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RAMADANOV CONJECTURE AND LINE BUNDLES OVER COMPACT HERMITIAN SYMMETRIC SPACES

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Abstract. We compute the Szegö kernels of the unit circle bundles of homogeneous negative line bundles over a compact Hermitian symmetric space. We prove that their logarithmic terms vanish in all cases and, further, that the circle bundles are not diffeomorphic to the unit sphere in $\mathbb{C}^n$ for Grassmannian manifolds of higher ranks. In particular they provide an infinite family of smoothly bounded strictly pseudo-convex domains on complex manifolds for which the log terms in the Fefferman expansion of the Szegö kernel vanish and which are not diffeomorphic to the sphere. The analogous results for the Bergman kernel are also obtained.

1. Introduction

Let Ω be a strongly pseudo-convex bounded domain in $\mathbb{C}^n$ with smooth boundary. The Bergman kernel has an expansion near the diagonal in terms of the defining function of the domain, with the leading term behaving like that of the Bergman kernel of the unit ball; see [3] and [8]. Similar result holds also for the Szegö kernel. However, there is in general also a logarithmic term in the expansion of the Bergman and Szegö kernel, and the study of the log term is of considerable interest for analytic and geometric motivations. Among other things there is the Ramadanov conjecture [25] which asserts that the answer to the following question is affirmative.

Question 1. Let Ω be a strongly pseudo-convex bounded domain in $\mathbb{C}^n$ with smooth boundary. Suppose that the Bergman kernel has no logarithmic term. Is the domain biholomorphic to the unit ball in $\mathbb{C}^n$?

For certain special cases, such as domains in $\mathbb{C}^2$, domains with transversal or rotational symmetries, etc., this has been proved to be true; see [5], [12], [13], [7], [24] and [17], and [1] for a real-variable version.

There is also an obvious analogue of the conjecture for the Szegö, instead of Bergman, kernel, and one may also consider smoothly bounded strictly pseudoconvex domains in complex manifolds. In this setup, in particular, the following special case of the Ramadanov conjecture was formulated in [22].

Question 2. Let $S(L^*)$ be the disc bundle of a negative line bundle over a simply-connected Kähler manifold M. Suppose that the Szegö kernel of $S(L^*)$ has no log term. Is the circle bundle diffeomorphic to the sphere?

In this paper, we will consider the generating positive line bundle over a compact Hermitian symmetric space M and compute the corresponding Szegö kernel. As a consequence we will see that the answer to the last question is negative. In fact,

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the simplest counterexamples are the powers $L^* = \mathcal{L}^{*m}$, $m > 1$, of the tautological bundle $\mathcal{L}^*$ over the complex projective space $\mathbb{C}P^n$: then the Szegö kernel of $S(\mathcal{L}^{*m})$ has no log term but $S(\mathcal{L}^{*m})$ is the lens space $S^{2n+1}/\mathbb{Z}_m$ which is not diffeomorphic to S^{2n+1} for $m > 1$. For compact symmetric spaces of higher rank, we even get counterexamples which are not diffeomorphic to $S^{2n+1}/\mathbb{Z}_m$ for any $m \geq 1$. Our results thus indicate that some topological conditions are probably needed to have an affirmative answer to Question 2.[1]

The analogous assertions for the Bergman kernel are also established. Here the simplest counterexample is the unit disc bundle $D(\mathcal{L}^*)$ of the above-mentioned tautological bundle $\mathcal{L}^*$ over $\mathbb{C}P^n$, $n \geq 1$, whose Bergman kernel has no log term but $D(\mathcal{L}^*)$ is not biholomorphic to the unit ball of $\mathbb{C}^{n+1}$ (even though its boundary $S(\mathcal{L}^*)$ is diffeomorphic to S^{2n+1}).

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2. Compact Hermitian symmetric spaces

We briefly recall some necessary facts on compact Hermitian symmetric spaces; see e.g. [15].

Let $\mathfrak{g}$ be a real simple Lie algebra of Hermitian type and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ has one-dimensional center $\mathbb{R}Z$. Let

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+, \tag{1}$$

where $\mathfrak{p}^\pm$ is the eigenspace of $\mathrm{ad}(Z)$ with eigenvalues $\pm i$.

Let $G^{\mathbb{C}}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and let G, K, $P^\pm$ be the analytic subgroups of $G^{\mathbb{C}}$ with Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{p}^\pm$. Now $P^+K^{\mathbb{C}}P^-$ is a dense subset of $G^{\mathbb{C}}$. For $g \in G^{\mathbb{C}}, z \in \mathfrak{p}^+$ we let $g \cdot z$ and $\mathcal{K}(g : z)$ be the $\mathfrak{p}^+$ and $K^{\mathbb{C}}$ components of $g \exp(z)$, respectively. Namely,

$$g \exp(z) = \exp(g \cdot z)\mathcal{K}(g : z)p_- \tag{2}$$

for some $p_- \in P^-$. Under the above action the G-orbit $G \cdot 0 = G/K$ of $z = 0 \in \mathfrak{p}^+$ is a bounded domain in $\mathfrak{p}^+$, which is the so-called Harish-Chandra realization of G/K.

Let G^* be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$. Then $M = G^*/K$ is a compact Hermitian symmetric space. Furthermore $M = G^{\mathbb{C}}/K^{\mathbb{C}}P^-$, and under the identification of $\mathfrak{p}^+$ with P^+, the space $\mathfrak{p}^+$ is imbedded in M as a dense subset. (In fact, the complement of $\mathfrak{p}^+$ in M is a complex submanifold of smaller dimension.)

Denote $\mathcal{K}(g) = \mathcal{K}(g : 0)$. From (2) we have

$$g = p_+\mathcal{K}(g)p_-, \tag{3}$$

for $g \in P^+K^{\mathbb{C}}P^-$.

For $z, w \in \mathfrak{p}^+$ we let $\mathcal{K}(z, \bar{w})$ be the $K^{\mathbb{C}}$-part of $\exp(-\bar{w})\exp(z)$ as in (3), namely $\mathcal{K}(z, \bar{w}) = \mathcal{K}(\exp(-\bar{w})\exp(z))$. The *Bergman operator* $B(z, w)$ is defined by

$$B(z, w) = \mathrm{ad}_{\mathfrak{p}^+} \mathcal{K}(z, \bar{w}).$$

There exists an irreducible polynomial $h(z, w)$, called the *Jordan canonical polynomial*, and an integer p, the *genus* of G/K (see (4) below), such that $\det B(z, w) = h(z, w)^p$; see e.g. [21, §4.15–4.17 and §7.4].

[1]We remark that there exist *unbounded* pseudoconvex domains in $\mathbb{C}^2$ with smooth boundary, as well as bounded pseudoconvex domains in $\mathbb{C}^2$ with *rough* boundary, for which the Ramadanov conjecture for the Szegö kernel is known to fail; cf. Remark 1.2 in [16].

We normalize the K-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}^+$ by

$$(z,w) = -\frac{1}{2p}\operatorname{tr}\left(\mathrm{Ad}_{\mathfrak{p}^+}(z)\,\mathrm{Ad}_{\mathfrak{p}^+}(\bar{w})\right)$$

where as before

$$p = (r-1)a + 2 + b \tag{4}$$

is the *genus* of M. Note that the complex dimension of M is $n = r + \frac{r(r-1)}{2}a + rb$. We let ω the G^*-invariant Kähler form on M normalized so that the volume of M with respect to ω^n is 1. In terms of the local coordinates $z = \sum_j z_j e_j \in \mathfrak{p}^+$ (where $\{e_j\}$ is any orthonormal basis of $\mathfrak{p}^+$), viewing ω^n as a measure on $\mathfrak{p}^+$, we have

$$\omega(z)^n = C_0 \frac{1}{h(z,-z)^p}\, dm(z), \tag{5}$$

where dm stands for the Lebesgue measure on $\mathbb{C}^n$. Further, the complement of $\mathfrak{p}^+$ in M has zero measure with respect to ω^n.

Finally we recall the Gindikin Gamma function and a version of the generalized Pochhammer symbol defined by

$$\Gamma_M(c) = \prod_{j=1}^{r} \Gamma(c - \frac{a}{2}(j-1)),$$

$$(\!(c)\!)_s = \frac{\Gamma_M(c+s)}{\Gamma_M(c)},$$

where r is the rank of M.

3. Szegö and Bergman kernels for the disc bundle

Let L be the homogeneous line bundle over $M = G^*/K$ induced by the representation $k \mapsto (\det(\mathrm{Ad}k))^{1/p}$, $k \in K$. (There exists a single-valued branch of the root of the determinant function so that this indeed defines a one-dimensional representation of K.) The bundle L^p is then the top exterior product $\wedge^n T^{(1,0)}$ of the holomorphic tangent bundle over G^*/K. Using the local coordinates $\mathfrak{p}^+ \ni z \to \exp(iz) \in G^*/K$, we have that the fiber metric in L^p is given by

$$\|\partial_1 \wedge \cdots \wedge \partial_n\|_z^2 = h(z,-z)^{-p}.$$

Denoting by $e(z)$ a local holomorphic section of L so that $e(z)^p = \partial_1 \wedge \cdots \wedge \partial_n$ we see that the metric on L is given by

$$\|e(z)\|_z^2 = h(z,-z)^{-1}.$$

Let $D = D(L^*) = \{\xi \in L^*; \|\xi\| < 1\}$ and $S(L^*) = \partial D$ be the unit disc and the unit circle bundle of the dual bundle L^* of L, respectively. A local defining function for $S(L^*)$ is given by

$$\rho(z, \lambda e^*(z)) = |\lambda|^2 h(z,-z) - 1, \quad \lambda \in \mathbb{C},\ z \in \mathfrak{p}^+,$$

where $e^*(z)$ is the local section of L^* dual to $e(z)$. The circle bundle $S(L^*)$ is a CR-manifold, with the CR-structure defined by ρ, and the disc bundle D is strictly pseudoconvex, namely the Hessian $\partial\bar{\partial}\rho$ is positive definite on the holomorphic tangent space of $S(L^*)$.

The manifold $S(L^*)$ is actually a compact homogeneous space of $G^* \times S^1$, with $S^1 = \{e^{i\theta}\}$ acting on L^* fiberwise. Let π be the projection $S(L^*) \to M$. We let $d\sigma$ be the measure

$$d\sigma = \pi^*(\omega^n) \wedge \frac{d\theta}{2\pi} \tag{6}$$

which is also the unique $G^* \times S^1$-invariant probability measure on $S(L^*)$. Here ω is the Kähler form given above.

Let ν be a non-negative integer and consider the space $\mathcal{H}^\nu$ of all holomorphic functions ϕ on L^* satisfying

$$\phi(e^{i\theta}\xi) = e^{\nu i\theta}\phi(\xi), \qquad \xi \in L^*, \theta \in \mathbb{R}.$$

Note that owing to the holomorphy of f this implies that even

$$\phi(\lambda\xi) = \lambda^\nu \phi(\xi), \qquad \xi \in L^*, \lambda \in \mathbb{C}.$$

As M is compact, any such function is, in particular, automatically square-integrable over $S(L^*)$ as well as over D.

Identifying $\mathfrak{p}^+$ with a dense open subset of M of full measure as described in the previous section, and using the local trivializing section $e^*(z)$ as before, the correspondence $\mathfrak{p}^+ \times \mathbb{C} \ni (z,\lambda) \longleftrightarrow \xi = (z, \lambda e^*(z)) \in L^*$ sets up a bijection between a dense open subset of D of full measure and the Hartogs domain

$$\Omega = \{(z,\lambda) \in \mathfrak{p}^+ \times \mathbb{C} : \ |\lambda|^2 h(z,-z) < 1\}.$$

The functions ϕ in $\mathcal{H}^\nu$ then correspond to square-integrable (with respect to (5) and the Lebesgue measure in λ) holomorphic functions $\tilde{\phi}$ on Ω satisfying $\tilde{\phi}(z,\lambda) = \lambda^\nu f(z)$ for some entire function f on $\mathfrak{p}^+$. [2] The norm of ϕ in $L^2(d\sigma)$ thus equals to

$$\|f\|_\nu^2 = \int_{\mathfrak{p}^+} |f(z)|^2 h(z,-z)^{-\nu} \omega(z)^n.$$

The space $A^2_\nu(\mathfrak{p}^+)$ of all entire functions f on $\mathfrak{p}^+$ for which this norm is finite carries a representation of G^*:

$$g \in G^* : f(z) \mapsto f(g^{-1}z) J_{g^{-1}}(z)^{-\nu/p},$$

where $J_{g^{-1}}$ is the complex Jacobian and p is the genus defined in (4).

The function $h(z,-z)$ on $\mathfrak{p}^+$ thus transforms according to

$$\begin{aligned} h(g(z), -g(z))^\nu &= \frac{h(z,-z)^\nu h(w,-w)^\nu}{|h(z,-w)^\nu|^2} \\ &= h(z,-z)^\nu \ |J_{g^{-1}}(z)|^{2\nu/p}, \end{aligned}$$

for $g \in G^*$ such that $g(0) = w$.

Lemma 3.1. *The reproducing kernel for the space $A^2_\nu(\mathfrak{p}^+)$ is given by*

$$\frac{((\nu + p - \frac{n}{r}))_{\frac{n}{r}}}{((p - \frac{n}{r}))_{\frac{n}{r}}} \ h(z,-w)^\nu.$$

Proof. It follows from the transformation rule of $h(z,-w)$ under G^* (see e.g. [26]) that the reproducing kernel is

$$c_\nu h(z,-w)^\nu.$$

We evaluate the constant, which is given by the norm square of the function 1,

$$c_\nu^{-1} = \int_M \|e^\nu\|^2 \omega^n = C_0 \int_{\mathfrak{p}^+} h(z,-z)^{-\nu} h(z,-z)^{-p} \, dm(z).$$

[2] It is clear that any function in $\mathcal{H}^\nu$ must be of this form when restricted to the above local chart. Conversely, any square integrable holomorphic function on Ω as above automatically extends to a holomorphic function on all of D. Indeed, since the complement of $\mathfrak{p}^+$ in M is a proper complex submanifold (see e.g. the discussion in §2 in Berezin [4]), making a suitable change of coordinates it is enough to show that any square-integrable holomorphic function on the punctured polydisc $\mathbb{D}^{n-1} \times (\mathbb{D} \setminus \{0\})$ extends to a holomorphic function on the whole $\mathbb{D}^n$. This "L^2-version of the removable singularity theorem" is then easily proved by looking at the Laurent expansion in z_n, cf. the proof for $n = 1$ in [2].

In terms of the polar coordinates (see [10]) we have

$$c_\nu^{-1} = C_0 C \int_{(\mathbb{R}^+)^r} \prod_{j=1}^r (1+t_j^2)^{-\nu-p} \prod_{j=1}^r t_j^{1+2b} \prod_{1\le i<j\le r} |t_i^2 - t_j^2|^a \, dt_1 \dots dt_r,$$

with some constant C independent of ν. Changing variables to $t_j^2 = x_j(1-x_j)^{-1}$, $j = 1, \dots, r$, we find that

$$c_\nu^{-1} = \frac{C_0 C}{2^r} \int_{(0,1)^r} \prod_{j=1}^r (1-x_j)^\nu \prod_{j=1}^r x_j^b \prod_{1\le i<j\le r} |x_i - x_j|^a \, dx_1 \dots dx_r,$$

which in turn can be expressed in terms of the Gindikin Gamma function [10], viz.,

$$c_\nu^{-1} = C' \frac{\Gamma_M(\nu + p - \frac{n}{r})}{\Gamma_M(\nu+p)} = \frac{C'}{((\nu + p - \frac{n}{r}))_{\frac{n}{r}}},$$

with some constant C' independent of ν. Taking $\nu = 0$ and recalling that ω^n was normalized to have total mass one, i.e. $c_0 = 1$, gives $C' = ((p - \frac{n}{r}))_{\frac{n}{r}}$. This completes the proof. □

Denote by

$$\rho(x, \alpha; y, \beta) = \alpha\bar\beta h(x, -y) - 1$$

the sesqui-holomorphic extension of the defining function ρ.

Theorem 3.2. *The Szegö kernel of the disc bundle D is given, in local coordinates $\alpha e^*(z) \mapsto (z, \alpha) \in \mathfrak{p}^+ \times \mathbb{C}$, $|\alpha|^2 h(z, -z) < 1$, by*

$$K(x, \alpha; y, \beta) = \sum_{\nu=0}^\infty \frac{((\nu + p - \frac{n}{r}))_{\frac{n}{r}}}{((p - \frac{n}{r}))_{\frac{n}{r}}} h(x, -y)^\nu (\alpha\bar\beta)^\nu. \tag{7}$$

It has an expansion in terms of the defining function ρ as

$$K(x, \alpha; y, \beta) = c_0 \rho(x, \alpha; y, \beta)^{-n-1} + c_1 \rho(x, \alpha; y, \beta)^{-n} + \dots + c_n \rho(x, \alpha; y, \beta)^{-1} \tag{8}$$

where $c_0 = (-1)^{n+1} \dfrac{n!}{((p - \frac{n}{r}))_{\frac{n}{r}}}$ and c_j are some real constants.

Proof. It is clear that the space $\mathcal{H}^\nu$, $\nu = 0, 1, 2, \dots$, are pairwise orthogonal subspaces of $L^2(d\sigma)$, and that their closed span is the Hardy space $H^2(D)$. (In fact, $H^2(D) = \oplus_{\nu=0}^\infty \mathcal{H}^\nu$ is just the Fourier decomposition of $H^2(D)$ into irreducible components with respect to the action of S^1.) Consequently, the Szegö kernel — the reproducing kernel of $H^2(D)$ — is the sum of the reproducing kernels of the spaces $\mathcal{H}^\nu$ over all ν, which gives (7). Since $((\nu + p - \frac{n}{r}))_{\frac{n}{r}}$ is always a monic polynomial in ν of degree n — hence, a linear combination of the expressions $\frac{(\nu+1)\dots(\nu+k)}{k!} = \frac{(k+1)_\nu}{\nu!}$, $k = 0, 1, 2, \dots, n$ — and

$$\sum_{\nu=0}^\infty \frac{(k+1)_\nu}{\nu!} h(x, -y)^\nu \, (\alpha\bar\beta)^\nu = [-\rho(x, \alpha; y, \beta)]^{-k-1},$$

the formula (8) follows. □

Recall that the Bergman space of a complex manifold of dimension n is in general defined as the space of all holomorphic $(n, 0)$-forms f such that

$$(-i)^n \int f \wedge \bar f < +\infty. \tag{9}$$

The Bergman kernel is then, by definition, the (n, n)-form

$$\sum_m f_m \wedge \bar f_m,$$

where $\{f_m\}$ is any orthonormal basis of the Bergman space, with respect to the inner product $\langle f, g\rangle$ obtained by replacing $\bar f$ in (9) by $\bar g$. The sum is independent of the choice of the basis, etc.; see e.g. [19]. Of course, if the manifold is just a domain in $\mathbb{C}^n$, then by the identification

$$f(z)\, dz_1 \wedge \cdots \wedge dz_n \longleftrightarrow f(z) \tag{10}$$

of $(n, 0)$-forms with functions one recovers the usual definition of the Bergman space and Bergman kernel of domains in $\mathbb{C}^n$.

We have now a complete analogue of Theorem 3.2 also for the Bergman kernel.

Theorem 3.3. *The Bergman kernel of the disc bundle D is given, in local coordinates $\alpha e^*(z) \mapsto (z, \alpha) \in \mathfrak{p}^+ \times \mathbb{C}$, $|\alpha|^2 h(z, -z) < 1$, by*

$$K(x, \alpha; y, \beta) = K^*(x, \alpha; y, \beta)\, dx_1 \wedge \cdots \wedge dx_n \wedge d\alpha \wedge d\bar y_1 \wedge \cdots \wedge d\bar y_n \wedge d\bar\beta,$$

where

$$K^*(x, \alpha; y, \beta) = \frac{1}{\pi} \sum_{\nu=0}^{\infty} (\nu + 1) \frac{((\nu + p - \frac{n}{r}))_{\frac{n}{r}}}{((p - \frac{n}{r}))_{\frac{n}{r}}}\, h(x, -y)^{\nu+1} (\alpha\bar\beta)^\nu. \tag{11}$$

It has an expansion in terms of the defining function ρ as

$$\frac{K^*(x, \alpha; y, \beta)}{h(x, -y)} = c_0 \rho(x, \alpha; y, \beta)^{-n-2} + \cdots + c_{n+1} \rho(x, \alpha; y, \beta)^{-1} \tag{12}$$

where $c_0 = (-1)^{n+2} \frac{1}{\pi} \frac{(n+1)!}{((p - \frac{n}{r}))_{\frac{n}{r}}}$ and c_j are some real constants.

Proof. In the local coordinates, we can still identify the $(n, 0)$-forms with functions via (10), and thus the Bergman space on D can be identified with the space of all functions holomorphic and square-integrable on Ω, i.e. with the usual Bergman space on the Hartogs domain $\Omega \subset \mathbb{C}^{n+1}$. [3] Using again the Fourier decomposition with respect to the S^1-action, together with the fact that now the norm of a function from $\mathcal{H}^\nu$, $\nu = 0, 1, 2, \ldots$, equals

$$\begin{aligned}
\|\phi\|^2_{L^2(\Omega)} &= \int_{\mathfrak{p}^+} \int_{|\lambda|^2 < 1/h(z,-z)} |\lambda|^{2\nu}\, |f(z)|^2\, d\lambda \wedge d\bar\lambda \wedge \omega(z)^n \\
&= \frac{\pi}{\nu + 1} \int_{\mathfrak{p}^+} |f(z)|^2\, h(z, -z)^{-\nu-1}\, \omega(z)^n,
\end{aligned}$$

and, consequently, the reproducing kernel of $\mathcal{H}^\nu$ with respect to this norm equals

$$\frac{\nu + 1}{\pi} \frac{((\nu + 1 + p - \frac{n}{r}))_{\frac{n}{r}}}{((p - \frac{n}{r}))_{\frac{n}{r}}}\, h(z, -w)^{\nu+1},$$

we get the first formula (11) in the theorem. The second formula (12) follows from it in the same way as in Theorem 3.2. □

With trivial modifications, the last two theorems extend also to the unit circle bundles $S(L^{*\mu})$ and the corresponding unit disc bundles $D_\mu = D(L^{*\mu})$ of the higher powers $L^{*\mu}$ of L^*, $\mu = 0, 1, 2, \ldots$ (one just needs to replace ν by $\nu\mu$ everywhere in the proofs.)

Theorem 3.4. *The Szegö kernel of the disc bundle D_μ is given, in local coordinates $\alpha e^*(z) \mapsto (z, \alpha) \in \mathfrak{p}^+ \times \mathbb{C}$, $|\alpha|^2 h(z, -z)^\mu < 1$, by*

$$K(x, \alpha; y, \beta) = \sum_{\nu=0}^{\infty} \frac{((\mu\nu + p - \frac{n}{r}))_{\frac{n}{r}}}{((p - \frac{n}{r}))_{\frac{n}{r}}}\, h(x, -y)^{\mu\nu} (\alpha\bar\beta)^\nu.$$

[3] Again, any such $(n, 0)$-form automatically extends to be holomorphic even on the whole D, i.e. also on the complement of Ω in D; see the footnote before Lemma 3.1 on page 4.

It has an expansion in terms of the sesqui-holomorphically extended defining function $\rho(x,\alpha;y,\beta)=\alpha\bar\beta h(x,-y)^\mu-1$ *as*

$$K(x,\alpha;y,\beta)=c_0\rho(x,\alpha;y,\beta)^{-n-1}+c_1\rho(x,\alpha;y,\beta)^{-n}+\cdots+c_n\rho(x,\alpha;y,\beta)^{-1}$$

where $c_0=(-1)^{n+1}\dfrac{n!\mu^n}{((p-\frac{n}{r}))_{\frac{n}{r}}}$ *and* c_j *are some real constants.*

Theorem 3.5. *The Bergman kernel of the disc bundle* D_μ *is given, in local coordinates* $\alpha e^*(z)\mapsto(z,\alpha)\in\mathfrak{p}^+\times\mathbb{C}$, $|\alpha|^2h(z,-z)^\mu<1$, *by*

$$K(x,\alpha;y,\beta)=K^*(x,\alpha;y,\beta)\,dx_1\wedge\cdots\wedge dx_n\wedge d\alpha\wedge d\bar y_1\wedge\cdots\wedge d\bar y_n\wedge d\bar\beta,$$

where

$$K^*(x,\alpha;y,\beta)=\frac{1}{\pi}\sum_{\nu=0}^{\infty}(\nu+1)\frac{((\mu\nu+p-\frac{n}{r}))_{\frac{n}{r}}}{((p-\frac{n}{r}))_{\frac{n}{r}}}\,h(x,-y)^{\mu\nu+1}(\alpha\bar\beta)^\nu.$$

It has an expansion in terms of the sesqui-holomorphically extended defining function $\rho(x,\alpha;y,\beta)=\alpha\bar\beta h(x,-y)^\mu-1$ *as*

$$\frac{K^*(x,\alpha;y,\beta)}{h(x,-y)}=c_0\rho(x,\alpha;y,\beta)^{-n-2}+\cdots+c_{n+1}\rho(x,\alpha;y,\beta)^{-1}\tag{13}$$

where $c_0=(-1)^{n+2}\dfrac{1}{\pi}\dfrac{(n+1)!\mu^n}{((p-\frac{n}{r}))_{\frac{n}{r}}}$ *and* c_j *are some real constants.*

The case of $\mu=p$ is of special interest, since in that case the $G^*\times S^1$-invariant probability measure (6) on $S(L^{*\mu})$ coincides with the surface measure used to get a holomorphically invariant Szegö kernel, namely

$$\sigma\wedge d\rho=J[\rho]^{1/(n+2)}\,dV,$$

where dV denotes the volume element in $\mathfrak{p}^+\times\mathbb{C}$ and $J[\rho]$ stands for the Monge-Ampére determinant

$$J[\rho]=(-1)^{n+1}\det\begin{bmatrix}\rho & \partial\rho\\ \bar\partial\rho & \partial\bar\partial\rho\end{bmatrix};$$

see e.g. [18]. Indeed, a short computation shows that $\sigma\wedge d\rho$ equals $h(z,-z)^{\mu-p}\,dV$ (up to an immaterial constant factor), while $J[\rho]=\mu^n h(z,-z)^{\mu-p}$; so they coincide when $\mu=p$. Thus for $\mu=p$ Theorem 3.5 concerns the invariant Szegö kernel occurring in the theory of holomorphic invariants.

The last four theorems yield abundant examples of smoothly bounded strictly pseudoconvex domains in complex manifolds for which the Szegö kernel as well as the Bergman kernel contain no log-term in their boundary singularity. From the point of view of the Ramadanov conjecture, it remains to verify that these domains are not biholomorphic to the ball. Since by Fefferman's 1974 result [11] any such biholomorphism extends smoothly to the boundaries, it is enough to show that the circle bundle $S(L^{*\mu})$ is not diffeomorphic to the unit sphere S^{2n+1}.

Recall that the simplest examples of compact Hermitian symmetric spaces are the Grassmann manifolds $U(l)/U(k)\times U(l-k)$, $1\le k\le l-k$. (They are the compact duals to the Cartan domains $I_{k,l-k}$ — the unit balls $SU(k,l-k)/S(U(k)\times U(l-k))$ of complex $k\times(l-k)$ matrices.) For $k=1$, the Grassmannians $M=U(l)/U(1)\times U(l-1)$ are just the complex projective spaces $M=\mathbb{C}P^n$, $n=l-1$, and then the cosphere bundle $S(L^*)$ actually is CR-equivalent to the sphere $S^{2n+1}=U(l)/U(l-1)$: the bundle L is the hyperplane bundle, L^* is the tautological bundle, and the mapping from the sphere S^{2n+1} to $S(L^*)$ is given by $z\mapsto(\mathbb{C}z,z)$. Similarly, the cosphere bundle $S(L^{*m})$ is CR-equivalent to the lens space $S^{2n+1}/\mathbb{Z}_m$, the isomorphism now being given by the mapping $z\mapsto(\mathbb{C}z,\otimes^m z)$ from the sphere S^{2n+1} which induces a diffeomorphism from $S^{2n+1}/\mathbb{Z}_m$ onto $S(L^{*m})$ (see e.g. [20, p. 542]).

From Theorems 3.2–3.5 we thus arrive at the following counterexamples to the manifold version of the Ramadanov conjecture (Question 2).

Corollary 3.6. *Let $M = \mathbb{C}P^n$ be the complex projective n-space, $n \geq 1$, and L the positive line bundle as defined in the beginning of this section (namely, L is the hyperplane bundle, i.e. the dual of the tautological bundle). Then the log-term vanishes in the Szegö kernel of the circle bundles $S(L^{*\mu})$, and $S(L^{*\mu})$ is not diffeomorphic to the sphere S^{2n+1} if $\mu > 1$.*

Proof. The only thing we need to prove is that $S(L^{*\mu}) \cong S^{2n+1}/\mathbb{Z}_\mu$ is not diffeomorphic to $S^{2n+1} \equiv S^{2n+1}/\mathbb{Z}_1$ for $\mu > 1$. However, this is immediate for instance from the cohomology groups (see e.g. [6, Example 18.5] or [14, Example 2.43, p. 144])

$$H^j(S^{2n+1}/\mathbb{Z}_\mu, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, 2n+1, \\ \mathbb{Z}_\mu, & j = 1, 3, \dots, 2n-1, \\ 0 & \text{otherwise}, \end{cases} \tag{14}$$

since the cohomology rings are diffeomorphic invariants. □

It is not difficult to see that for Grassmannians of higher rank, we even get counterexamples which are not diffeomorphic to any lens space $S^{2n+1}/\mathbb{Z}_m$.

Corollary 3.7. *Let $M = U(l)/U(k)\times U(l-k)$ $(1 < k \leq l-k)$ be the Grassmannian of higher rank $k > 1$ and complex dimension $n = k(l-k)$. Let L be the positive line bundle as defined in the beginning of this section. (Namely, L is the determinant bundle of the hyperplane bundle.) Then the log-term vanishes in the Szegö kernel of the circle bundles $S(L^{*\mu})$, $\mu \geq 1$, and $S(L^{*\mu})$ is not diffeomorphic to any lens space $S^{2n+1}/\mathbb{Z}_m$.*

Proof. We use the Gysin exact sequence [23, Theorem 12.2] [14, p. 437 ff.] of the circle bundle $E := S(L^{*\mu})$ over M (all cohomology groups are over $\mathbb{R}$):

$$\dots \to H^{2j-1}(E) \to H^{2j-2}(M) \to H^{2j}(M) \to H^{2j}(E) \to \cdots .$$

If E were diffeomorphic to $S^{2n+1}/\mathbb{Z}_m$, then by (14) we would have

$$H^j(E) = \begin{cases} \mathbb{R}, & j = 0, 2n+1, \\ 0 & \text{otherwise}. \end{cases}$$

From the Gysin sequence it would thus follow that

$$H^{2j-1}(M) = 0, \quad H^{2j-2}(M) \cong H^{2j}(M), \qquad j = 1, \dots, n. \tag{15}$$

On the other hand, it is known that the Poincaré series of the cohomology ring $H^*(M)$ is given by (see e.g. [6, Chapter IV, Proposition 23.1])

$$\frac{(1-t^2)\cdots(1-t^{2l})}{(1-t^2)\cdots(1-t^{2k})(1-t^2)\cdots(1-t^{2(l-k)})}.$$

Thus (15) can happen only for $k = 1$. □

The lowest-dimensional counterexample to the Ramadanov conjecture for the Szegö kernel of circle bundles, namely Question 2, supplied by Corollary 3.6 thus occurs for the circle bundles $S(L^{*m})$, $m > 1$, of powers of the tautological bundle over the Gauss sphere $\mathbb{C}P^1$ (so that $S(L^{*m})$ has real dimension 3), while that supplied by Corollary 3.7 — i.e. not diffeomorphic to the lens spaces — for the Grassmannian with $k = l - k = 2$ (i.e. with $S(L^*)$ of real dimension 9).

Finally, we also have the corresponding assertions for the Bergman, instead of the Szegö, kernel.

Corollary 3.8. *Let $M = U(l)/U(k)\times U(l-k)$ $(1 \le k \le l-k)$ be the Grassmannian of rank $k \ge 1$ and complex dimension $n = k(l-k)$. Let L be the positive line bundle as defined in the beginning of this section. Then the log-term vanishes in the Bergman kernel of the corresponding disc bundles D_μ, $\mu \ge 1$, and D_μ is not biholomorphic to the unit ball $\mathbb{B}^{n+1}$ of $\mathbb{C}^{n+1}$.*

Proof. That D_μ is not biholomorphic to $\mathbb{B}^{n+1}$ if $k > 1$ or $\mu > 1$ follows from the last two corollaries since its boundary $\partial D_\mu = S(L^{*\mu})$ is then not diffeomorphic to $\partial\mathbb{B}^{n+1} = S^{2n+1}$. We claim that D_μ is still not biholomorphic to $\mathbb{B}^{n+1}$ even if $k = \mu = 1$, i.e. for L^* the tautological line bundle over $M = \mathbb{C}P^n$ (even though $S(L^*)$ then is diffeomorphic to S^{2n+1}). Indeed, a short computation using (13) shows that the zero section of $D(L^*)$ is then a totally geodesic submanifold with respect to the Bergman metric; since any biholomorphism is automatically an isometry with respect to Bergman metrics, it would follow that the image of the zero section is a compact submanifold of the unit ball which is totally geodesic with respect to the Bergman metric. However, no such submanifold can exist, since every geodesic in the ball with respect to the Bergman metric reaches the boundary (the geodesics through the origin are just straight lines, and the ball is homogeneous). Thus $D(L^*)$ cannot be biholomorphic to the ball. (This is in apparent contrast with the situation for domains in $\mathbb{C}^n$, where by the recent theorem of Chern and Ji [9], any smoothly-bounded simply connected domain whose boundary is locally spherical must be biholomorphic to the ball.) This completes the proof. □

In particular, for $n = \mu = 1$ the disk bundle D over the Gauss sphere $\mathbb{C}P^1$ provides a two-dimensional counterexample to the manifold version of the Ramadanov conjecture for the Bergman kernel (Question 1).

In view of the above results, it seems somewhat natural to pose the following modified version of the Ramadanov conjecture.

Question 3. Suppose that the Szegö or Bergman kernel of a domain in a complex manifold has no log term in its boundary singularity. Is the domain then always biholomorphic to the unit disc bundle $D(L^*)$ for some positive line bundle L over a compact Hermitian symmetric space M?

We conclude by remarking that there is a well-known intimate relationship between *functions* on the dual disc bundle $D \subset L^*$ and *sections* of the tensor powers L^ν of the original line bundle L. Namely, let $\mathcal{L}^\nu$, $\nu = 0, 1, 2, \dots$, stand for the space of all functions f on L^* satisfying

$$f(e^{i\theta}\xi) = e^{\nu i\theta} f(\xi), \qquad \xi \in L^*, \theta \in \mathbb{R}.$$

Then the natural mapping $s \mapsto \tilde{s}$,

$$\tilde{s}(\xi) := \langle s, \xi^{\otimes\nu} \rangle, \qquad \xi \in L^*,$$

sets up a bijection between functions $\tilde{s} \in \mathcal{L}^\nu$ and sections s of L^ν; further, s is holomorphic if and only if $\tilde{s}$ is (i.e. if and only if $\tilde{s}$ belongs to the space $\mathcal{H}^\nu$). In this way, some of the results in this paper can be recast in the language of reproducing kernels of Bergman spaces of sections of the powers L^ν of the line bundle L. We omit the details.

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