)),$$

where the particular factors

$$f(\Xi_{\bullet|g}|v(t)) = \frac{\Gamma(\sum_v V_{v|g;t})}{\prod_v \Gamma(V_{v|g;t})} \prod_v \Xi_{v|g}^{V_{v|g;t}-1}$$

are distributed by the Dirichlet distribution. In each time step, only one of these factors is updated—that one with corresponding past data $g = g_{t-1}$. The other factors remain unchanged.

As it holds

$$\mathcal{D}_{\mathrm{KL}}(f_1(x)f(y) \| f_2(x)f(y)) = \mathcal{D}_{\mathrm{KL}}(f_1(x) \| f_2(x)),$$

thus

$$\mathcal{D}_{\mathrm{KL}}(f(\Xi|v(t)) \| f(\Xi|v(t-1))) = \mathcal{D}_{\mathrm{KL}}(f(\Xi_{\bullet|g_t}|v(t)) \| f(\Xi_{\bullet|g_t}|v(t-1))) \tag{34}$$

is a divergence of two Dirichlet distributions. Now, the divergence of the two Dirichlet distributions derived in (Kárný *et al.*, 2005) can be used in (34) and the stopping rule (33) then yields

$$U_{c;t} = -\ln \frac{V_{v_t|g_t;t-1}}{\sum_v V_{v|g_t;t-1}} + \frac{\partial}{\partial V_{v_t|g_t;t}} \ln \Gamma\left(V_{v_t|g_t;t}\right) - \frac{\partial}{\partial \sum_v V_{v|g_t;t}} \ln \Gamma(\sum_v V_{v|g_t;t}). \tag{35}$$

At the stopping time T, determined by (33), a stabilized MC model is obtained.

For the stopping purposes only first order, $\eta = 1$, Markov chain is used. Its steady state probability $\mathbf{P}(v_t = 0)$ of state number zero in (31) can be used for obtaining the value of Z_{c_P}. The steady state p evaluation for Markov chain with certain parameters Ξ requires calculation of vector p such that $\sum_{i=1}^{3} p_i = 1$ and $\Xi p = p$.

For uncertain Ξ with Dirichlet distribution it is difficult to calculate the distribution of steady state p. Also another problem arises when using smooth optimization technique for constraints satisfaction measure evaluated from only finite number of samples. Thus the Markov chain is evaluated only for stopping purposes.

Properties of the Stationarity Measures Illustrated on an Example To show the properties of the stationarity measures $U_{c;t}$ (35) and $U_{o;t}$ (23) using ARX and MC stopping models, a simple illustrative experiment is presented. The results can be seen in Figure 2. The data used for

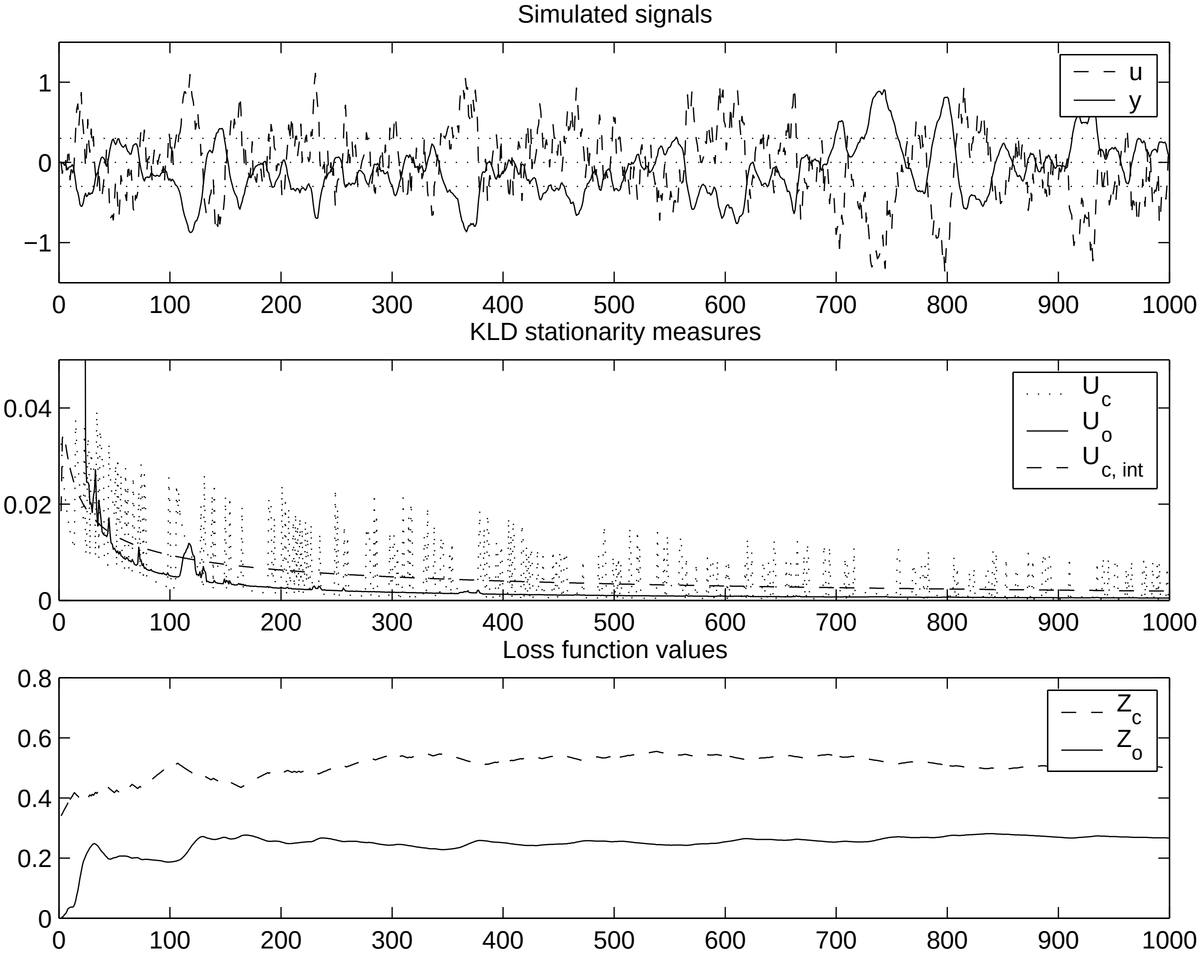

Fig. 2: Properties of stationarity measures. The symbols u and y denote inputs and outputs, U_c and U_o are the stationarity measures obtained by Markov chain and logarithm ARX model approximations. U_c^{int} is interpolation of U_c. Z_c and Z_o are controller quality functions of constraint violation and output error. Horizontal axis represents the time.

the evaluation of the loss and constraint functions were generated using the linear system with transfer function

$$\frac{0.00468 + 0.00438z^{-1}}{1 - 1.81z^{-1} + 0.8178z^{-2}}$$

which was driven by zero mean white noise with variance one. This model was obtained by discretization of a simple continuous model with transfer function

$$\frac{1}{(1+s)^2}$$

with sampling period 0.1.

The squares of the generated output samples were used as a partial quality function for the stopping by using ARX model stabilization, see (20) with $W = 1$. The stationarity measure $U_{o;t}$ for the ARX model is seen in the second part of the figure and the evolution of the mean value estimation is in its third part.

The constraining interval $[-0.3, 0.3]$ is used on the generated data to obtain the discrete three-state indicator (31) for the purpose of stopping through the MC model (32). The resulting stationarity measure $U_{c;t}$ and the corresponding estimation of probability of the state zero are shown in the second and third part of the figure.

It can be seen that the measure $U_{c;t}$ is rather fuzzy. This complicates the decision whether to stop simulation, because the rule to stop whenever the measure is below the threshold is quite unsatisfactory as the several next samples immediately increase the value above this threshold. To solve this problem, an interpolation is performed using the approximation by the following model

$$U^{\text{int}}_{c,t} = a_0 + a_1 t^{-1/2} + a_2 t^{-1}, \tag{36}$$

were the coefficients are obtained by linear regression. The interpolated measure, denoted by $U^{\text{int}}_{c;t}$, is shown in the figure. The interpolation is, up to a tiny peak close to the origin, satisfactory for the stopping purposes.

It is possible to think about stopping for the interpolating regression, too, and trigger the stopping when the interpolated measure is below the threshold as well as the interpolation itself has been stabilized.

The threshold for the measures $U^{\text{int}}_{c;t}$ and $U_{o;t}$ need not be of the same value. The stopping models are different and have a different number of identified parameters. From the particular example presented in this section a reasonable threshold for $U^{\text{int}}_{c;t}$ is roughly 0.008 and for $U_{o;t}$ it is 0.004.

5. CONCLUSIONS

This paper described the effective quality function evaluation using the stopping rules. Of course the controller tuning contains the optimization method which uses the quality evaluated to search for better tuning parameters. The optimization method is described elsewhere (Novák, 2005), but because it is done numerically. The quality functions are evaluated many times. Thus the effectiveness of their calculation is necessary.

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