Thermalization in a model of neutron star.

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Bernard Ducomet [a] and Šárka Nečasová [b]

[a] *Département de Physique Théorique et Appliquée, CEA/DAM Ile de France*
BP 12, F-91680 Bruyères-le-Châtel, France
E-mail: bernard.ducomet@cea.fr

[b] *Mathematical Institute AS ČR*
Žitná 25, 115 67 Praha 1, Czech Republic
E-mail: matus@math.cas.cz

SUMMARY

We consider an initial boundary value problem for the equation describing heat conduction in a spherical model of neutron star considered by Lattimer et al. We estimate the asymptotic decay of the solution, which provides a plausible estimate for a "thermalization time" for the system.

1 Introduction

We consider a simple model used by Lattimer et al. to describe the evolution of temperature [7] in neutron stars. It supposes a linear dependence of the specific heat as a function of temperature and assumes that a mechanical equilibrium is reached, so the problem reduces to the study of large time asymptotic for a Fast Diffusion Equation satisfied by the temperature.

In fact this model supposes that the mechanical structure of the medium does not change as the star cools and only considers the heat transfer problem for the temperature θ. It amounts to suppose that we make $v \equiv 0$ and $\rho \equiv \rho_S$ in the complete thermo-mechanical system (Navier-Stokes system if viscosity is taken into account), together with frozen internal energy $e(\rho,\theta) \equiv e(\rho_S,\theta)$, specific heat $c_V(\rho,\theta) \equiv c_V(\rho_S,\theta)$ and thermal conductivity $\kappa(\rho,\theta) \equiv \kappa(\rho_S,\theta)$.

Then the resulting problem reduces to solve an IBV problem for a Fast Diffusion Equation [9], for which, one can prove global existence, uniqueness and precise large-time asymptotic.

So we consider the quasilinear parabolic problem for the temperature θ

$$e(\theta)_t = q(\theta,\theta_x)_x, \tag{1}$$

in the domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$.

In (1), e is the internal energy of the medium $e(\theta) = \frac{A}{2(\beta-1)} \frac{\theta^2}{\eta^{1-\beta}}$, where $A > 0$ and $\beta \geqslant 0$, and the heat flux q is defined as $q(\theta) = \kappa(\theta)\ \frac{r^4}{\eta}\ \theta_x$. In these definitions, $\eta, r \in W^{1,\infty}(\Omega)$ are given functions associated to a "frozen system" defined on Ω (see [2] for physical motivations) such that

$$0 < \underline{\eta} \leqslant \eta(x) \leqslant \overline{\eta} \quad \text{and} \quad 0 < R_0 \leqslant r(x) \leqslant R_1, \tag{2}$$

where $\underline{\eta}, \overline{\eta}, R_0, R_1$ are positive constants.

We suppose that the thermal conductivity satisfies

$$\kappa(\theta) > 0 \quad \text{and} \quad s \to \kappa(s) \in C^1(\mathbb{R}_+). \tag{3}$$

In the original model of [7], the following choice is made: $\kappa(\theta) = \frac{A_m}{\theta^m \eta^\alpha}$ for $A_m > 0$, with the two possibilities $(m,\alpha) = (1,1)$ or $(m,\alpha) = (0, 2/3)$.

We consider the boundary conditions

$$q|_{x=0} = 0, \;\; \theta|_{x=M} = \theta_\Gamma, \tag{4}$$

for $t > 0$, with $\theta_\Gamma > 0$, and initial conditions

$$\theta|_{t=0} = \theta^0(x) \qquad \text{on } \Omega. \tag{5}$$

We study weak solutions for the above problem with properties

$$\theta \in L^\infty([0,T], L^2(\Omega)), \quad \sqrt{\rho}\,\theta_x \in L^\infty([0,T], L^2(\Omega)). \tag{6}$$

where $Q_T := \Omega \times (0,T)$.

We also assume the following conditions on the data:

$$\theta^0 \in L^2(\Omega), \quad \inf_{\Omega} \theta^0 > 0. \tag{7}$$

We look for a weak solution θ such that, for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$

$$\int_{Q_T} \left[\phi_t e + \frac{\kappa r^4 \theta_x}{\eta} \phi_x\right] \, dx \, dt = \int_{\Omega} \phi(0,x) \, \theta^0(x) \, dx. \tag{8}$$

Then our first result is the following

Theorem 1 *Suppose that the initial data satisfy (7) and that T is an arbitrary positive number.*

Then the problem (1)(4)(5) possesses at least one global weak solution satisfying (6) together with properties (8). Moreover, the solution is unique.

For that purpose, we first prove a classical existence result in the Hölder category. We denote by $C^\alpha(\Omega)$ the Banach space of real-valued functions on Ω which are uniformly Hölder continuous with exponent $\alpha \in (0,1)$, and $C^{\alpha,\alpha/2}(Q_T)$ the Banach space of real-valued functions on $Q_T := \Omega \times (0,T)$ which are uniformly Hölder continuous with exponent α in x and $\alpha/2$ in t. The norms of $C^\alpha(\Omega)$ (resp. $C^{\alpha,\alpha/2}(Q_T)$) will be denoted by $\|\cdot\|_\alpha$ (resp. $|||\cdot|||_\alpha$).

Theorem 2 *Suppose that the initial data satisfy*

$$\left(\theta^0, \theta^0_x, \theta^0_{xx}\right) \in \left(C^\alpha(\Omega)\right)^3,$$

for some $\alpha \in (0,1)$. Suppose also that $\theta^0(x) > 0$ on Ω, and that the compatibility conditions

$$\theta^0_x(0) = 0 \quad \text{and } \theta^0(M) = \theta_\Gamma,$$

hold. Then, there exists a unique solution $\theta(x,t)$ with $\theta(x,t) > 0$ to the initial-boundary value problem (1)(4)(5) on $Q = \Omega \times \mathbb{R}_+$ such that for any $T > 0$

$$(\theta, \theta_x, \theta_t, \theta_{xx}) \in \left(C^\alpha(Q_T)\right)^4, \quad \text{and } \theta_{xt} \in L^2(Q_T).$$

Then Theorem 1 will be obtained from Theorem 2 through a regularization process.

Finally we prove

Theorem 3 *Suppose that the initial data satisfy (7). Then the solution of the problem (1)(4)(5) follows the following large time behavior:*

1. *There exist positive constants T_{as}, C and λ such that for $t \geqslant T_{as}$*

$$\|\theta - \theta_\Gamma\|_{C(\Omega)} \leqslant C e^{-\lambda t}. \tag{9}$$

2. *Let $m \in [0,1]$. There exists $T_m > 0$ a positive constant C_m and a function U_m such that for any $t > T_m$*

$$\lim_{t \to \infty} \left\|\theta - \theta_\Gamma - \mathcal{C}_m e^{-\lambda_m t} U_m(y)\right\|_{H^1_0(\omega)} = 0. \tag{10}$$

After the previous result, we get a pointwise (rough) estimate of the cooling time T_c as the inverse of the constant λ in (9), which is of qualitative nature as it depends on the initial data and the physical constants of the problem. The major improvement shown in (10) is to get a more precise behaviour of the type $\theta - \theta_S - \mathcal{C}_m e^{-\lambda_m t} U_m \to 0$, with a constant λ_m independent of the initial data. Moreover the pair (λ_m, U_m) is obtained as the solution of a precise eigenvalue problem (see Propositions 3 and 4 below).

Remark 1 *In a more general setting, it is also interesting to consider the complete problem where temperature is coupled to density and velocity fluctuations through a thermo-mechanical system and to solve the associated compressible Navier-Stokes system. The simplest description of such a model is achieved in [2] in which asymptotic stability is proved. Unfortunately more severe constraints on the growth of the thermal conductivity are required.*

The plan of the article is as follows: in section 2 we give a priori estimates sufficient to prove in section 3 global existence of a unique solution, then we give in section 4 the precised asymptotic behaviour of the solution for large time, leading to a plausible notion of "thermalization time".

2 A priori estimates

We first suppose that the solution is classical in the following sense

$$\theta \in C^1([0,T], C^0(\Omega)) \cap C^0([0,T], C^2(\Omega)), \quad \theta > 0. \tag{11}$$

Our first task is to prove the following regularity result

Theorem 4 *Suppose that the initial-boundary value problem (1)(4)(5) has a classical solution described by Theorem 2. Then the solution* (θ, θ_x) *is bounded in the Hölder space* $C^{1/3,1/6}(Q_T)$

$$|||\theta|||_{1/3} + |||\theta_x|||_{1/3} \leqslant C(T),$$

where C *depends on* T*, the physical data of the problem and the initial data. Moreover there exist two positive numbers* $\underline{\theta}$ *and* $\overline{\theta}$ *such that*

$$0 < \underline{\theta} \leqslant \theta \leqslant \overline{\theta}.$$

Let N and T be arbitrary positive numbers. In all the following, we denote by $C = C(N)$ or $K = K(N)$ various positive non-decreasing functions of N, which may possibly depend on the physical constants but not on T.

Lemma 1 *Under the following condition on the data*

$$\left\|\theta^0\right\|_{L^1(\Omega)} \leqslant N, \tag{12}$$

1. *The following energy-entropy identity holds*

$$\frac{d}{dt}\int_\Omega \frac{A}{2(\beta-1)}\ \eta^{\beta-1}\ (\theta-\theta_\Gamma)^2\ dx + \int_\Omega \frac{\kappa(\theta) r^4}{\eta\theta^2}\ \theta_x^2 dx = 0. \tag{13}$$

2. *The following estimate holds*

$$\left\| \frac{(\theta-\theta_\Gamma)^2}{\eta^{1-\beta}} \right\|_{L^\infty(0,T;L^1(\Omega))} \leqslant K(N), \tag{14}$$

Proof: Adding (1) to the same equation multiplied by $-\frac{\theta_\Gamma}{\theta}$ and integrating by parts on Ω, we get

$$\frac{d}{dt}\int_\Omega \frac{A}{2(\beta-1)}\ \eta^{\beta-1}\ (\theta^2 - 2\theta\theta_\Gamma)\ dx = \int_\Omega \left(1-\frac{\theta_\Gamma}{\theta}\right)\left(\kappa\frac{r^4}{\eta}\theta_x\right)_x dx = -\theta_\Gamma \int_\Omega \kappa\frac{r^4}{\eta\theta^2}\theta_x^2 dx, \tag{15}$$

which gives (13), from which (14) follows □

Proposition 1 *Under the previous condition on the data, there exists positive constants* $\underline{\eta}$ *and* $\overline{\eta}$ *depending only on* N *such that*

$$0 < \underline{\theta} \leqslant \theta(x,t) \leqslant \overline{\theta} \ \ \textit{for}\ (t,x) \in Q_T. \tag{16}$$

Proof: Let us consider the symmetrized problem

$$e(\tilde{\theta})_t = q(\tilde{\theta}, \tilde{\theta}_x)_x, \tag{17}$$

in the symmetric domain $\tilde{\Omega} := (-M, M)$.

We consider the boundary conditions

$$\tilde{\theta}\Big|_{x=\pm M} = \theta_\Gamma, \tag{18}$$

for $t > 0$, with $\theta_\Gamma > 0$, and initial conditions

$$\tilde{\theta}\Big|_{t=0} = \tilde{\theta}^0(x) \qquad \text{on } \tilde{\Omega}, \tag{19}$$

with $\tilde{\theta}^0(x) = \theta^0(x)$ if $x \geqslant 0$ and $\tilde{\theta}^0(x) = \theta^0(-x)$ if $x \leqslant 0$.

Applying the maximum principle to this problem, we find

$$0 < \underline{\theta} := \min\{\theta_\Gamma, \inf_{x\in\Omega} \theta^0(x)\} \leqslant \theta(x,t) \leqslant \max\{\theta_\Gamma, \sup_{x\in\Omega} \theta^0(x)\} =: \overline{\theta},$$

which is (16) $\square$

Now, following [2], let us define the positive quantities

$$Y(t) := \max_{0\leqslant s\leqslant t} \int_\Omega \theta_x^2 \; dx \text{ and } X(t) := \int_{Q_t} \theta_s^2 \; dx \; ds.$$

Lemma 2 *The following inequality holds*

$$X(t) + Y(t) \leqslant C, \quad \textit{for any } t \in [0, T]. \tag{20}$$

Proof: Defining $K(\theta) := \int_1^\theta \frac{r^4 \kappa(s)}{\eta} \; ds$, multiplying (1) by K_t and integrating by parts on Ω, we get

$$\int_\Omega \frac{r^4}{\eta} \kappa^2 \theta_t^2 ds = \int_\Omega K_{xx} K_t \; dx = -\int_\Omega K_{xt} K_x \; dx = -\frac{d}{dt} \int_\Omega K_x^2 \; dx.$$

Integrating on $(0, T)$, we get

$$\sup_{(0,T)} \int_\Omega e_\theta(\theta) \; \frac{r^4}{\eta} \; \kappa(\theta) \; \theta_t^2 dx + \int_{Q_T} \frac{r^8}{\eta^2} \; \kappa^2(\theta) \; \theta_x^2 \; dx \; dt \leqslant C.$$

As r, η and θ are bounded after (2) and Proposition 1, the estimate (20) follows $\square$

Corollary 1 *For any* $T > 0$

$$\max_{[0,T]} \int_\Omega \theta_x^2 \; dx \leqslant K \quad \textit{and} \quad \max_\Omega \theta_x^2 \in L^1(0, T). \tag{21}$$

Proof: From Lemma 2, we see that $X(t) \leqslant C$, for any $t \leqslant T$. The first inequality (21) follows directly. The second estimate is a direct consequence of Lemma 2 $\square$

Proposition 2 *The following bounds hold*

$$\max_{[0,T]} \|\theta_t\|_{L^2(\Omega)} \leqslant C(T), \quad \|\theta_{xt}\|_{L^2(Q_T)} \leqslant C(T), \quad \max_{[0,T]} \|\theta_{xx}\|_{L^2(\Omega)} \leqslant C(T). \tag{22}$$

$$\max_{Q_T} \theta_x^2 \leqslant C(T). \tag{23}$$

Proof:

1. Rewriting the equation (1) as

$$e_\theta \theta_t = q_x,$$

differentiating this equation with respect to t (this can be made rigorous as previously), and multiplying by $e_\theta\theta_t$, we get

$$\int_\Omega \left(\frac{1}{2}\ e_\theta^2\theta_t^2\right)_t\ dx = \int_\Omega e_\theta\theta_t q_{xt}\ dx$$

$$= -\int_\Omega (e_{\theta\eta}\eta_x\theta_t + e_{\theta\theta}\theta_x\theta_t + e_\theta\theta_{tx})\left(\kappa_\theta\frac{r^4}{\eta}\theta_t\theta_x + \frac{\kappa r^4}{\eta}\theta_{xt}\right)\ dx$$

$$= -\int_\Omega e_{\theta\eta}\kappa_\theta\frac{r^4}{\eta}\ \eta_x\theta_t^2\theta_x\ dx - \int_\Omega e_{\theta\eta}\frac{\kappa r^4}{\eta}\ \eta_x\theta_t\theta_{xt}\ dx - \int_\Omega e_{\theta\theta}\kappa_\theta\frac{r^4}{\eta}\ \theta_x^2\theta_t^2\ dx$$

$$-\int_\Omega e_{\theta\theta}\frac{\kappa r^4}{\eta}\ \theta_x\theta_t\theta_{xt}\ dx - \int_\Omega e_\theta\kappa_\theta\frac{r^4}{\eta}\ \theta_x\theta_t\theta_{tx}\ dx - \int_\Omega e_\theta\frac{\kappa r^4}{\eta}\ \theta_{xt}^2\ dx =: A.$$

Integrating on $(0,t)$ for $0 \leqslant t \leqslant T$, we find that, for two positive constant α and β

$$\alpha \max_{[0,T]}\int_\Omega \theta_t^2 dx + \beta\int_{Q_T}\theta_{tx}^2 dx\ dt \leqslant \int_\Omega \left(\frac{1}{2}\ e_\theta^2\theta_t^2\right)_t (x,0)\ dx + \sum_{k=1}^5 E_k, \tag{24}$$

where the E_k correspond to the various contributions corresponding to the integrand in A.

Let us estimate all of these terms, using previous bounds (see (2) and (20)) and Cauchy-Schwarz inequality.

$$|E_1| \leqslant C\int_{Q_T}|\theta_t^2\theta_x|\ dx\ dt \leqslant C\int_0^T \max_\Omega \theta_t^2\left(1+\int_\Omega \theta_x^2 dx\right)\ dt$$

$$\leqslant C\int_0^T \max_\Omega \theta_t^2 dt \leqslant C\left(\int_{Q_T}\theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_2| \leqslant C\int_{Q_T}|\theta_t\theta_{xt}|\ dx\ dt \leqslant \varepsilon\int_{Q_T}\theta_{xt}^2 dx\ dt + C_\varepsilon\int_{Q_T}\theta_t^2 dx\ dt$$

$$|E_3| \leqslant C\int_{Q_T}|\theta_t^2\theta_x^2|\ dx\ dt \leqslant C\int_0^T \max_\Omega \theta_t^2\ dt \leqslant C + C\left(\int_{Q_T}\theta_{xt}^2 dx\ dt\right)^{1/2}.$$

$$|E_4| \leqslant C\int_{Q_T}|\theta_t\theta_{xt}|\ dx\ dt \leqslant \varepsilon\int_{Q_T}\theta_{xt}^2 dx\ dt + C_\varepsilon\int_{Q_T}\theta_t^2\ dx\ dt.$$

$$|E_5| \leqslant \varepsilon\int_{Q_T}\theta_{xt}^2 dx\ dt + C_\varepsilon\left(1+\left(\int_{Q_T}\theta_{xt}^2\ dx\ dt\right)^{1/2}\right).$$

Finally, plugging all these estimate into (24) for ϵ small enough, we get

$$\alpha \max_{[0,T]}\int_\Omega \theta_t^2 dx + \frac{1}{2}\ \beta\int_{Q_T}\theta_{tx}^2 dx\ dt \leqslant C + C\left(\int_{Q_T}\theta_{xt}^2 dx\ dt\right)^{1/2},$$

which implies the first two estimates (22).

2. Equation (1) rewrites

$$\theta_{xx} = \frac{4\eta}{r^3}\ \theta_x + \frac{\eta}{\kappa r^4}\ e_\theta\theta_t + \frac{1}{\eta}\ \eta_x\theta_x - \frac{\eta\kappa'}{\kappa}\theta_x^2.$$

Taking the square and integrating on Ω, we get

$$\int_\Omega \theta_{xx}^2\ dx \leqslant C\int_\Omega \left(\theta_x^4 + \theta_x^2 + \theta_t^2 + \theta_x^2\right)\ dx.$$

Using Corollary 1, together with the first bound (22), we can bound the right-hand side, which provide us with the last estimate (22).

3. The inequality (23) follows after (22) $\square$

Proof of Theorem 4

From Proposition 2 we have

$$|\theta(x,t)-\theta(x,t')| \leqslant |t-t'|^{1/2}\left(\int_0^T \theta_t^2\ dt\right)^{1/2} \leqslant C|t-t'|^{1/2}\left(\int_0^T\int_\Omega \theta_{xt}^2\ dx\ dt\right)^{1/2} \leqslant C|t-t'|^{1/2},$$

and after Proposition 2

$$|\theta(x,t)-\theta(x',t)| \leqslant C|x-x'|^{1/2}\left(T\cdot\max_{[0,T]}\int_\Omega \theta_t^2\ dx + \int_0^T\int_\Omega \theta_{xt}^2\ dx\right) \leqslant C|x-x'|^{1/2},$$

so we find that $\theta \in C^{1/2,1/4}(Q_T)$. As we have also after Propositions 2

$$|\theta_x(x,t)-\theta_x(x',t)| \leqslant |x-x'|^{1/2}\left(\int_\Omega \theta_{xx}^2\ dt\right)^{1/2} \leqslant |x-x'|^{1/2},$$

we conclude using an interpolation argument of [6], that $\theta_x \in C^{1/3,1/6}(Q_T)$, which ends the proof of Theorem 4 $\square$

3 Existence and uniqueness of a solution

1. We recall the classical Leray- Schauder fixed point theorem

Theorem 5 *Let $\mathcal{B}$ be a banach space and suppose that $P : [0,1]\times\mathcal{B} \to \mathcal{B}$ has the following properties:*

- *i) For any fixed $\lambda \in [0,1]$ the map $P(\lambda,.) : \mathcal{B} \to \mathcal{B}$ is completely continuous.*
- *ii) For every bounded subset $\mathcal{S} \subset \mathcal{B}$ the family of maps $P(.,\chi) : [0,1] \to \mathcal{B},\ \chi \in \mathcal{S}$ is uniformly equicontinuous.*
- *iii) There is a bounded subset $\mathcal{S}$ of $\mathcal{B}$ such that any fixed point in $\mathcal{B}$ of $P(\lambda,.), \lambda \in [0,1]$ is contained in $\mathcal{S}$.*
- *iv) $P(0,.)$ has precisely one fixed point in $\mathcal{B}$.*

Then, $P(1,.)$ has at least one fixed point in $\mathcal{B}$.

In our case $\mathcal{B}$ will be Banach space of functions $\theta \in \mathcal{B}$ on Q_T with $\theta, \theta_x \in C^{1/3,1/6}(Q_T)$ with the norm

$$|||\theta|||_{\mathcal{B}} := |||\theta|||_{1/3} + |||\theta_x|||_{1/3}.$$

For $\lambda \in [0,1]$ we define $P(\lambda,.)$ as the map which carries $\{\tilde\theta\} \in \mathcal{B}$ into $\{\theta\} \in \mathcal{B}$ by solving the problem

$$\tilde e_\theta(\tilde u,\tilde\theta)\theta_t - \frac{\tilde\kappa(\tilde\eta,\tilde\theta)\tilde r^4}{\tilde\eta}\ \theta_{xx} = \left(\frac{\tilde\kappa(\tilde\eta,\tilde\theta)\tilde r^4}{\tilde\eta}\right)_\eta \tilde\theta_x\eta_x + \frac{\tilde\kappa_\theta(\tilde\eta,\tilde\theta)}{\tilde\eta}\tilde\theta_x^2, \tag{25}$$

together with the boundary conditions

$$\theta_x|_{x=0} = 0,\ \ \theta|_{x=M} = \theta_\Gamma, \tag{26}$$

for $t > 0$, and initial conditions

$$\theta(x,0) = (1-\lambda)\theta_\Gamma + \lambda\theta_0(x). \tag{27}$$

We can consider the parabolic equation (64) and apply the classical Schauder-Friedmann estimates

$$\|\theta_x\|_{1/3} + \|\theta\|_{1/3} \leqslant C\|\tilde{\theta}_x\|_{1/3}.$$

It implies that $P(\lambda, .) : \mathcal{B} \to \mathcal{B}$ is well defined and continuous.

Using a priori estimates from Section 2 it follows that for any $\tilde{\theta}$ from any fixed bounded subset the family $P(., \{\tilde{\theta}\}) : [0,1] \to \mathcal{B}$ of mappings is uniformly equicontinuous.

Now, in order to verify (iii), we observe that any fixed point of P will initially satisfy original problem, therefore θ cannot escape from $[\underline{\theta}, \overline{\theta}]$ up to time T. This fact follows clearly from Theorem 4. To check (iv) we see by inspection that the unique fixed point of $P(0, .)$ is given by $\theta(x,t) = \theta_\Gamma$.

All the previous facts allow us to apply Theorem 5, which imply the existence of classical solutions of (1)(4)(5) in $\Omega \times (0, t^*)$. This ends the existence part of the proof of Theorem 2.

2. Let us now prove the existence of a weak solution. From previous results it follows

- θ_k converge to θ in $L^2(0, t^*, C^0(\Omega))$ strongly and in $L^2(0, t^*, H^1(\Omega))$ weakly ,
- $\theta_k \to \theta$ a.e. in $\Omega \times [0, t^*]$ and in $L^\infty(0, t^*; L^2(\Omega))$ weakly,

Then from the previous computations, it follows that

$$\frac{\kappa_k r_k^2 (\theta_k)_x}{\eta_k} \to A_4 \quad \text{weakly in } L^2(0, t^*, H^1(\Omega)).$$

Then applying similar technique as in [3] it follows that

$$A_4 = \frac{\kappa r^2(\theta)_x}{\eta} \quad \text{in } L^2(0, t^*, H^1(\Omega)),$$

which ends the proof of existence.

3. Finally let us prove uniqueness of the solution. Let θ_i, $i = 1, 2$ be two solutions of (1), and let us consider the differences: $T = \theta_1 - \theta_2$. Dividing the equation (1) by e_θ, we can rewrite it as

$$\theta_t = \frac{q_x}{e_\theta} = \left(\frac{\kappa r^4}{\eta e_\theta} \, \theta_x \right)_x + \frac{\kappa r^4 e_{\theta\theta}}{\eta e_\theta^2} \, \theta_x^2 =: (a_1(\theta)\theta_x)_x + a_2(\theta)\theta_x^2,$$

where one checks easily that the coefficients a_j satisfy $|a_j(\theta_1) - a_j(\theta_2)| \leqslant C|T|$, for $j = 1, 2$.

Subtracting this equation written for θ_1 from the same for θ_2 multiplying by T and integrating by parts we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega T^2 dx + \int_\Omega a_1(\theta_1) T_x^2 \, dx \leqslant C \int_\Omega (T^2 + |TT_x|) dx \leqslant \varepsilon \int_\Omega T_x^2 \, dx + C_\varepsilon \int_\Omega T^2 dx.$$

Choosing ε small enough, we get

$$\frac{d}{dt} \int_\Omega T^2 dx \leqslant C\|T\|_2^2,$$

which clearly implies uniqueness and ends the proof of Theorem 2..

4 Asymptotic behaviour

4.1 Asymptotic stability

Let us begin with the very simple proof of the first part of Theorem 3.

From (13), we have

$$\frac{d}{dt} \int_\Omega \eta^{\beta-1} \, (\theta - \theta_\Gamma)^2 \, dx + \frac{2(\beta-1)}{A} \, \inf_\Omega \, \frac{\kappa(\theta) r^4}{\eta^\beta \theta^2} \int_\Omega \eta^{\beta-1} \theta_x^2 dx = 0. \tag{28}$$

Using the estimate $(\theta-\theta_\Gamma)^2 \leqslant M\int_\Omega \theta_x^2 dx$ we have

$$\frac{d}{dt}\ X(t) + \frac{2(\beta-1)}{A}\ \inf_\Omega\ \frac{\kappa(\theta)r^4}{\eta^\beta\theta^2}\ X(t) \leqslant 0, \tag{29}$$

where $X(t) := \int_\Omega \eta^{\beta-1}\ (\theta-\theta_\Gamma)^2\ dx$.

This implies

$$X(t) \leqslant X(0)\ e^{-\lambda t},$$

with $\lambda = \frac{2M(\beta-1)}{A}\ \inf_\Omega\ \frac{\kappa(\theta)r^4}{\eta^\beta\theta^2}$, and finally

$$\|\theta-\theta_\Gamma\|_{L^2(\Omega)} \leqslant C\ e^{-\lambda t}.$$

Going back to (28), we have

$$\frac{2(\beta-1)}{A}\ \inf_\Omega\ \frac{\kappa(\theta)r^4}{\eta^\beta\theta^2}\int_\Omega \eta^{\beta-1}\theta_x^2 dx \leqslant \left|\frac{d}{dt}\ X(t)\right|,$$

then using (29)

$$\int_\Omega \theta_x^2 dx \leqslant C\left|\frac{d}{dt}\ X(t)\right| \leqslant C\ e^{-\lambda t}.$$

We conclude that $\|\theta-\theta_\Gamma\|_{H_0^1(\Omega)} \leqslant C\ e^{-\lambda t}$ and the pointwise estimate (9) follows, which ends the proof of Theorem 3.

4.2 Precised asymptotics

We split the second part of Theorem 3 in two parts, depending the value of m.

We consider the problem (1), with

$$e(\theta) = \frac{A}{2(1-\beta)}\frac{\theta^2}{\eta^{1-\beta}}, \quad \kappa(\theta) = \frac{A_m}{\theta^m\eta^\alpha},$$

for $A, A_m > 0$, with the two possibilities considered by Lattimer et al. in [7]:

$$(\beta, m, \alpha) = (2/3, 1, 1)\ \ (\text{Case I}), \ \text{ or }\ (\beta, m, \alpha) = (2/3, 0, 2/3)\ \ (\text{Case II}).$$

Depending the value of m, we consider two cases for the (known) variable coefficients in 1. We note

$$a(x) := \frac{A}{4(\beta-1)\eta^{1-\beta}(x)},$$

and

$$b_m(x) := \left|\begin{array}{ll} \dfrac{A_m r^4(x)}{(1-m)\eta^{1+\alpha}} & \text{if } m \neq 1, \\ \dfrac{2A_1 r^4(x)}{\eta^{1+\alpha}} & \text{if } m = 1. \end{array}\right.$$

Setting $u := \theta^2$, $u_\Gamma \equiv \theta_\Gamma^2$, $u^0(x) \equiv (\theta^0)^2(x)$ and using the previous symmetry $x \to y = -x$, (with $a(x) = a(-x)$, $b_m(x) = b_m(-x)$ and $u^0(x) = u^0(-x)$ for $x \in (-M, 0)$), the previous problem rewrites on $\omega \equiv (-M, M)$

$$\left\{\begin{array}{l} a(y)u_t = \left(b_m(y)\left(u^{\frac{1-m}{2}}\right)_y\right)_x, \\ \\ u_x|_{y=-M} = u_\Gamma,\ \ u|_{y=M} = u_\Gamma, \\ \\ u|_{t=0} = u^0(y) \qquad \text{for } y \in \omega, \end{array}\right. \tag{30}$$

for $m \neq 1$, and

$$\begin{cases} a(y)u_t = \Big(b_1(y)\,(\log u)_y\Big)_y\,, \\ \\ u_y\big|_{y=-M} = u_\Gamma,\ \ u\big|_{y=M} = u_\Gamma, \\ \\ u\big|_{t=0} = u^0(y) \qquad \text{for } y \in \omega, \end{cases} \tag{31}$$

for $m = 1$.

The following conditions on the data:

$$u^0 \in L^2(\omega), \quad \inf_\omega u^0 > 0 \tag{32}$$

4.3 Precised symptotics in the case $m = 1$

Using the transformation $v = \log\left(\frac{u}{u_\Gamma}\right)$, problem (31) becomes

$$\begin{cases} a(y)\,(e^v)_t = (b_1(y)v_y)_y\,, \\ v\big|_{y=-M} = v\big|_{y=M} = 0, \\ v\big|_{t=0} = v^0(y) \qquad \text{for } y \in (\omega), \end{cases} \tag{33}$$

with $v^0 = \log\left(\frac{u^0}{u_\Gamma}\right)$. If one expects as in [1], that v is small for large time, a plausible guess for the asymptotic behavior is $v(y,t) \sim Ce^{-\lambda_1 t}U_1(y)$, where the eigenpair $(\lambda_1, U_1(y))$ is the solution of the eigenvalue problem

$$\begin{cases} -\big(b_1(y)U_{1y}\big)_y = \lambda_1 a(y)U_1, \\ U_1\big|_{y=-M} = U_1\big|_{y=M} = 0, \end{cases} \tag{34}$$

where $\lambda_1 > 0$ is the smallest eigenvalue of the operator $-a^{-1}\frac{d}{dy}\left(b_1\frac{d}{dy}\cdot\right)$, with zero boundary conditions.

Proposition 3 *There exists $T > 0$ and a positive constant C_1 such that for any $t > T$ and any $\varepsilon > 0$, the solution v of the problem (33) satisfies*

$$\lim_{t\to\infty} \left\|v(y,t) - \mathcal{C}e^{-\lambda_1 t}U_1(y)\right\|_{H_0^1(\omega)} = 0.$$

Moreover, one has the upper bound

$$\mathcal{C} \leqslant \left[\frac{\int_{-M}^0 b_1^{-1}dy \int_0^M b_1^{-1}dx}{\int_\omega b_1^{-1}\ dy}\,\frac{\int_\omega b_1({v^0}_y)^2\ dy}{\int_\omega b_1 U_{1y}^2\ dy}\right]^{1/2},$$

and, provided that $v^0 \leqslant 1$, the lower bound

$$\mathcal{C} \geqslant \frac{\int_\omega a(e^{v^0} - 1)U_1\ dy}{\int_\omega aU_1\ dy}.$$

Proof: The proof essentially follows the arguments of [1], so we only sketch the main steps.

1. Using $v\big|_{y=-M} = v\big|_{y=M} = 0$ and Cauchy-Schwarz inequality, one has

$$v^2(y,t) = \left(\int_{-M}^y b_1^{1/2}b_1^{-1/2}v_y\ dy\right)^2 \leqslant \int_{-M}^y b_1^{-1}dy \int_{-M}^y b_1 v_y^2 dy,$$

and also

$$v^2(y,t) = \left(\int_y^M b_1^{1/2}b_1^{-1/2}v_y dy\right)^2 \leqslant \int_y^M b_1^{-1}\ dy \int_{-M}^y b_1 v_y^2\ dy.$$

Adding these inequalities, we obtain

$$v^2(y,t) \leqslant \zeta^2(t) := C_1 \int_\omega b_1 v_y^2 \, dy, \tag{35}$$

where $C_1 = \frac{\int_\omega b_1^{-1} dy \int_0^M b_1^{-1} dy}{\int_\omega b_1^{-1} dy}$.

¿From (35), we see that $v \leqslant \zeta$, uniformly in t, or $e^{v-\zeta} \leqslant 1$, so

$$e^{-\zeta} \int_\omega a v^2 e^v \, dy \leqslant \int_\omega a v^2 \, dy,$$

and using once more (35)

$$\int_\omega A v^2 \, dy \leqslant C_2 \int_\omega b_1 v_y^2 \, dy, \tag{36}$$

where $C_2 := C_1 \int_\omega a \, dy$.

But, after Courant-Fisher theorem

$$\lambda_1 \leqslant \frac{\int_\omega b_1 v_y^2 \, dy}{\int_\omega a v^2 \, dy},$$

so we get

$$\lambda_1 \leqslant e^\zeta \frac{\int_\omega b_1 v_y^2 \, dy}{\int_\omega a v^2 e^v \, dy}. \tag{37}$$

Now, integrating by parts and using Cauchy-Schwarz, we see that

$$\left(\int_\omega b_1 v_y^2 \, dy \right)^2 \leqslant \left(\int_\omega (b_1 v_y)_y \, v \, dy \right)^2 \leqslant \left(\int_\omega v a^{1/2} e^{\frac{v}{2}} (b_1 v_y)_y \, a^{-1/2} e^{-\frac{v}{2}} \, dy \right)^2$$

$$\leqslant \int_\omega a v^2 e^v \, dy \times \int_\omega \left[(b_1 v_y)_y \right]^2 a^{-1} e^{-v} \, dy.$$

Plugging into (37)

$$\lambda_1 \leqslant C_1 \, \frac{e^\zeta}{\zeta^2} \int_\omega \left[(b_1 v_y)_y \right]^2 a^{-1} e^{-v} \, dy. \tag{38}$$

As a direct computation gives

$$\frac{d}{dt} \, \zeta^2(t) = 2C_1 \int_\omega b_1 v_y v_{yt} \, dy = -2C_1 \int_\omega (b_1 v_y)_y \, v_t \, dy,$$

we have finally

$$\frac{d\zeta}{dt} = -\frac{2C_1}{\zeta} \int_\omega \left[(b_1 v_y)_y \right]^2 a^{-1} e^{-v} \, dy. \tag{39}$$

Putting into (38), we get the differential inequality

$$-e^\zeta \, \frac{\zeta'}{\zeta} \geqslant \lambda_1,$$

which, as in [1] can be integrated and provides the estimate

$$|\zeta(t)| \leqslant \zeta(0) e^{-\lambda_1 t}. \tag{40}$$

2. Changing v into $u = e^{\lambda_1 t} v$ gives the following equation satisfied by u

$$u_t = \lambda_1 u + a^{-1} e^{-v} \left(b_1 u_y \right)_y. \tag{41}$$

As we know that $v \to 0$ as $t \to \infty$, it is sufficient to prove that $u_t \to 0$ as $t \to \infty$ in order to prove that $u \sim Cte \times U_1$ for large time.

Multiplying (41) by $ae^v u$ and integrating by parts, we get

$$\int_\omega ae^v uu_t \ dx = \int_\omega Ae^v u^2 \ dy - \lambda_1 \int_\omega b_1 u_y^2 dy =: -J(u).$$

As the second contribution in the right-hand side is bounded by $C_1^{-1}\zeta(0)$, after (40) and the first one is bounded by $e^{\zeta(0)} C_1^{-1}\zeta(0)$, after (36), we find that The functional $J(u)$ is bounded.

Computing the derivative of J gives

$$\frac{d}{dt} \ J(u) = 2\lambda_1 e^{-\lambda_1 t} \int_\omega uu_y^2 \ dy - 2 \int_\omega ae^v u_t^2 \ dy, \tag{42}$$

where the first integral in the right-hand side is bounded by a $K > 0$ after (40).

Supposing that $\|u_t\|_{L^2(\omega)}$ does not tend to zero as $t \to \infty$ amounts to have, for a $c > 0$, the inequality

$$2 \int_\omega ae^v u_t^2 \ dy \geqslant c > 0,$$

which implies, after (42)

$$\frac{d}{dt} \ J \leqslant -c + 2\lambda_1 e^{-\lambda_1 t} K,$$

and after integrating

$$J \leqslant -ct + 2K,$$

which contradicts the boundedness of J. Thus we have proved that

$$\lim_{t\to\infty} \|u_t\|_{L^2(\omega)} = 0.$$

Then finally, as $t \to \infty$

$$u \sim \mathcal{C} U_1, \quad \text{in } H^1(\omega).$$

3. In the same way as [1], we can give rough estimates for the constant $\mathcal{C}$.

After (35), we have

$$\zeta(t) = C_1^{1/2} \left(\int_\omega b_1 v_y^2 \ dy \right)^{1/2} \sim C_1^{1/2} e^{-\lambda_1 t} \mathcal{C} \left(\int_{-1}^1 b_1 U_{1y}^{\ 2} \ dy \right)^{1/2},$$

then

$$e^{\lambda_1 t}\zeta(t) \to \mathcal{C} \left(\int_\omega b_1 U_{1y}^{\ 2} \ dy \right)^{1/2} \leqslant \zeta(0),$$

which gives the upper bound

$$\mathcal{C} \leqslant \zeta(0) \left(\int_\omega b_1 U_{1y}^{\ 2} \ dy \right)^{-1/2}.$$

To get a lower bound, we multiply (33) by U_1, integrate by parts and use (34). We get

$$\frac{d}{dt} \int_\omega a(e^v - 1) U_1 \ dy = \int_\omega U_1 (b_1 v_y)_y \ dy = \int_\omega v (b_1 U_{1y})_y \ dy = -\lambda_1 \int_\omega a U_1 v \ dy$$

$$\geqslant -\lambda_1 \int_\omega a U_1 (e^v - 1) \ dy,$$

which can be integrated, giving

$$\int_\omega a(e^v - 1) U_1 \ dy \geqslant C_0 e^{-\lambda_1 t},$$

where $C_0 := \int_\omega a(e^{v^0} - 1) U_1 \ dy$.

Replacing v by its asymptotic form, we get the required lower bound, which ends the proof of Proposition 3 $\square$

4.4 Precised asymptotics in the case $m \neq 1$

Let us consider problem (30). As we know after Theorem 3 that $u - u_\Gamma$ is small for large time, we set

$$u(y,t) - u_\Gamma = z(y,t) + \text{large order terms},$$

where z is small in a suitable sense.

Plugging in (30), we see that z satisfies approximately the linear problem

$$\begin{cases} a(y)z_t = \left(B_m(y)z_y\right)_y, \\ z|_{y=-M} = 0, \;\; z|_{y=M} = 0, \\ z|_{t=0} = u^0(y) - u_\Gamma \qquad \text{for } y \in \omega, \end{cases} \tag{43}$$

up to small corrections, where $B_m(y) := \frac{1-m}{2}\, u_\Gamma^{\frac{1-m}{2}}\, b_m(y)$.

So we expect that the asymptotic behavior of u is $|u(y,t) - u_\Gamma| \sim Ce^{-\lambda_m t}U_m(y)$, where the eigenpair $(\lambda_m, U_m(y))$ is a solution of the eigenvalue problem

$$\begin{cases} -\left(B_m(y)U_{my}\right)_y = \lambda_m a(y)U_m, \\ U_m|_{y=-M} = U_m|_{y=M} = 0. \end{cases} \tag{44}$$

It is well known (see for example [4]) that this problem admits an increasing sequence of eigenvalues $0 < \lambda_m^{(1)} < \lambda_m^{(2)} < \cdots < \lambda_m^{(n)} < \cdots$ tending to $+\infty$ for $n \to \infty$, then $z(y,t) \sim C_m e^{-\lambda_m^{(1)} t}U_m(y)$ for t large.

The precise result is as follows

Proposition 4 *There exists a positive constant C_m such that the solution v of the problem (30) satisfies*

$$\lim_{t\to\infty} \left\| v(y,t) - v_\Gamma - C_m e^{-\lambda_m^{(1)} t}U_m(y) \right\|_{H_0^1(\omega)} = 0.$$

Proof:

As the method is in the same style as in the case $m = 1$, we only sketch the proof.

1. We use the transformation $v = u^{\frac{1-m}{2}} - u_\Gamma^{\frac{1-m}{2}}$, the problem (30) becomes

$$\begin{cases} a(y)\phi(v)v_t = \left(b_m(y)v_y\right)_y, \\ v|_{y=-M} = v|_{y=M} = 0, \\ v|_{t=0} = v^0(y) \qquad \text{for } y \in (\omega), \end{cases} \tag{45}$$

where $\phi(v) := \frac{2}{1-m}\,(v + v_\Gamma)^{\frac{1+m}{1-m}}$, $v_\Gamma := u_\Gamma^{\frac{1-m}{2}}$ and $v^0(y) = (u^0(y))^{\frac{1-m}{2}} - u_\Gamma^{\frac{1-m}{2}}$.

Defining the function $\zeta_m(t) := \int_\omega b_m v_y^2\, dy$ with (negative) derivative $\zeta_m'(t) := 2\int_\omega b_m v_y v_{yt}\, dy$, integrating by parts and using (45) we get

$$\zeta_m'(t) := -2\int_\omega a^{-1}(\phi(v))^{-1} \left[\left(b_m v_y\right)_y\right]^2 \, dy. \tag{46}$$

Using Cauchy-Schwarz inequality, we have also

$$\zeta_m^2(t) \leqslant \int_\omega a^{-1}(\phi(v))^{-1} \left[\left(b_m v_y\right)_y\right]^2 \, dy \times \int_\omega a\phi(v)\; v^2\; dy.$$

Using Taylor's formula, we modify the last integral

$$\int_\omega a\phi(v)\; v^2\; dy = \phi(0)\int_\omega av^2\; dy + \int_\omega \int_0^1 av^3\phi'(sv)\; ds\; dy.$$

After Proposition 1, v is uniformly bounded, then using once more Cauchy-Schwarz, we have

$$\int_\omega a\phi(v)\ v^2\ dy \leqslant \phi(0)\int_\omega av^2\ dy + C\zeta_m^{1/2}(t).$$

Plugging into (46) and using the Courant-Fisher's bound

$$\lambda_m^{(1)} = \inf_{w\in H_0^1(\omega)} \frac{\int_\omega B_m w_y^2\ dy}{\int_\omega aw^2\ dy},$$

we get the inequality

$$\zeta_m \leqslant -\frac{\zeta_m'}{2}\left[\frac{1}{\lambda_m^{(1)}} + C\zeta_m^{1/2}\right].$$

Integrating this differential inequality, and using the boundedness of ζ_m (see Lemma 2), we obtain the estimate

$$|\zeta_m(t)| \leqslant Ke^{-\lambda_m^{(1)}t}. \tag{47}$$

2. Changing v into $u = e^{\lambda_m^{(1)}t}v$ gives the following equation satisfied by u

$$u_t = \lambda_m^{(1)}u + a^{-1}\phi^{-1}\left(b_m u_y\right)_y. \tag{48}$$

As we know that $v \to 0$ as $t \to \infty$, it is sufficient to prove that $u_t \to 0$ as $t \to \infty$ in order to prove that $u \sim Cte \times U_m$ for large time.

Defining the following functional on $H_0^1(\omega)$

$$J_m(u) := \lambda_m^{(1)}\int_\omega a\phi(v)\ u^2\ dy - \int_\omega b_m u_y^2\ dy \equiv \int_\omega a\phi(v)\ uu_t\ dx,$$

and computing its derivative, one get

$$\frac{d}{dt}\ J_m(u) := \int_\omega a\phi(v)\ u_t^2\ dy - 2\lambda_m^{(1)}\int_\omega b_m uu_y^2\ dy.$$

One checks that J_m is bounded, by a $K > 0$ after (47). Studying the derivative $\frac{d}{dt}\ J_m(u)$ we conclude as in the case $m = 1$ that

$$\lim_{t\to\infty}\|u_t\|_{L^2(\omega)} = 0.$$

Then finally, as $t \to \infty$

$$u \sim \mathcal{C}U_m, \quad \text{in } H^1(\omega).$$

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