

# Notes on switching lemmas

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We prove three switching lemmas, for random restrictions for which variables are set independently; for random restrictions where variables are set in blocks (both due to Håstad [3]); and for a distribution appropriate for the bijective pigeonhole principle [2, 4]. The proofs are based on Beame's version [1] of Razborov's proof of the switching lemma in [5], except using families of weighted restrictions rather than families of restrictions which are all the same size. This follows a suggestion of Beame in [1]. The result is something between Håstad's and Razborov's methods of proof. We use probabilistic arguments rather than counting ones, in a similar way to Håstad, but rather than doing induction on the terms in our formula with an inductive hypothesis involving conditional probability, as Håstad does, we explicitly build one function to bound the probabilities for the whole formula.

## 1 A restriction which sets variables independently

Let  $F$  be an  $r$ -DNF, that is, a disjunction of conjunctions (which we will usually call "terms") where each conjunction has size  $r$  or less. Suppose the variables in  $F$  come from a set  $X$  of size  $n$ .

Fix a probability  $p$ . Define a distribution  $\mathcal{R}$  of partial restrictions to the variables by choosing a restriction  $\rho$  as follows: independently for each  $x \in X$ , set  $x$  to 0 with probability  $\frac{1-p}{2}$ ; to 1 with probability  $\frac{1-p}{2}$ ; or to  $*$  (meaning "leave it unset") with probability  $p$ .

The weight  $|\rho|$  of a restriction  $\rho \in \mathcal{R}$  is its probability of being chosen from  $\mathcal{R}$ . So if  $\rho$  has exactly  $a$  many 1s,  $b$  many 0s and  $c$  many  $*$ s, then the weight of  $\rho$  is  $(\frac{1-p}{2})^{a+b}p^c$ . The weight  $|S|$  of a set of restrictions  $S \subseteq \mathcal{R}$  is the probability that a random restriction from  $\mathcal{R}$  belongs to  $S$ , or, equivalently, the sum of the weights of the restrictions in  $S$ .

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The canonical tree  $T(F, \rho)$  is defined by the following decision procedure: Look through  $F$  for a term  $C$  such that  $C \upharpoonright \rho \neq 0$ . If there is no such term, then halt and output “0”. Otherwise let  $C_1$  be the first such term. Let  $\beta_1$  list the variables that appear starred in  $C_1 \upharpoonright \rho$ . Query all these variables in order, and let the assignment  $\pi_1$  be given by the replies. If  $\rho\pi_1$  satisfies  $C_1$  (or if  $\beta_1$  was empty and  $C_1$  was already satisfied in  $\rho$ ) halt and output “1”. Otherwise repeat this step starting with  $\rho\pi_1$  in place of  $\rho$ , looking for a term  $C_2$  which is the first such that  $C_2 \upharpoonright \rho\pi_1 \neq 0$  etc., until we run out of terms.

Note that this tree correctly decides  $F \upharpoonright \rho$ , in that if  $\pi$  is given by the answers along a branch in the tree, then the label at the end of the branch (called the “output” above) is the value of  $F \upharpoonright \rho\pi$ .

**Lemma 1** *Fix a number  $s > 0$ . Let  $S$  be the set of restrictions  $\rho$  in  $\mathcal{R}$  for which  $T(F, \rho)$  has height  $s$  or greater. Then  $|S| \leq (9pr)^s$  (assuming  $p < 1/9$ ).*

**Proof** We will bound the weight of  $S$  by defining an injection from  $S$  into a set of small weight (roughly speaking) and then arguing about how this map changes weights. So let  $\rho \in S$  and let  $\pi$  be the first path in  $T(F, \rho)$  with length  $s$  or greater.

Let  $C_1, \dots, C_k, \beta_1, \dots, \beta_k$  and  $\pi_1, \dots, \pi_k$  be the terms, unset variables and assignments to them encountered along  $\pi$ , as far as the  $s$ th query in  $\pi$ , from the construction of  $T(F, \rho)$ . It may be that the  $s$ th query in  $\pi$  occurred in the middle of querying the variables  $\beta_k$ , in which case we trim  $\beta_k$  and  $\pi_k$  to only include the variables mentioned in the first  $s$  queries in  $\pi$ .

For each  $i = 1, \dots, k$  let  $\sigma_i$  be the (unique) assignment to  $\beta_i$  which is consistent with  $C_i \upharpoonright \rho\pi_1 \dots \pi_{i-1}$  (for  $i < k$ , this will actually satisfy  $C_i \upharpoonright \rho\pi_1 \dots \pi_{i-1}$ ). Note that the  $\beta_i$ s are all disjoint so that  $\rho\sigma_1 \dots \sigma_k$  is a well-defined restriction. Let  $\sigma$  be  $\sigma_1 \dots \sigma_k$ .

We will code each  $\beta_i$  as a string  $\beta'_i$  of  $|\beta_i|$  numbers, each less than  $2r$ , by recording, for each variable in  $\beta_i$ , its location in  $C_i$  (a number less than  $r$ ) and whether it is the last variable in  $\beta_i$  (one bit). We will code the whole tuple  $\beta_1, \dots, \beta_k$  as the concatenation  $\beta' = \beta'_1 \dots \beta'_k$  of these strings. So over  $S$  there are  $(2r)^s$  possible different strings  $\beta'$ .

We will code each  $\pi_i$  simply as a string  $\pi'_i$  of  $|\beta_i|$  bits, one for each variable in  $\beta_i$ .  $\pi'$  will be the concatenation  $\pi'_1 \dots \pi'_k$ . So there are  $2^s$  possible strings  $\pi'$ .

We now define a map  $\theta : S \rightarrow \mathcal{R} \times (2r)^s \times 2^s$  by

$$\theta : \rho \mapsto (\rho\sigma, \beta', \pi').$$

We claim that this is an injection. To see this, suppose we are given  $(\rho\sigma, \beta', \pi')$ . We may recover  $\rho$  as follows: First, we can easily recover all

strings  $\beta'_i$  and  $\pi'_i$ . Now let  $C'_1$  be the first term in  $F$  such that  $C'_1 \upharpoonright \rho\sigma \neq 0$ .  $C'_1$  cannot come before  $C_1$ , and by construction  $C_1$  is not falsified by  $\rho\sigma$ . So we must have  $C'_1 = C_1$ . From  $\beta'_1$  and  $C_1$  we can recover  $\beta_1$ , and from this and  $\pi'_1$  we can recover  $\pi_1$ .  $\sigma_1$  and  $\pi_1$  were assignments to the same variables, so we can construct a restriction  $\rho\sigma[\pi_1/\sigma_1] = \rho\pi_1\sigma_2 \dots \sigma_k$ . Let  $C'_2$  be the first term in  $F$  such that  $C'_2 \upharpoonright \rho\pi_1\sigma_2 \dots \sigma_k \neq 0$ . Then as above,  $C'_2$  must equal  $C_2$ , and we can recover  $\beta_2$  and  $\pi_2$  and carry on in the same way. Once we have recovered all the  $\beta_i$ s we know exactly what changed between  $\rho$  and  $\rho\sigma$  and can recover  $\rho$ .

Now temporarily fix some values of  $\beta'$  and  $\pi'$  and let  $S_{\beta',\pi'}$  be the subset of  $S$  consisting of all  $\rho$ s to which  $\theta$  assigns these values.

Restricted to  $S_{\beta',\pi'}$ , the first component  $\theta_1 : \rho \mapsto \rho\sigma$  of  $\theta$  is an injection  $S_{\beta',\pi'} \rightarrow \mathcal{R}$ , so the weight  $|\theta_1[S_{\beta',\pi'}]|$  of its image is the sum of the individual weights  $|\theta_1(\rho)|$  over all  $\rho \in S_{\beta',\pi'}$ . But  $\rho\sigma$  sets exactly  $s$  variables that were unset in  $\rho$ , so  $|\rho\sigma| = p^{-s}(\frac{1-p}{2})^s|\rho|$ . Hence

$$|\theta_1[S_{\beta',\pi'}]| = \left(\frac{1-p}{2p}\right)^s |S_{\beta',\pi'}|.$$

But  $\theta_1[S_{\beta',\pi'}]$  is a subset of  $\mathcal{R}$  so has weight  $\leq 1$ , so  $|S_{\beta',\pi'}| \leq (\frac{2p}{1-p})^s$ .

Finally  $S$  is the union of the sets  $S_{\beta',\pi'}$  over all possible strings  $\beta', \pi'$ . So

$$|S| \leq (2r)^s 2^s \left(\frac{2p}{1-p}\right)^s = \left(\frac{8pr}{1-p}\right)^s$$

giving the result. □

## 2 A restriction which sets variables in blocks

Let  $F$  be an  $r$ -DNF in variables  $X$ , as above. Suppose that  $X$  is partitioned into a family  $\mathcal{B}$  of disjoint blocks. We assume there is some fixed ordering on the variables in each block.

Fix probabilities  $p$  and  $q$  (in the usual application we may take  $p = q$ , but it is useful to keep them separate to keep track of what is happening in the proof). Define a distribution  $\mathcal{R}$  of partial restrictions by choosing a restriction  $\rho$  in two stages, as follows: First, independently for each  $x \in X$ , set  $x$  to 1 with probability  $1 - p$ , otherwise leaving it starred. Then, independently for each block  $B \in \mathcal{B}$  (ignoring any blocks which are already set to all 1), with probability  $1 - q$  set all starred variables in  $B$  to 0 (in which case we call  $B$  a 0-block), otherwise leaving them all starred (in which case we call  $B$  a \*-block).

The weight of a restriction  $\rho$  is the product of the weights of its blocks. If a block has  $a$  many 1s and  $b > 0$  many non-1s, its weight is  $(1-p)^a p^b (1-q)$

if it is a 0-block and  $(1-p)^a p^b q$  if it is a \*-block. If a block is all 1s (in which case the terms 0-block and \*-block become meaningless) then its weight is  $(1-p)^{|B|}$  (although in applications there are unlikely to be any such blocks).

The restriction  $g(\rho)$  extends  $\rho$  further: for each \*-block in  $\rho$ , it sets every starred variable, except for the first one, to be 1.

The canonical tree  $T(F, \rho)$  is defined by the following decision procedure: Look through  $F$  for a term  $C$  such that  $C \upharpoonright \rho \neq 0$ . If there is no such term, then halt and output “0”. Otherwise let  $C_1$  be the first such term. Let  $\beta_1$  list the blocks  $B$  such that a starred variable from  $B$  appears in  $C_1 \upharpoonright \rho$ , in the order in which they appear. For  $B \in \beta_1$ , query the single starred variable in  $B \upharpoonright g(\rho)$ , regardless of whether or not this is the variable appearing in  $C_1 \upharpoonright \rho$ . Let  $\pi_1$  be the complete assignment to all the blocks in  $\beta_1$  given by  $g(\rho)$  together with all the replies to the queries. That is, under  $\rho\pi_1$  each block  $B \in \beta_1$  will be set to 1 everywhere, except that the variable that was queried may be set to either 0 or 1, depending on the reply. If  $\rho\pi_1$  satisfies  $C_1$  (or if  $C_1$  was already satisfied in  $\rho$ ) halt and output “1”. Otherwise repeat this step starting with  $\rho\pi_1$  in place of  $\rho$ , etc.

Note that along any branch in the tree no block is queried more than once, and that the moment a block is queried all variables in that block are given a 0 or 1 value.

The tree  $T(F, \rho)$  correctly decides  $F \upharpoonright g(\rho)$ , in that if  $\pi$  is given by the answers along a branch in the tree, then the label at the end of the branch is the value of  $F \upharpoonright g(\rho)\pi$ .

**Lemma 2** *Fix a number  $s > 0$ . Let  $S$  be the set of restrictions  $\rho$  in  $\mathcal{R}$  for which  $T(F, \rho)$  has height  $s$  or greater. Then  $|S| \leq (13qr)^s$  (assuming  $p < 1/2r$  and  $q < 1/13$ ).*

**Proof** The proof is similar to the proof of the previous lemma. Let  $\rho \in S$  and let  $\pi$  be the first path in  $T(F, \rho)$  with length  $s$  or greater.

Let  $C_1, \dots, C_k, \beta_1, \dots, \beta_k$  and  $\pi_1, \dots, \pi_k$  be the terms, blocks and assignments encountered along  $\pi$ , as far as the  $s$ th query in  $\pi$ , from the construction of  $T(F, \rho)$ . If necessary, trim  $\beta_k$  and  $\pi_k$  to only include the blocks mentioned in the first  $s$  queries in  $\pi$ .

For  $i = 1, \dots, k$  let  $\gamma_i$  list the starred variables from the blocks in  $\beta_i$  which appear positively in  $C_i \upharpoonright \rho\pi_1 \dots \pi_{i-1}$ .

For  $i = 1, \dots, k$  let  $\sigma_i$  be the following assignment. For each block  $B \in \beta_i$ ,  $\sigma_i$  sets every starred (under  $\rho$ ) variable in  $B$  that appears positively in  $C_i \upharpoonright \pi_1 \dots \pi_{i-1}$  to 1, and sets all remaining starred variables in  $B$  to 0. Note that this is consistent with  $C_i \upharpoonright \pi_1 \dots \pi_{i-1}$ , and also that  $\sigma_i$  sets exactly the same variables as  $\pi_i$  does, so that the different  $\sigma_i$ s are disjoint and  $\rho\sigma_1 \dots \sigma_k$  is a well-defined restriction. Let  $\sigma$  be  $\sigma_1 \dots \sigma_k$ .

As before, we will code  $\beta_1, \dots, \beta_k$  as a string  $\beta' = \beta'_1 \dots \beta'_k$ , by recording the location of the first starred variable from a block. There are  $(2r)^s$  possible strings  $\beta'$ .

As before, we will code  $\pi_1, \dots, \pi_k$  as a string  $\pi' = \pi'_1 \dots \pi'_k$  of  $s$  bits.

We will code each  $\gamma_i$  as a string  $\gamma'_i$  of  $r$  bits, one for each literal in  $C_i$ , recording whether that literal is in  $\gamma_i$ . We code the whole tuple  $\gamma_1, \dots, \gamma_k$  as the concatenation  $\gamma' = \gamma'_1 \dots \gamma'_k$  of these strings. There are at most  $2^{rs}$  many possible  $\gamma'$ s.

We define  $\theta : S \rightarrow \mathcal{R} \times (2r)^s \times 2^s \times 2^{rs}$  by

$$\theta : \rho \mapsto (\rho\sigma, \beta', \pi', \gamma').$$

To see that this is an injection, suppose we are given  $(\rho\sigma, \beta', \pi', \gamma')$ . We can recover  $C_1$  and  $\beta_1$  just as in the previous lemma, and from  $C_1$  and  $\gamma'_1$  we can recover  $\gamma_1$ . But now for each block  $B \in \beta_1$ ,  $\gamma_1$  tells us exactly what we changed to go from  $\rho \upharpoonright B$  to  $\rho\sigma_1 \upharpoonright B$ . We can undo the changes, and recover  $\rho \upharpoonright B$ , by setting all variables in  $B$  mentioned in  $\gamma_1$  to  $*$  and setting all 0s in  $\rho\sigma_1 \upharpoonright B$  to  $*$ . Then we can recover  $\pi_1 \upharpoonright B$  by setting all but the first  $*$  to 1 and setting the first  $*$  according to  $\pi'_1$ . Then we continue for the rest of the terms, similarly to the previous lemma.

Now temporarily fix some values of  $\beta'$ ,  $\pi'$  and  $\gamma'$  and let  $S_{\beta', \pi', \gamma'}$  be the subset of  $S$  consisting of all  $\rho\sigma$  to which  $\theta$  assigns these values. Let  $\theta_1$  be as before, and suppose that  $\gamma'$  records  $m$  many variables in total.

For  $\rho \in S_{\beta', \pi', \gamma'}$ , going from  $\rho$  to  $\rho\sigma$  changes  $m$  many non-1s into 1s, which increases the weight by a factor of  $(\frac{1-p}{p})^m$ . It also changes  $s$  many  $*$ -blocks, each one being changed into either a 0-block, increasing the weight by a factor of  $\frac{1-q}{q}$ , or into an all-1 block, increasing the weight by a factor of  $\frac{1}{q}$ . So the total increase is  $\geq (\frac{1-p}{p})^m (\frac{1-q}{q})^s$ . Hence

$$|S_{\beta', \pi', \gamma'}| \leq \left(\frac{p}{1-p}\right)^m \left(\frac{q}{1-q}\right)^s.$$

For  $\rho \in S$  there are at most  $rs$  variables that could be recorded in  $\gamma'$ . So we can calculate

$$\begin{aligned} |S_{\beta', \pi'}| &= \sum_{\gamma'} |S_{\beta', \pi', \gamma'}| = \sum_{m=0}^{rs} \sum_{|\gamma'|=m} |S_{\beta', \pi', \gamma'}| \\ &\leq \sum_{m=0}^{rs} \binom{rs}{m} \left(\frac{p}{1-p}\right)^m \left(\frac{q}{1-q}\right)^s \\ &= \left(1 + \frac{p}{1-p}\right)^{rs} \left(\frac{q}{1-q}\right)^s \\ &\leq e^{\frac{prs}{1-p}} \left(\frac{q}{1-q}\right)^s \end{aligned}$$

$$\leq \left( \frac{3q}{1-q} \right)^s$$

where the last inequality holds if  $p < \frac{1}{2r}$ .

Finally

$$|S| \leq (2r)^s 2^s \left( \frac{3q}{1-q} \right)^s = \left( \frac{12qr}{1-q} \right)^s$$

giving the result. Clearly the constant could be improved by putting some more conditions on  $p$  and  $q$ .  $\square$

### 3 A restriction for the pigeonhole principle

Let  $F$  be an  $r$ -DNF in variables  $P = \{p_{xy} : x \in n+1, y \in n\}$ , where we take  $p_{xy}$  to express that “pigeon  $x$  goes to hole  $y$ ”.

Fix a probability  $q$ . Define a distribution  $\mathcal{R}$  of partial injections of  $n+1$  into  $n$  by choosing a partial injection  $\rho$  as follows: choose the range of  $\rho$  by putting each hole into the range independently at random with probability  $(1-q)$ , then choose uniformly at random from all possible injections from the set of pigeons into this range (we do this “backwards” to exclude the possibility of having to find holes for  $n+1$  pigeons). If  $\rho$  sets exactly  $m$  pigeons, then the weight of  $\rho$  is  $(1-q)^m q^{n+1-m} \frac{(n+1-m)!}{(n+1)!}$ .

We will identify a partial injection  $\rho$  with the partial assignment to the variables  $P$  in which, for each pigeon  $x$  which is sent to a hole  $y$  by  $\rho$ :  $p_{xy}$  is set to true,  $p_{xy'}$  is set to false for each  $y' \neq y$ , and  $p_{x'y}$  is set to false for each  $x' \neq x$ .

To define the canonical tree  $T(F, \rho)$  for  $F$  with respect to such a  $\rho$ , we first pre-process  $F$ . For each term in  $F$  we replace each negative literal  $\neg p_{xy}$  with the disjunction  $\bigvee_{y' \neq y} p_{xy'}$  and then distribute out so that the formula is once again an  $r$ -DNF (which may now possibly be  $n^r$  times larger). We then remove any term which asserts that two pigeons go to one hole or that one pigeon goes to two holes. We call our new  $r$ -DNF  $F'$ .

The tree will again be given by a decision procedure. This time the queries are not to the values of propositional variables. Instead we can either name a pigeon and query which hole it goes to, or name a hole and query which pigeon goes to it. We may assume that the replies always form a partial injection (assume that the tree halts in some error state if a branch becomes so long that there are no free holes or pigeons available to reply to a query).

Now look for the first term  $C$  in  $F'$  such that  $C \upharpoonright \rho \neq 0$ . If there is no such term, halt and output “0”. Otherwise let  $C_1$  be the first such term. Let  $\beta_1$  list the literals  $p_{xy}$  that appear in  $C_1 \upharpoonright \rho$ . For each such  $p_{xy}$ , query which

hole pigeon  $x$  goes to, and then query which pigeon goes to hole  $y$ . Let  $\pi_1$  be the partial injection given by the replies to these queries. As before, if  $\rho\pi_1$  satisfies  $C_1$  halt and output “1”, and otherwise repeat this step starting with  $\rho\pi_1$  in place of  $\rho$ , etc.

Let  $\pi$  be a branch in the tree, which we will identify with the partial injection given by the replies along that branch. If  $\pi$  ends with the output “1” at the leaf, then by construction  $\pi$  must satisfy some term  $C'$  in  $F'$  and hence must also satisfy the term  $C$  in  $F$  from which  $C$  arose. Conversely, if  $\pi$  ends with the output “0”, then there is no extension of  $\pi$  to a partial injection  $\alpha$  which satisfies any term  $C$  in  $F$ . For suppose such an  $\alpha$  and  $C$  existed. Then  $\alpha$  would also satisfy some term  $C'$  in  $F'$  arising from  $C$ ; but then  $C' \upharpoonright \rho$  cannot be 0, contradicting the construction of the tree.

**Lemma 3** *Fix a number  $s > 0$ . Let the probability  $q$  be chosen so that  $128r^2n^3q^4 < 1$  (the constant here could easily be improved). Let  $S$  be the set of partial injections  $\rho \in \mathcal{R}$  for which  $T(F, \rho)$  has height  $s$  or greater. Then the weight  $|S|$  is exponentially small in  $s$ .*

**Proof** Let  $l = 2qn$ . By the Chernoff bound, for all but an exponentially small (in  $n$ ) number of exceptions, every  $\rho \in S$  leaves fewer than  $l$  pigeons and  $l$  holes unset. Hence we may assume in what follows that every  $\rho \in S$  has this property.

Let  $\rho \in S$  and let  $\pi$  be the first path in  $T(F, \rho)$  with length  $s$  or greater.

Let  $C_1, \dots, C_k, \beta_1, \dots, \beta_k$  and  $\pi_1, \dots, \pi_k$  be the terms, literals and replies encountered along  $\pi$ , as far as the  $s$ th query in  $\pi$ , from the construction of  $T(F, \rho)$ . If necessary, trim  $\beta_k$  and  $\pi_k$  to only include the blocks mentioned in the first  $s$  queries in  $\pi$ .

For each  $i = 1, \dots, k$  let  $\sigma_i$  be the partial injection which is specified by the literals in  $\beta_i$ . Note that every pigeon and hole in  $\sigma_i$  also occurs in  $\pi_i$ , because in the tree we query both the pigeons and the holes occurring in  $\beta_i$ . Hence for  $j > i$ ,  $\sigma_i$  is consistent with  $\sigma_j$ , because the pigeons and holes occurring in  $\sigma_j$  are disjoint from those in  $\pi_i$  and hence disjoint from those in  $\sigma_i$  (recall that  $\beta_j$  lists the literals  $p_{xy}$  in  $C_j \upharpoonright \rho\pi_1 \dots \pi_{j-1}$ ). Let  $\sigma$  be  $\sigma_1 \dots \sigma_k$ .

As before, we will code  $\beta_1, \dots, \beta_k$  as a string  $\beta' = \beta'_1 \dots \beta'_k$ , by recording the locations of the literals  $\beta_i$  in each term  $C_i$ . There are  $(2r)^s$  possible strings  $\beta'$ .

Let  $A$  and  $B$  be respectively the set of pigeons and the set of holes left unset in  $\rho\sigma$ . We know that  $|A|, |B| \leq l$ . We will code  $\pi_i$  as follows. For each  $p_{xy}$  in  $\beta_i$ , pigeon  $x$  was queried in the branch  $\pi$  and was either assigned to hole  $y$  or to some hole  $y' \neq y$ . In the second case, such a  $y'$  must be in  $B$  because from this point onwards, neither the tree nor  $\sigma$  can assign any other pigeon to hole  $y'$ . Hence we may code which hole was assigned using a single

bit and a number less than  $l$ . Similarly hole  $y$  was assigned either pigeon  $x$  or some pigeon from  $A$ , and we code this in a similar way. Let  $\pi'_i$  be the string coding all replies in  $\pi_i$  in this way, and let  $\pi'$  be the concatenation  $\pi'_1 \dots \pi'_k$ . There are  $(2l)^s$  possible strings  $\pi'$ .

We define  $\theta : S \rightarrow \mathcal{R} \times (2r)^s \times (2l)^s$  by

$$\theta : \rho \mapsto (\rho\sigma, \beta', \pi').$$

This is an injection, for suppose we are given  $(\rho\sigma, \beta', \pi')$ . We know  $A$  and  $B$  immediately. We can recover  $C_1$  and  $\beta_1$  as before, and from  $\beta_1$ ,  $A$  and  $B$  we can recover  $\pi_1$ . Then we continue as before.

Now temporarily fix some values of  $\beta'$  and  $\pi'$  and let  $S_{\beta', \pi'}$  be the subset of  $S$  to which  $\theta$  assigns these values. Let  $\theta_1$  be as before.

For  $\rho \in S_{\beta', \pi'}$ , going from  $\rho$  to  $\rho\sigma$  sets at least  $s/2$  more pigeons. Adding one more pigeon to a partial injection of size  $m$  changes its weight by a factor of  $\frac{1-q}{q} \frac{1}{n+1-m}$ . In our case  $n - m$  is always smaller than  $l$ , so the factor is at least  $\frac{1-q}{ql}$ . Hence the total increase is at least  $(\frac{1-q}{ql})^{s/2}$ , and thus  $|S_{\beta', \pi'}| \leq (\frac{ql}{1-q})^{s/2}$  which we can bound by  $(2ql)^{s/2}$  if we assume that  $q < 1/2$ .

Finally, recalling that  $l = 2qn$ , we have

$$\begin{aligned} |S| &\leq (2r)^s (2l)^s (2ql)^{s/2} = (4r^2 \cdot 16q^2 n^2 \cdot 2q^2 n)^{s/2} \\ &= (128r^2 n^3 q^4)^{s/2}, \end{aligned}$$

which gives the result. □

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