Overlap index, overlap functions and migrativity

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Abstract— *In this work we study overlap degrees expressed in terms of overlap functions. We present the basic properties that from our point of view must satisfy these overlap functions. We study a construction method, we analyze which t-norms are also overlap functions and we prove that if we apply particular aggregations to such functions we recover the overlap index between fuzzy sets as defined by Dubois, and the consistency index of Zadeh. We also consider some properties that can be required to overlap functions, as* k*-Lipschitzianity or migrativity .*

Keywords— Overlap degree, Overlap function, Overlap index, t-norm, Migrativity.

1 Overlap function. Definition and properties

Zadeh's fuzzy sets theory has been very useful for solving problems which are described by imprecise models and with a large amount of noise. In particular, this theory is very appropriate to study the problem of identifying the objects in an image (see [18, 26]).

To separate the object from the background in an image, the first thing to do is to represent the object by means of a fuzzy set and the background by means of another one. The success of the separation method lies on the correct choice of those fuzzy sets, which do not need to be disjoint in the sense of Ruspini [27] (see [1, 2, 17]).

To build these sets it is necessary to know the exact property that characterizes the pixels belonging to the object (background). This property determines the expression of the membership function associated to the fuzzy set representing the object (background)(see [8, 9]). Usually, this function is not precisely known. There are some pixels for which the expert is sure they belong to the object or the background, but there are other pixels for which the expert hesitates. It is for these last ones that the value of the membership function is not accurately known.

So, suppose that for a given an image, we ask an expert to assign to each pixel of intensity q the following values:
$\mu_B(q)$, representing the membership of the pixel to the background.
$\mu_O(q)$, representing the membership of the pixel to the object.

In Fig.1 we show the two membership functions provided by the expert to represent an image in an L gray-level scale. We can deduce that, for intensities less than q_i, the expert is sure that the pixels do not belong to the object. For intensities greater than q_j he is sure that the pixels do not belong to the background. However, for intensities between q_i and q_j the expert is not sure about the membership of the pixels, with intensity q_k corresponding to the maximum lack of knowledge.

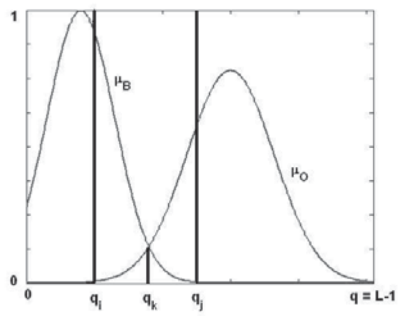

Figure 1: Overlap between two functions

From this analysis we deduce that the overlap degree between the two functions can be understood as a representation of the lack of knowledge of the expert when he has to settle if a given pixel belongs to the background or to the object. So we can define the overlap degree between $\mu_B(q)$ and $\mu_O(q)$ by means of an overlap function

$$G_S : [0,1] \times [0,1] \to [0,1]$$

such that:
(G_S0) G_S depends only on $\mu_B(q)$ and $\mu_O(q)$.
(G_S1) G_S is symmetric. The overlap degree does not depend on the order we consider the membership degrees.
(G_S2)
$G_S(\mu_B(q), \mu_O(q)) = 0$ if and only if
$\mu_B(q) = 0$ or $\mu_O(q) = 0$, (i.e., $\min(\mu_B(q), \mu_O(q)) = 0$) .
(G_S3)
$G_S(\mu_B(q), \mu_O(q)) = 1$ if and only if
$\mu_B(q) = 1$ and $\mu_O(q) = 1$, (i.e., $\min(\mu_B(q), \mu_O(q)) = 1$).
(G_S4) If the membership degrees increase, so does the overlap degree.
(G_S5) Continuity. The overlap degree between the two memberships associated to a given pixel must not react chaotically

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under small variations of the values of the membership to the background or to the object.

Apart from these five necessary properties, we consider it is also natural to require the following migrative property.
(G_S6) Migrativity. If we decrease $\mu_B(q)$ in a proportion $\alpha \in (0,1]$, the overlap degree G_S should decrease in the same amount as if we decrease $\mu_O(q)$ in the same proportion α. That is:

$$G_S(\alpha\mu_B(q), \mu_O(q)) = G_S(\mu_B(q), \alpha\mu_O(q)) \text{ for all } \alpha \in (0,1].$$

Definition 1 *A mapping $G_S : [0,1]^2 \to [0,1]$ is an overlap function if and only if it satisfies $(G_S0) - (G_S5)$. If G_S satisfies (G_S6) for any $0 < \alpha \leq 1$, we say that it is a migrative overlap function.*

Let's denote by $\mathcal{G}$ the set of overlap functions in the sense of Definition 1. Then the following result is immediate.

Theorem 1 $(\mathcal{G}, \leq_{\mathcal{G}})$ *with the ordering $\leq_{\mathcal{G}}$ defined for $G_1, G_2 \in \mathcal{G}$ by*

$$G_1 \leq_{\mathcal{G}} G_2 \text{ if and only if } G_1(x,y) \leq G_2(x,y) \quad (1)$$

for all $x, y \in [0,1]$ is a lattice.

It is clear that the lattice $(\mathcal{G}, \leq_{\mathcal{G}})$ is not complete, neither has it top or bottom elements. On the other hand, it is closed under the action of appropriate aggregation functions, as shown next.

Definition 2 *An aggregation function of dimension n ([7, 11, 20, 16]) is a mapping $M : [0,1]^n \to [0,1]$ satisfying (see also [12, 3]):*
$M1$. $M(0, \cdots, 0) = 0$ *and* $M(1, \ldots, 1) = 1$.
$M2$. *For any $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n) \in [0,1]^n$, if $x_i \leq y_i$ for any $i \in \{1, \cdots, n\}$, then $M(x_1, \cdots, x_n) \leq M(y_1, \cdots, y_n)$; that is, M is monotone increasing in all its arguments.*

Theorem 2 *Let M be a binary aggregation function without zero divisors (that is, $M(x,y) = 0$ implies $min(x,y) = 0$) and such that $M(x,y) = 1$ only if $\max(x,y) = 1$. Then, $M(G_1, G_2) \in \mathcal{G}$ for any $G_1, G_2 \in \mathcal{G}$.*

2 Construction

The following theorem provides both a characterization and a construction method of overlap functions.

Theorem 3 *A mapping $G_S : [0,1]^2 \to [0,1]$ is an overlap function if and only if it can be written as*

$$G_S(x,y) = \frac{f(x,y)}{f(x,y) + h(x,y)} \quad (2)$$

for some $f, h : [0,1]^2 \to [0,1]$ such that
1) *f and h are symmetric;*
2) *f is non decreasing and h is non increasing;*
3) $f(x,y) = 0$ *if and only if* $\min(x,y) = 0$;
4) $h(x,y) = 0$ *if and only if* $\min(x,y) = 1$;
5) *f and h are continuous;*

Example 1.
1) If $f(x,y) = \min(x,y)$ and $h(x,y) = \max(1-x, 1-y)$, then $G_S(x,y) = \min(x,y)$ is an overlap function.
2) If we take $f(x,y) = \sqrt{x.y}$ and $h(x,y) = \max(1-x, 1-y)$, then the construction proposed in Theorem 3 provides an overlap function.
3) If $f(x,y) = \sqrt{x \cdot y}$ and $h(x,y) = 1 - x \cdot y$, expression

$$G_S(x,y) = \frac{\sqrt{x \cdot y}}{\sqrt{x \cdot y} + 1 - x \cdot y} \quad (3)$$

is an overlap function.

Corollary 1 *In the setting of Theorem 3, assume that G_S can be expressed in two different ways:*

$$G_S(x,y) = \frac{f_1(x,y)}{f_1(x,y) + h_1(x,y)} = \frac{f_2(x,y)}{f_2(x,y) + h_2(x,y)} \quad (4)$$

for any $x, y \in [0,1]$. Let M be a binary continuous aggregation function that is homogeneous of order one, that is, such that

$$M(\lambda x, \lambda y) = \lambda M(x,y) \quad (5)$$

for any $x, y \in [0,1]$ and any $\lambda \geq 0$ such that $\lambda x, \lambda y \in [0,1]$. Then, if we define $f(x,y) = M(f_1(x,y), f_2(x,y))$ and $h(x,y) = M(h_1(x,y), h_2(x,y))$ it also holds that

$$G_S(x,y) = \frac{f(x,y)}{f(x,y) + h(x,y)} \,. \quad (6)$$

Proof. First observe that $f_i = h_i \frac{G_S}{1-G_S}$ for $i = 1, 2$. By the homogeneity condition on M, also $f = h \frac{G_S}{1-G_S}$ and the result follows.

2.1 Specific case: t-norms

In this subsection we study under which conditions we can assure a t-norm satisfies the properties required to an overlap function.

We know that a t-norm is a commutative, associative, increasing mapping $T : [0,1]^2 \to [0,1]$ such that $T(x,1) = x$ for any $x \in [0,1]$. So t-norms satisfy (G_S1) and (G_S4). They also satisfy (G_S3), since if $T(x,y) = 1$, as $T(x,y) \leq \min(x,y)$, it must be $x = y = 1$. The reciprocal is direct, taking into account that 1 is the neutral element of any t-norm. So it is only required to study condition (G_S2). Observe that when dealing with t-norms, the necessary condition in this property coincides with the definition of positive t-norm. Besides, the analysis of the conditions under which a t-norm satisfies (G_S2) leads to the following classification result.

Theorem 4 *If a t-norm T is an overlap function, then T is of one of the following three types:*
1) $T = \min$;
2) *T is strict;*
3) *T is the ordinal sum of the family $\{([a_m, b_m], T_m)\}$, with all the T_m continuous Archimedean t-norms and such that if for some m_0 we have $a_{m_0} = 0$ then necessarily T_{m_0} is a strict t-norm.*

Proof. By hypothesis, T is continuous. From the classification of continuous t-norms given in page 11 of [15] (see [25],

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[21]), we know that for a continuous t-norm T there are three possibilities:
1.- $T = \min$;
2.- T is Archimedean;
3.- There exists a family $\{([a_m, b_m], T_m)\}$ such that T is the ordinal sum of this family in the sense of [15].

As, by hypothesis, T is an overlap function, (G_S2) holds. If T is also Archimedean, we have that T is strict.

Suppose now that T is the ordinal sum of the family $\{([a_m, b_m], T_m)\}$; that is:

$$T(x,y) = \begin{cases} a_m + (b_m - a_m)T_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}) \\ \qquad \text{if } (x,y) \in [a_m, b_m]^2 \text{ ;} \\ \min(x,y) \text{ otherwise.} \end{cases} \quad (7)$$

We know that for any t-norm T, if $\min(x,y) = 0$ then $T(x,y) = 0$. Since our t-norm is an overlap function, also the reciprocal is true. So, if $T(x,y) = 0$ two things can happen:
a) (x,y) does not belong to any $[a_m, b_m]^2$. Then we have $T(x,y) = \min(x,y)$ for that (x,y).
b) (x,y) belongs to $[a_m, b_m]^2$. As $T(x,y) = 0 = a_m + (b_m - a_m)T_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m})$, we have $a_m = 0$ and $b_m \neq 0$ since otherwise the interval would be $[0,0]$ and $x = y = 0$. We know that T verifies (G_S2) and if $T(x,y) = 0$, then $T_m(\frac{x}{b_m}, \frac{y}{b_m}) = 0$, so T_m also verifies (G_S2). Hence, the continuous and Archimedean t-norm T_m associated to the interval $[0, b_m]$ also satisfies (G_S2), so it is strict.

Example 2. 1) In the construction of the following overlap function we use item 3) of Theorem 4 . We take as t-norm for the corresponding interval $[0, b_m]$ the product, which is strict, continuous and Archimedean.

$$G_S(x,y) = \begin{cases} 2xy \text{ if } (x,y) \in [0, 0.5]^2 \text{ ;} \\ \min(x,y) \text{ in other case.} \end{cases} \quad (8)$$

2) In the construction of the following overlap function we take the Lukasiewicz and the product t-norms (see page 84 in [19]). Observe that now we do not consider any interval of the type $[0, b_m]$.

$$G_S(x,y) = \begin{cases} 0.1 + 2.5(x-0.1)(y-0.1) \text{ if } (x,y) \in [0.1, 0.5]^2; \\ 0.7 + \max(x+y-1.6, 0) \text{ if } (x,y) \in [0.7, 0.9]^2; \\ \min(x,y) \text{ in other case.} \end{cases} \quad (9)$$

3) The following t-norm does not satisfy (G_S2). This is due to the fact that in $[0, 0.25]^2$ we take Lukasiewicz t-norm, which is continuous and Archimedean, but it is not strict.

$$T(x,y) = \begin{cases} \max(x+y-0.25, 0) \text{ if } (x,y) \in [0, 0.25]^2 \\ \min(x,y) \text{ in other case.} \end{cases} \quad (10)$$

3 Overlap functions and k-Lipschitzianity

In this section we consider a particular type of overlap functions, satisfying a sort of stronger continuity. We start adapting the definition of k-Lipschitz functions to the overlap function case.

Definition 3 *Let $k \geq 1$. An overlap function G_S is k-Lipschitz if for any $x, y, z, t \in [0,1]$ it holds*

$$|G_S(x,y) - G_S(z,t)| \leq k(|x-z| + |y-t|) \text{ .} \quad (11)$$

It is worth remarking that the usual definition of k-Lipschitziniaty allows any value of k greater than zero. But, in the case of overlap functions, just by taking $x = y = z = 1$ and $t = 0$ the restriction to $k \geq 1$ becomes justified.

The set of k-Lipschitz overlap functions is bounded and its supremum can be easily determined, as the next result shows.

Theorem 5 *Let $k \geq 1$. Then the supremum of the set of k-Lipschitz overlap functions is given by the mapping* $\min(kx, ky, 1)$*, whereas the infimum is given by* $\max(kx + ky - 2k + 1, 0)$*.*

Proof. Suppose that $G_S(x,y) > \min(kx, ky, 1)$ for some $x, y \in [0,1]$. Since $G_S(x,y) \leq 1$, this means that $\min(kx, ky, 1) = kx$ or $\min(kx, ky, 1) = ky$. In the first case, $y = t = 1$ and $z = 0$ in Eq. (11), we arrive at

$$kx < G_S(x,1) \leq kx \quad (12)$$

which is a contradiction. The second case is analogous. On the other hand, by defining for $\epsilon > 0$ the mappings

$$\max(x \cdot y, (1-\epsilon))(\min(kx, ky, 1)) \quad (13)$$

we get a sequence of overlap functions which converges uniformly to $\min(kx, ky, 1)$ as $\epsilon \to 0$. The proof for the lower bound is similar.

The mapping $\max(kx + ky - 2k + 1, 0)$ is never an overlap function. On the contrary, although in general, the mapping $\min(kx, ky, 1)$ for $k > 1$ and $x, y \in [0,1]$ such that $kx, ky \in [0,1]$ does not define an overlap function, (since by taking $x = y = \frac{1}{k}$ we see that it does not fulfill condition (G_S3)), $\min(x,y)$ is an overlap function, so we have the following corollary.

Corollary 2 *The mapping $\min(x,y)$ is the strongest 1-Lipschitz overlap function, in the sense that for any other 1-Lipschitz overlap function G_S the inequality*

$$G_S(x,y) \leq \min(x,y) \quad (14)$$

holds for any $x, y \in [0,1]$.

For associative k-Lipschitz overlap functions we have the next result which can be derived from [23, 24].

Theorem 6 *If G_S is an associative k-Lipschitz overlap function, then G_S is a t-norm of the form given in Theorem 4, where each involved strict t-norm T (see item 2) or item 3)) has a k-convex additive generator t, i.e.,*

$$t(y + k\epsilon) - t(y) \leq t(x + \epsilon) - t(x) \quad (15)$$

for all $0 \leq y \leq x < 1$ and $\epsilon \in]0, \min(1-x, (1-y)/k)[$.

4 Overlap index. Construction from overlap functions

In this section we are going to build overlap indexes by aggregating overlap degrees. We start recalling the concepts of overlap index and consistency. Then we justify why most of the overlap indexes in the literature do not satisfy one of the four conditions required by Dubois and Prade. Next we show the properties we have to demand to aggregation functions so

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that, when applied to the previously studied overlap degrees, we get overlap indexes. We finish showing how to construct such indexes and, in particular, how to recover the two most used expressions.

We denote by $F(U)$ the set of all fuzzy sets over the finite, non empty referential U ($Cardinal(U) = n$). We are going to represent the fuzzy sets over U in the following way:

$$A = \{(u, \mu_A(u)) | u \in U\} \tag{16}$$

In 1978 Zadeh [29] presented the natural extension to the fuzzy set theory of the classical concept of overlap, which he called consistency:

$$O(A, B) = \sup_{i=1}^{n} (\min(\mu_A(u_i), \mu_B(u_i))). \tag{17}$$

Clearly, $O(A, B) = 0$ if A and B are completely disjoint, and $O(A, B) = 1$ if there is $u_i \in U$ such that $\mu_A(u_i) = \mu_B(u_i) = 1$.

In 1982 Dubois and Prade [13] presented the following axiomatization for the overlap index:

Definition 4 *An overlap index is a function $O(A, B)$ from $F(U) \times F(U)$ on the unit interval such that:*
$(O1)$ $O(A, B) = 0$ *if and only if A and B have disjoint supports;*
$(O2)$ $O(A, B) = 1$, *if* ($\mu_A(u_i) = 0$ *or* $\mu_B(u_i) = 1$) *or* ($\mu_A(u_i) = 1$ *or* $\mu_B(u_i) = 0$)*;*
$(O3)$ $O(A, B) = O(B, A)$*;*
$(O4)$ *If* $B \leq C$*, then* $O(A, B) \leq O(A, C)$*.*

Condition $(O2)$ in this definition presents the advantage of that, if A is not fuzzy, then $O(A, A) = 1$. But Dubois, Ostasiewicz and Prade in [14] settled the following:

1. For subnormal fuzzy sets (i.e., $\mu_A(u_i) < 1$ and $\mu_B(u_i) < 1$ for any $u_i \in U$), $(O2)$ *must be ignored*.

2. The ROC index(see [13]) does not fulfill $(O2)$.

It is also interesting to notice that if $A = \{(u_i, \mu_A(u_i) = 0) | u_i \in U\}$, then $\min(A, A) = \{(u_i, \mu_{\min(A,A)}(u_i) = 0) | u_i \in U\}$ and from $(O1)$, $O(A, A) = 0$. If we also impose $(O2)$, then $O(A, A) = 1$. So we get a contradiction.

Due to all these considerations, usually only conditions $(O1)$,$(O3)$ and $(O4)$ from Def. 4 are required to overlap indexes.

In the following theorem we present a construction method of overlap indexes, by means of aggregation functions.

Theorem 7 *Let $M : [0, 1]^n \to [0, 1]$ be an aggregation function being idempotent and such that $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \ldots = x_n = 0$.*
Let $G_S : [0, 1]^2 \to [0, 1]$ be a mapping and consider:
$O : F(U) \times F(U) \to [0, 1]$ *defined as*
$O(A, B) = M_{i=1}^{n}(G_S(\mu_A(u_i), \mu_B(u_i)))$
Then the following items hold:
$i)$ *O verifies* $(O1)$ *if and only if* G_S *verifies* (G_S2)*;*
$ii)$ *O verifies* $(O3)$ *if and only if* G_S *verifies* (G_S1)*;*
$iii)$ *If* G_S *verifies* (G_S4)*, then O verifies* $(O4)$*.*

Example 3.

1. $O(A, B) = \max_{i=1}^{n} G_S(\mu_A(u_i), \mu_B(u_i))$

2. $O(A, B) = \left(\frac{1}{n} \sum_{i=1}^{n} (G_S(\mu_A(u_i), \mu_B(u_i))^\beta \right)^{\frac{1}{\beta}}, \beta \neq 0$

Notes for the Example

1. If in 1. we take $G_S(x, y) = \min(x, y)$, then we recover Zadeh's consistency index (see [29]).

2. If there is a single u_i such that $G_S(\mu_A(u_i), \mu_B(u_i)) = 1$, then, by 1. we have $O(A, B) = 1$. This fact of a single element making $O(A, B) = 1$ suggest us to use 2. instead of 1.

3. Expression 2. satisfies:

   $O(A, B) = 1$ if and only if $G_S(\mu_A(u_i), \mu_B(u_i)) = 1$

   for all $i = 1, \ldots, n$. In these conditions, if $G_S(x, y) = \min(x, y)$ or $G_S(x, y) = x \cdot y$ or $G_S(x, y) = \sqrt{x \cdot y}$, then $O(A, B) = 1$ if and only if

   $$A = B = \{(u_i, \mu_A(u_i) = \mu_B(u_i) = 1) | u_i \in U\} \tag{18}$$

   so 2., as most of the expressions of overlap indexes (see [4, 5, 6, 14, 13]), does not fulfill condition $(O2)$ in Definition 2.

Corollary 3 *In the setting of Theorem 7, if we demand M to satisfy $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \ldots = x_n = 1$, then, if G_S verifies (G_S3) we have*
$O(A, B) = 1$ *if and only if* $A = B = \{(u_i, \mu_A(u_i) = 1) | u_i \in U\}$.

5 Construction of migrative overlap functions

As we have already said, migrativity seems to be quite a natural property to be required to overlap functions. In [10], an in-deep study of the migrativity property is carried on for general aggregation functions. In this paper we use the following results that are proved there.

Lemma 1 *Let $H : [0, 1]^2 \to [0, 1]$ be a binary function. Then H is migrative if and only if $H(x, y) = H(1, x \cdot y)$, for all $x, y \in [0, 1]$.*

Lemma 2 *A function $H : [0, 1]^2 \to [0, 1]$ is migrative if and only if there exists $h : [0, 1] \to [0, 1]$ such that $H(x, y) = h(x \cdot y)$, for all $x, y \in [0, 1]$.*

In this section we present a characterization theorem of migrative overlap functions. Clearly there are overlap functions that are not migrative (for instance, those in Ex. 1). In the following results we prove that there exist also overlap functions which are migrative.

Theorem 8 *Let $H : [0, 1]^2 \to [0, 1]$ be a migrative mapping (not necessarily an overlap function). Then*

$$H(x, 1) = H(\sqrt{x}, \sqrt{x}) \tag{19}$$

for each $x \in [0, 1]$.

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Proof. By Lemmas 1 and 2 there exists a mapping $h : [0,1] \to [0,1]$ such that $H(x,1) = h(x) = h(\sqrt{x} \cdot \sqrt{x}) = H(\sqrt{x}, \sqrt{x})$.

Theorem 9 *A mapping* $G_S : [0,1] \to [0,1]$ *is a migrative overlap function if and only if there exists a non decreasing function* $g : [0,1] \to [0,1]$ *satisfying* $g^{-1}((0,1)) = (0,1)$ *such that*

$$G_S(x,y) = g(x \cdot y). \quad (20)$$

Proof. (Necessity) Since G_S is migrative, by Lemma 2 we know that there exists a function $g : [0,1] \to [0,1]$ such that $G_S(x,y) = g(x \cdot y)$ for all $x, y \in [0,1]$. As G_S is an overlap function, g is not decreasing and continuous. Besides G_S satisfies $(G_S 2)$ and $(G_S 3)$, so:

$$g(x) = g(x \cdot 1) = G_S(x,1) = 0 \text{ if and only if } x = 0 \quad (21)$$

$$g(x) = g(x \cdot 1) = G_S(x,1) = 1 \text{ if and only if } x = 1 \quad (22)$$

(Sufficiency) By Lemma 2 we have that $G_S(x,y) = g(x,y)$ satisfies $(G_S 6)$. From the migrativity it is clear that $(G_S 1)$ holds. On the other hand:

$$G_S(x,y) = 0 = g(x \cdot y) \text{ if and only if } x \cdot y = 0 \quad (23)$$

$$G_S(x,y) = 1 = g(x \cdot y) \text{ if and only if } x \cdot y = 1 \quad (24)$$

Clearly, G_S satisfies $(G_S 4)$ and $(G_S 5)$ since g is non decreasing and continuous.

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