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IMAGE FEATURES INVARIANT WITH RESPECT TO BLUR

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Abstract—The paper is devoted to the feature-based description of blurred images acquired by a linear shift-invariant imaging system. The proposed features are invariant with respect to blur (this means with respect to the system point spread function), are based on image moments and are calculated directly from the blurred image. This way, we are able to describe the original image without the PSF identification and image restoration. In many applications (such as in image recognition from a database) our approach is much more effective than the traditional "blind-restoration" one. The derivation of the blur invariants is the major theoretical result of the paper. Several experiments are presented to illustrate the efficiency of the invariants for blurred image description. Stability of the invariants with respect to additive random noise and boundary effect is also discussed and is shown to be sufficiently high.

Blurred image   Image restoration   Linear imaging system   Symmetric blur
Image moments   Blur invariants

1. INTRODUCTION

One of the most frequent tasks in image processing is restoration, recognition or other processing of an image which was captured by an imperfect imaging system. The acquired image usually represents the scene in an unsatisfactory manner. Since real imaging systems as well as imaging conditions are imperfect, an observed image represents only a degraded version of the original scene. Blur is introduced into the image during the imaging process by such factors as diffraction, lens aberration, motion of the scene, wrong focus and atmospheric turbulence. The recent difficulties with the Hubble Space Telescope show the necessity of having appropriate tools for dealing with blurred images.

The widely accepted standard linear model[1] describes the imaging process by convolution of an unknown original (or ideal) image $f(x, y)$ with the space-invariant point spread function (PSF) $h(x, y)$:

$$g(x, y) = (f * h)(x, y) \tag{1}$$

where $g(x, y)$ represents the observed image. The PSF $h(x, y)$ describes the imaging system and in our case it is assumed to be unknown.

The classical "blind-restoration" approach consists of the following two steps:

- Estimation of the PSF $h(x, y)$;
- Estimation of the ideal image $f(x, y)$ via restoration of the blurred image $g(x, y)$.

Both of these steps have been dealt with extensively in the literature during the last two decades.

One group of methods for PSF identification is based on the investigation of zero patterns in the spectral domain[2–4] or spike patterns in the cepstral domain.[5] Another group of methods is based on modeling of the image by a stochastic process. The original image is modeled as an autoregressive (AR) process and the blur as a moving average (MA) process. The blurred image is then modeled as a mixed autoregressive moving average (ARMA) process and the MA process identified by this model is considered as a description of the PSF. In this way the problem of PSF estimation is transformed into the problem of determining the parameters of an ARMA model.[6–11]

After the PSF has been identified, the original image can then be estimated via restoration of the blurred image by an inverse filter, a Wiener filter or by any other similar technique [see[1,12] or[15] for a survey].

As a rule the above mentioned approach to image restoration is very complicated and time-consuming. In many cases, we do not need to know the original image itself; we only need to know some representation of it (a typical example is a recognition of a blurred image against a database). However, such a representation should be independent of the imaging system and should really describe the original image, not the degraded one. In this paper, we present a set of features for image description which are invariant with respect to blur (this means that the feature values of $g(x, y)$ do not depend on $h(x, y)$ and that they are the same as the feature values of $f(x, y)$). Image recognition may then be accomplished via classification in the feature space.

In this way, we get rid of the necessity of the PSF identification and image restoration.

It should be noted that a similar approach has been used many times for the description of images with *geometric* degradations [see[14] for instance] but our work presents the first attempt to apply a feature-based technique in the case of *radiometric* degradations given by equation (1).

The blur-invariant features introduced in this paper are based on image moments. In Section 2 we deal with ordinary and central moments of a blurred image and we express them as functions of moments of an ideal image and the PSF. Then our attention will be focused on symmetric blur, which involves long-term atmospheric blur and out-of-focus blur as special cases. Section 3 is devoted to the derivation of invariant features. An original algorithm for derivation of the invariants is presented and the invariants up to the 7th order are shown in explicit form. In Section 4, their invariance and discriminability are demonstrated by experiments. Stability of the invariants with respect to additive random noise as well as to boundary effects is discussed in Section 5.

2. MOMENTS OF A BLURRED IMAGE

The two-dimensional $(p+q)$th order *ordinary moment* $m_{pq}^{(f)}$ of an image $f(x,y)$ is defined by the integral

$$m_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\,dx\,dy. \qquad (2)$$

The point $(x_t^{(f)}, y_t^{(f)})$ given by the relations

$$x_t^{(f)} = \frac{m_{10}^{(f)}}{m_{00}^{(f)}},$$

$$y_t^{(f)} = \frac{m_{01}^{(f)}}{m_{00}^{(f)}},$$

is called *the center of gravity* or *centroid* of the image $f(x,y)$. The $(p+q)$th order *central moment* $\mu_{pq}^{(f)}$ is then defined as

$$\mu_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(f)})^p (y - y_t^{(f)})^q f(x,y)\,dx\,dy. \qquad (3)$$

The following theorem expresses the moments of the blurred image given by equation (1) in terms of moments of the original image and the PSF.

Theorem 1. Let $f(x,y)$ be a function describing an original image and $h(x,y)$ a shift-invariant PSF of a linear imaging system. The functions $f(x,y)$ and $h(x,y)$ are assumed to be piecewise continuous and non-zero only on bounded supports. Let $g(x,y)$ be a blurred image given by the convolution

$$g(x,y) = (f*h)(x,y).$$

Then the relation

$$\mu_{pq}^{(g)} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(f)}\mu_{p-k,q-j}^{(h)}$$

holds for every p and q.

Proof.

$$\begin{aligned}
\mu_{pq}^{(g)} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(g)})^p (y - y_t^{(g)})^q g(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(f)} - x_t^{(h)})^p (y - y_t^{(f)} - y_t^{(h)})^q \\
&\quad \times (f*h)(x,y)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(f)} - x_t^{(h)})^p (y - y_t^{(f)} - y_t^{(h)})^q \\
&\quad \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(a,b) f(x-a, y-b)\,da\,db)\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(a,b)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(f)} - x_t^{(h)})^p \right. \\
&\quad \left. \times (y - y_t^{(f)} - y_t^{(h)})^q f(x-a, y-b)\,dx\,dy\right) da\,db \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(a,b)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - x_t^{(f)} + a - x_t^{(h)})^p \right. \\
&\quad \left. \times (y - y_t^{(f)} + b - y_t^{(h)})^q f(x,y)\,dx\,dy\right) da\,db \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(a,b)\left(\sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j} \right. \\
&\quad \left. \times (a - x_t^{(h)})^{p-k} (b - y_t^{(h)})^{q-j} \mu_{kj}^{(f)}\right) da\,db \\
&= \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j}\mu_{kj}^{(f)}\mu_{p-k,q-j}^{(h)} \qquad \square
\end{aligned}$$

3. SYMMETRIC BLUR INVARIANTS

In this Section, we derive moment-based image features which are independent of blur, i.e. independent of the type and parameters of $h(x,y)$. The feature B is called *blur invariant* if and only if $B^{(f)} = B^{(f*h)} \equiv B^{(g)}$ for every $h(x,y)$.

We consider *symmetric* blur only, this means that $h(x,y)$ is assumed to be symmetric with respect to both axes and both diagonals. More formally, $h(x,y)$ is assumed to satisfy the following conditions (conditions of symmetry):

$$h(x,y) = h(-x,y),$$

$$h(x,y) = h(y,x).$$

Note that every PSF with radial symmetry $h(x,y) = h(r)$ is a special case of symmetric blur as defined above. Therefore two very frequent types of blur—long-term atmospheric turbulence blur and out-of-focus blur—belong to our class of symmetric blur.

Moreover, the degradation system is assumed to be energy-preserving, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,dx\,dy = \mu_{00}^{(h)} = 1. \qquad (4)$$

Lemma 1. If $h(x, y)$ satisfies the conditions of symmetry then

- $\mu_{pq}^{(h)} = \mu_{qp}^{(h)}$ for every p and q;
- if p or q is odd, then $\mu_{pq}^{(h)} = 0$.

The proof of Lemma 1 is straightforward.

3.1. *Derivation of the invariants*

Blur invariants are supposed to be functions of central moments of an image, i.e. they should have the form $B = B(\mu_{00}, \mu_{20}, \ldots, \mu_{pq})$. By the order r of the invariant B we understand the order of the highest moment μ_{pq}, i.e. $r = p + q$.

Derivation of the invariants up to the 3rd order is almost trivial. It is quite easy to prove by means of Theorem 1 and Lemma 1 that $\mu_{00}, \mu_{11}, \mu_{20}-\mu_{02}, \mu_{12}, \mu_{21}, \mu_{03}$ and μ_{30} are invariant with respect to symmetric blur. Note that $\mu_{10} = \mu_{01} = 0$ for any image and therefore there are no first-order invariants.

We propose the following algorithm for the construction of symmetric blur invariants of order $r > 3$.

(1) Let $r > 3$ be the order of the desired invariant B. Let μ_{pq} be any central moment of order r. Then we start by setting

$$K = [(r-4)/2]$$

(the symbol $[x]$ denotes the integer part of x),

$$I_0 = \mu_{pq}$$

if p and/or q is odd and

$$I_0 = \mu_{pq} - \mu_{qp}$$

if p and q are even and $p \neq q$. If p is even and $p = q$, no invariant is generated by the moment μ_{pq}.

(2) **for** $n = 0$ **to** K

Define D_n as

$$D_n = I_n^{(g)} - I_n^{(f)}.$$

D_n has the form

$$D_n = \sum_{i=1}^{s_n} F_i(\mu^{(f)})\mu_{a_i b_i}^{(h)} + R_n(\mu^{(f)}, \mu^{(h)})$$

where $F_1, \ldots, F_{s_n}$ are functions of the central moments of the image $f(x, y)$ only and $\mu_{a_i b_i}^{(h)}$ are central moments of the highest order of $h(x, y)$. No moment of $h(x, y)$ of the same order is contained in $R_n(\mu^{(f)}, \mu^{(h)})$. The moments of $g(x, y)$ were evaluated by means of Theorem 1. It follows that

$$a_1 + b_1 = a_2 + b_2 = \cdots = a_{s_n} + b_{s_n} = 2(K - n + 1).$$

Due to the symmetry of $h(x, y)$, all a_i and b_i are even for each n and satisfy the relation $a_i \geq b_i$. Define I_{n+1} as

$$I_{n+1} = I_n - \frac{1}{\mu_{00}} \sum_{i=1}^{s_n} F_i(\mu)\mu_{a_i b_i}.$$

end for

(3) Set

$$B = I_{K+1}.$$

However, it is not clear whether every function B produced by this algorithm is really blur invariant. The answer is given by the following Lemma.

Lemma 2. Consider the situation in Step 2 for the last n (i.e. $n = K$). Then

- $s_K = 1$, $a_1 = 2$ and $b_1 = 0$. Therefore D_K has the form

$$D_K = F(\mu^{(f)})\mu_{20}^{(h)}.$$

- If the function $F(\mu)$ is a blur invariant, then the function B produced as a result of the above mentioned algorithm is also blur invariant.

Proof.

- Proof of the first assertion is straightforward.
-

$$B^{(g)} = I_{K+1}^{(g)} = I_K^{(g)} - \frac{F(\mu^{(g)})\mu_{20}^{(g)}}{\mu_{00}^{(g)}}$$

$$I_K^{(f)} - \frac{F(\mu^{(g)})(\mu_{00}^{(f)}\mu_{20}^{(h)} + \mu_{20}^{(f)})}{\mu_{00}^{(g)}} + F(\mu^{(f)})\mu_{20}^{(h)}$$

$$= I_K^{(f)} - \frac{F(\mu^{(f)})\mu_{20}^{(f)}}{\mu_{00}^{(f)}} = I_{K+1}^{(f)} = B^{(f)} \qquad \square$$

It follows from the algorithm description that the number n_r of independent invariants of order r is

$$n_r = r + 1$$

if r is odd (each moment μ_{pq} of order r generates one independent invariant) and

$$n_r = [(3r+2)/4]$$

if r is even (moments μ_{pq} and μ_{qp} generate dependent invariants if both p and q are even).

Note that the above mentioned formulae are not valid for $r = 0$ and $r = 1$:

$$n_0 = 1,$$

$$n_1 = 0.$$

3.2. *Metric space of the invariants*

Since invariants should serve as features for image similarity or dissimilarity assessment, it is sometimes inconvenient to use the invariants described above directly.

The invariants should be normalized in two ways: to be independent of the average gray-level value of the image and to have the same "weight" in an Euclidean metric space. To achieve this, we use *normalized blur invariants*

$$B' = \frac{B}{(N/2)^r \cdot \mu_{00}}$$

where N is the size of the image and r is the order of B. Moreover, as a result of this normalization, all the B' remain invariant even in the case of a non energy-preserving imaging system where assumption (4) does not hold.

The set $\mathscr{B}_r$ of invariants B' up to the rth order has

$$N_r = \sum_{k=2}^{r} n_k$$

elements. The N_r-dimensional metric space $(\mathscr{B}_r, \varrho)$ where ϱ is the Euclidean metric is a suitable feature space for evaluating image-to-image distances.

3.3. *Symmetric blur invariants in explicit form*

By applying the above described algorithm, we can construct blur invariants of any order and express them in explicit form. A set of invariants up to the 7th order is listed below:

- Zero-order.

$$B_0 = \mu_{00}.$$

- 1st order.

none.

- 2nd order.

$$B_1 = \mu_{11},$$
$$B_2 = \mu_{20} - \mu_{02}.$$

- 3rd order.

$$B_3 = \mu_{12},$$
$$B_4 = \mu_{21},$$
$$B_5 = \mu_{03},$$
$$B_6 = \mu_{30}.$$

- 4th order.

$$B_7 = \mu_{13} - \frac{3\mu_{20}\mu_{11}}{\mu_{00}},$$

$$B_8 = \mu_{31} - \frac{3\mu_{20}\mu_{11}}{\mu_{00}},$$

$$B_9 = \mu_{40} - \mu_{04} - \frac{6\mu_{20}B_2}{\mu_{00}}.$$

- 5th order.

$$B_{10} = \mu_{32} - \frac{(3\mu_{12} + \mu_{30})\mu_{20}}{\mu_{00}},$$

$$B_{11} = \mu_{23} - \frac{(3\mu_{21} + \mu_{03})\mu_{20}}{\mu_{00}},$$

$$B_{12} = \mu_{41} - \frac{6\mu_{21}\mu_{20}}{\mu_{00}},$$

$$B_{13} = \mu_{14} - \frac{6\mu_{12}\mu_{20}}{\mu_{00}},$$

$$B_{14} = \mu_{05} - \frac{10\mu_{03}\mu_{20}}{\mu_{00}},$$

$$B_{15} = \mu_{50} - \frac{10\mu_{30}\mu_{20}}{\mu_{00}}.$$

- 6th order.

$$B_{16} = \mu_{33} - \frac{9\mu_{22}\mu_{11}}{\mu_{00}} - \frac{3\mu_{20}}{\mu_{00}^2}(\mu_{00}(\mu_{31} + \mu_{13}) - 3\mu_{11}(\mu_{20} + \mu_{02})),$$

$$B_{17} = \mu_{42} - \mu_{24} + \frac{(\mu_{40} - 6\mu_{22})B_2}{\mu_{00}} - \frac{\mu_{20}}{\mu_{00}^2}(\mu_{00}B_9 + 6B_2^2),$$

$$B_{18} = \mu_{60} - \mu_{06} - \frac{15}{\mu_{00}}(\mu_{40}B_2 + \mu_{20}B_9),$$

$$B_{19} = \mu_{15} - \frac{5(\mu_{40}\mu_{11} + 2\mu_{20}B_7)}{\mu_{00}},$$

$$B_{20} = \mu_{51} - \frac{5(\mu_{40}\mu_{11} + 2\mu_{20}B_8)}{\mu_{00}}.$$

- 7th order.

$$B_{21} = \mu_{07} - \frac{35\mu_{03}\mu_{40}}{\mu_{00}} - \frac{21\mu_{20}B_{14}}{\mu_{00}},$$

$$B_{22} = \mu_{16} - \frac{15\mu_{12}\mu_{40}}{\mu_{00}} - \frac{15\mu_{20}B_{13}}{\mu_{00}},$$

$$B_{23} = \mu_{25} - \frac{5\mu_{21}\mu_{40}}{\mu_{00}} - \frac{10\mu_{03}\mu_{22}}{\mu_{00}} - \frac{\mu_{20}P_1}{\mu_{00}},$$

where

$$P_1 = \mu_{05} + 10\mu_{23} - \frac{30\mu_{21}\mu_{20}}{\mu_{00}} - \frac{10\mu_{03}(\mu_{20} + \mu_{02})}{\mu_{00}},$$

$$B_{24} = \mu_{34} - \frac{\mu_{30}\mu_{40}}{\mu_{00}} - \frac{18\mu_{12}\mu_{22}}{\mu_{00}} - \frac{3\mu_{20}P_2}{\mu_{00}},$$

where

$$P_2 = \mu_{14} + 2\mu_{32} - \frac{2\mu_{30}\mu_{20}}{\mu_{00}} - \frac{6\mu_{12}(\mu_{20} + \mu_{02})}{\mu_{00}},$$

$$B_{25} = \mu_{70} - \frac{35\mu_{30}\mu_{40}}{\mu_{00}} - \frac{21\mu_{20}B_{15}}{\mu_{00}},$$

$$B_{26} = \mu_{61} - \frac{15\mu_{21}\mu_{40}}{\mu_{00}} - \frac{15\mu_{20}B_{12}}{\mu_{00}},$$

$$B_{27} = \mu_{52} - \frac{5\mu_{12}\mu_{40}}{\mu_{00}} - \frac{10\mu_{30}\mu_{22}}{\mu_{00}} - \frac{\mu_{20}P_3}{\mu_{00}},$$

where

$$P_3 = \mu_{50} + 10\mu_{32} - \frac{30\mu_{12}\mu_{20}}{\mu_{00}} - \frac{10\mu_{30}(\mu_{20} + \mu_{02})}{\mu_{00}},$$

$$B_{28} = \mu_{43} - \frac{\mu_{03}\mu_{40}}{\mu_{00}} - \frac{18\mu_{21}\mu_{22}}{\mu_{00}} - \frac{3\mu_{20}P_4}{\mu_{00}},$$

where

$$P_4 = \mu_{41} + 2\mu_{23} - \frac{2\mu_{03}\mu_{20}}{\mu_{00}} - \frac{6\mu_{21}(\mu_{20} + \mu_{02})}{\mu_{00}}.$$

3.4. *Efficiency of the invariants*

The computing complexity of the blur invariants is determined by the computing complexity of the central moments μ_{pq}. However, in the case of digital image f_{ij} of the size $N \times N$ we have to use a discrete version of equation (3) for moment evaluation:

$$\mu_{pq} = \sum_{i=1}^{N} \sum_{j=1}^{N} (i - x_t)^p (j - y_t)^q f_{ij}. \qquad (5)$$

Direct evaluation of μ_{pq} requires $O(N^2)$ operations. Although several methods for fast moment computing have been published recently [see[14] for a survey], they are not applicable in the case of gray-level images.

It is well-known that the central moments μ_{pq} contain redundant information. This is caused by the fact that the basis $x^p y^q$ is not orthogonal. To overcome this, it would be possible to use any orthogonal basis for moment definition and to derive blur invariants using orthogonal moments (Zernike moments for instance).

3.5. *Uniqueness of the invariants*

By uniqueness of the invariants we mean the ability to distinguish (in the feature space) among "similar" images. The experimental results have shown the sufficiently high discriminability of the portrait photographs by means of the symmetric blur invariants. However, from the theoretical point of view, the set of invariants $\mathscr{B}_r$ is an incomplete feature system for every r. That means there may exist two different images $f_1(x, y)$ and $f_2(x, y)$ such that

$$B_j^{r(f_1)} = B_j^{r(f_2)} \quad j = 0, 1, 2, \ldots, N_r.$$

Fig. 1. Top row: (a) Lena original, (b) out-of-focus blur (PSF radius = 7 pixels), (c) out-of-focus blur (PSF radius = 15 pixels); middle row: atmospheric turbulence blur with additive Gaussian random noise, (d) STD = 0, (e) STD = 10, (f) STD = 30; bottom row: (g) averaging by the mask with a hole in the center, (h) averaging by the mask with negative values on the boundary, (i) Lisa original.

4. NUMERICAL EXPERIMENTS

In order to demonstrate the performance of the proposed blurred image features, a number of experiments were carried out. A major goal of the experiment described below was to prove the invariance of the features with respect to various degradation systems.

In this experiment, an image was successively corrupted by various types of symmetric blur (see Fig. 1). In the top row of Fig. 1 one can see the original Lena image of size 256 × 256 pixels, a slightly defocused image (support of the PSF had a radius 7 pixels) and a heavily defocused image (15 pixel-radius of the PSF support). In the middle row the Lena image is blurred by simulated atmospheric turbulence (the PSF was a Gaussian with 6 pixel-radius support) and corrupted by additive zero-mean Gaussian noise with standard deviation 0, 10 and 30, respectively. In the bottom row one can see the Lena image degraded by two special filters: averaging over a 33 × 33 neighbourhood with a 29 × 29 hole in the center and averaging over a 21 × 21 neighbourhood with negative values on the boundary. The last image—portrait of Lisa—was incorporated into the test set to show the discriminability of the invariants.

The invariants $B'_1, B'_2, \ldots, B'_{28}$ were calculated for each image [equation (5) was used for moment evaluation]. Their values are summarized in Table 1. The distances of each image from the original Lena image in 28D Euclidean space $(\mathscr{B}_7, \varrho)$ are shown in the last row of Table 1. It is clear from Table 1 that $B'_1, B'_2, \ldots, B'_{28}$ are invariant with respect to PSF, sufficiently stable under additive noise and object discriminatory. Stability of the invariants is discussed in detail in the following Section.

5. STABILITY OF THE BLUR INVARIANTS

Features which we want to use for image description should have the property of stability. Generally speaking, if two images are similar in some manner (in l_2 norm for instance), then their feature values should be similar too, and vice versa.

In this Section, we will discuss how the blur invariants B_i satisfy this criterion.

There are three different kinds of stability to be investigated in the case of blur invariants: stability under additive random noise, stability with respect to boundary effect and stability with respect to PSF distortions.

5.1. *Stability under additive random noise*

So far we have considered the noise-free model (1) only. Now let us consider an imaging model with

Table 1. The values of the invariants of the images in Fig. 1 and the distances of the images from the original Lena image (last row).

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
B'_1	0.0396	0.0396	0.0396	0.0396	0.0399	0.0393	0.0396	0.0396	−0.1187
B'_2	−0.0018	−0.0018	−0.0018	−0.0018	−0.0019	−0.0016	−0.0018	−0.0018	0.2366
B'_3	−0.0171	−0.0171	−0.0171	−0.0171	−0.0169	−0.0168	−0.0171	−0.0171	0.0291
B'_4	0.0162	0.0162	0.0162	0.0162	0.0160	0.0170	0.0162	0.0162	−0.0582
B'_5	0.0347	0.0347	0.0347	0.0347	0.0350	0.0352	0.0348	0.0347	0.0080
B'_6	−0.0835	−0.0835	−0.0835	−0.0835	−0.0834	−0.0845	−0.0835	−0.0835	0.1344
B'_7	−0.0363	−0.0363	−0.0362	−0.0363	−0.0366	−0.0356	−0.0363	−0.0363	0.1443
B'_8	−0.0355	−0.0355	−0.0355	−0.0355	−0.0356	−0.0352	−0.0355	−0.0355	0.1115
B'_9	0.0275	0.0275	0.0275	0.0275	0.0276	0.0276	0.0275	0.0275	−0.6116
B'_{10}	−0.0047	−0.0047	−0.0047	−0.0047	−0.0048	−0.0046	−0.0047	−0.0047	−0.0797
B'_{11}	−0.0249	−0.0249	−0.0249	−0.0249	−0.0247	−0.0255	−0.0249	−0.0249	0.0647
B'_{12}	−0.0353	−0.0353	−0.0353	−0.0353	−0.0349	−0.0368	−0.0353	−0.0353	0.1824
B'_{13}	0.0315	0.0315	0.0315	0.0315	0.0310	0.0311	0.0315	0.0315	−0.0658
B'_{14}	−0.1125	−0.1126	−0.1125	−0.1125	−0.1132	−0.1138	−0.1126	−0.1125	−0.0404
B'_{15}	0.2713	0.2713	0.2714	0.2713	0.2708	0.2741	0.2713	0.2714	−0.6950
B'_{16}	0.0283	0.0283	0.0283	0.0283	0.0285	0.0280	0.0283	0.0283	−0.1203
B'_{17}	0.0008	0.0008	0.0008	0.0008	0.0008	0.0006	0.0008	0.0008	−0.0144
B'_{18}	−0.1580	−0.1578	−0.1581	−0.1579	−0.1582	−0.1583	−0.1580	−0.1579	4.1227
B'_{19}	0.1089	0.1090	0.1089	0.1090	0.1101	0.1068	0.1090	0.1090	−0.6496
B'_{20}	0.1113	0.1113	0.1113	0.1113	0.1114	0.1102	0.1113	0.1113	−0.4703
B'_{21}	0.7357	0.7359	0.7356	0.7358	0.7395	0.7425	0.7361	0.7355	0.3932
B'_{22}	−0.1438	−0.1438	−0.1439	−0.1437	−0.1414	−0.1418	−0.1438	−0.1438	0.4256
B'_{23}	0.0852	0.0852	0.0852	0.0853	0.0845	0.0870	0.0852	0.0852	−0.2888
B'_{24}	0.0073	0.0073	0.0073	0.0073	0.0075	0.0072	0.0073	0.0073	0.1926
B'_{25}	−1.7723	−1.7721	−1.7726	−1.7723	−1.7676	−1.7860	−1.7722	−1.7727	6.8090
B'_{26}	0.1639	0.1638	0.1638	0.1639	0.1620	0.1701	0.1639	0.1639	−1.2782
B'_{27}	0.0174	0.0174	0.0174	0.0174	0.0175	0.0173	0.0174	0.0174	0.3733
B'_{28}	0.0566	0.0566	0.0566	0.0567	0.0562	0.0575	0.0566	0.0566	−0.1916
$\varrho[10^{-4}]$	-	3.7	3.4	1.8	71.0	173.1	4.2	4.3	98688

additive zero-mean random noise $n(x, y)$:

$$g(x, y) = (f * h)(x, y) + n(x, y). \tag{6}$$

Since the image $g(x, y)$ is then a random field, all its moments and all invariants can be viewed as random variables. It holds

$$E(\mu_{pq}^{(n)}) = E\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q n(x, y)\, dx\, dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q E(n(x, y))\, dx\, dy = 0$$

and

$$E(\mu_{pq}^{(g)}) = E(\mu_{pq}^{(f*h)}) + E(\mu_{pq}^{(n)}) = \mu_{pq}^{(f*h)},$$

where $E(X)$ denotes the mean value of the random variable X.

In practice, however, only a single image $g(x, y)$ (i.e. only one realization of a random field) is available in most cases. We obtain $\mu_{pq}^{(g)}$ but we are not able to estimate the mean values $E(\mu_{pq}^{(g)})$. Since the moments are computed by summation over the whole image, they should be affected by additive noise very little. This means that the moments $\mu_{pq}^{(g)}$ are supposed to be close to $E(\mu_{pq}^{(g)})$ and we can use $\mu_{pq}^{(g)}$ directly for the computation of the invariants B_i.

The accuracy of such a description is illustrated by the following experiment. Denote an image corrupted by additive Gaussian zero-mean noise $n(x, y)$ with standard deviation σ as $g_\sigma(x, y)$. The stability of the invariants is characterized by the distance $\varrho(f, g_\sigma)$ as a function of σ in $\mathscr{B}_r$ space. The results in the case of Lena image, σ values from 5 to 50 and $r = 7$ are shown in Fig. 2 (the values of ϱ displayed on the vertical axis are

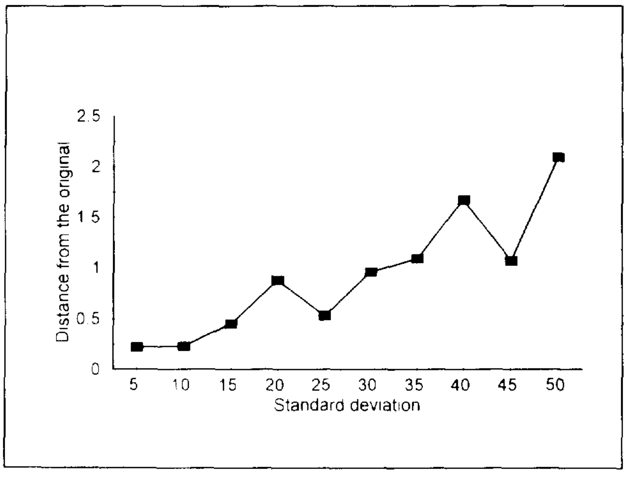

Fig. 2. Stability of the invariants with respect to additive Gaussian zero-mean random noise. Horizontal axis: standard deviation of noise; vertical axis: the distance (multiplied by 100) between the corrupted and original images in $\mathscr{B}_r$ space.

Fig. 3. Left: the original image; right: the image corrupted by additive Gaussian zero-mean random noise, STD = 50.

in units of 10^{-2}). One can see that the stability of the invariants is sufficiently high: even for the most corrupted image (which is very difficult to recognize by human vision, see Fig. 3) the distance from the original is about 0.02, whereas the distance between Lena and the other portrait image is usually higher than 1.

5.2. *Stability with respect to boundary effect*

We now consider a discrete form of equation (1). Provided that the size of the original image is $N \times N$ pixels and the size of the PSF support is $M \times M$ pixels, the correct size of the acquired image should be $(N + M - 1) \times (N + M - 1)$ pixels. However, in practice the value of M is unknown and the original and acquired images are assumed to have the same size. If $M \ll N$, the errors of invariant calculation caused by boundary effect are negligible. If M is relatively large (in case of heavy blur), the boundary effect might lead to more significant errors of the values of invariants. Stability of the invariants with respect to boundary effect is investigated in the following experiment, which is very similar to that described in Section 5.1.

Two portrait images of size 256 × 256 pixels were successively blurred by convolving with a square mask of size $M \times M$ with constant coefficients. The convolution was calculated by means of mirror extension on the image boundaries. The blurred image was cut off so that its size is the same as the size of the size of the original, independent of M. The distance between the original and blurred images in $\mathscr{B}_7$ space as a function of the mask size M is shown in Fig. 4 for the Lena as well as for the Eve images. Note that the distances between the most blurred images and the originals (see Fig. 5) are only 0.04 and 0.14, respectively, whereas the distance between the originals of Lena and Eve is about 4.

The results of this experiment show that the blur invariants are sufficiently stable under boundary effect.

5.3. *Stability under PSF distortions*

Stability of the invariants under two special cases of the PSF distortion should be investigated thoroughly in the near future:

- $h(x, y)$ does not satisfy the condition of symmetry exactly;
- $h(x, y)$ itself is random, i.e. it can be modeled as

$$h(x, y) = h_1(x, y) + n_1(x, y),$$

where $h_1(x, y)$ is the deterministic part of the PSF and $n_1(x, y)$ is zero-mean random noise. This model is usually used for description of a randomly vibrating imaging system.[13]

Although we have not carried out a representative experiment on this topic, stability under PSF distortions should be probably a little bit lower than in the two previous cases.

6. SUMMARY

The paper was devoted to the construction of image features invariant with respect to blur. The images were supposed to be formed by a linear shift-invariant imaging system, where the blur can be modeled by convolving an original image with a system point spread function.

A set of blur invariants based on image moments was introduced in this paper. The derivation of the invariants is a major theoretical result of the paper.

Invariance of the features as well as their ability to distinguish among different images was demonstrated experimentally. Stability of the invariants with respect to additive random noise and boundary effect was also discussed and was shown to be sufficiently high.

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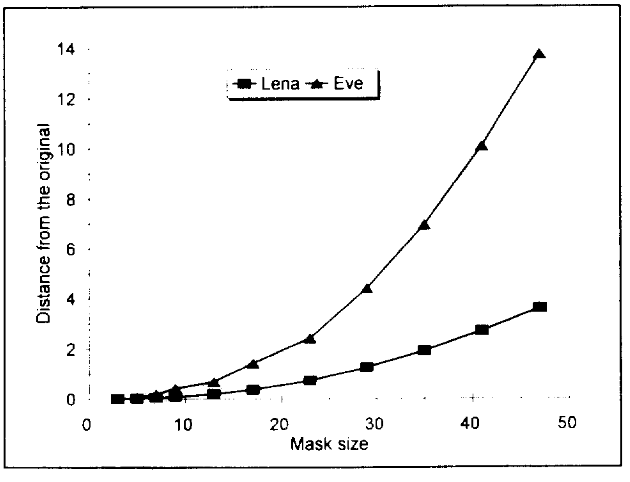

Fig. 4. Stability of the invariants with respect to boundary effect. Horizontal axis: the mask size; vertical axis: the distance (multiplied by 100) between blurred and original images in $\mathscr{B}_7$ space.

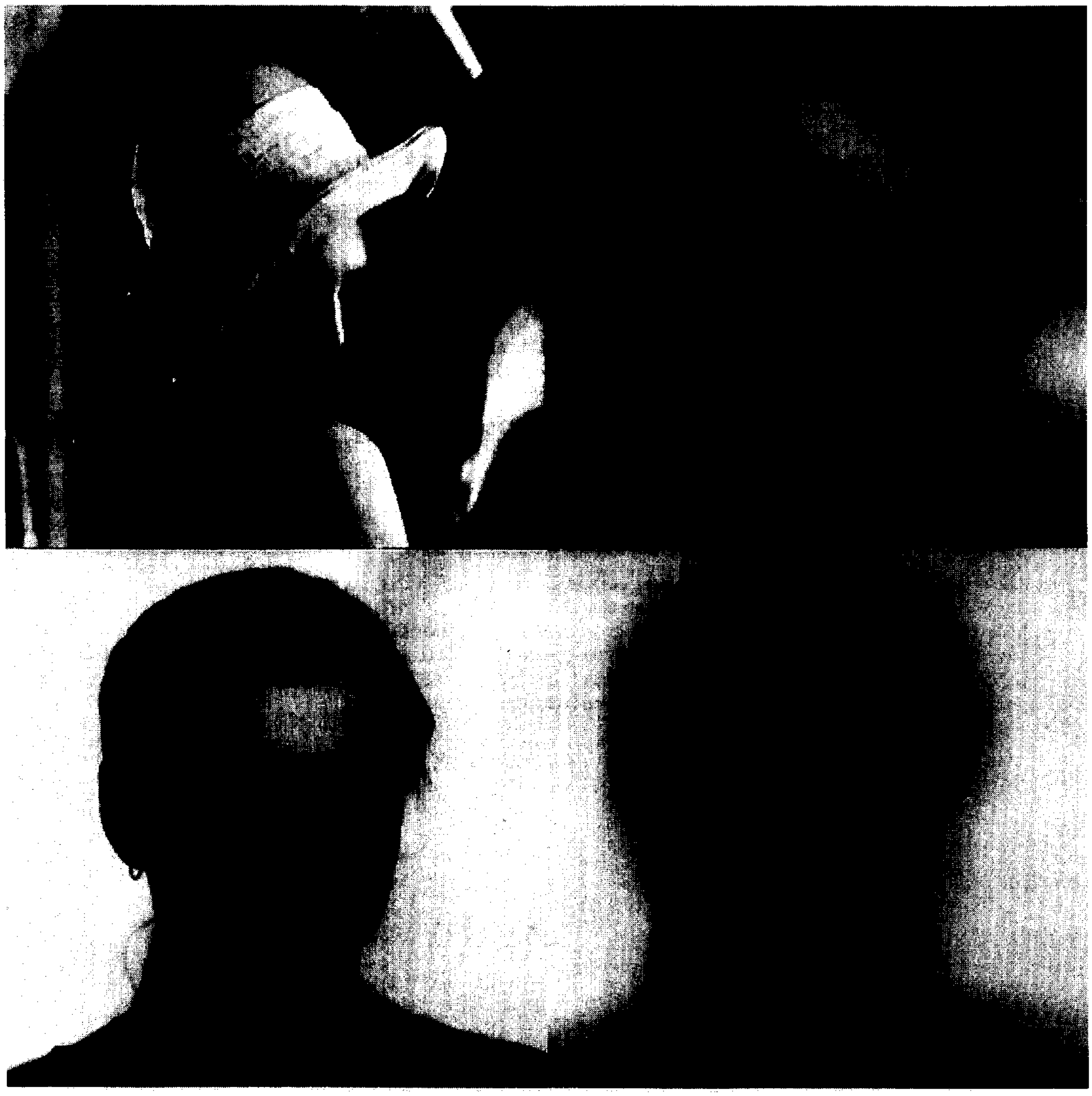

Fig. 5. Left: original images; right: blurred images (the mask size was $M = 47$).

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