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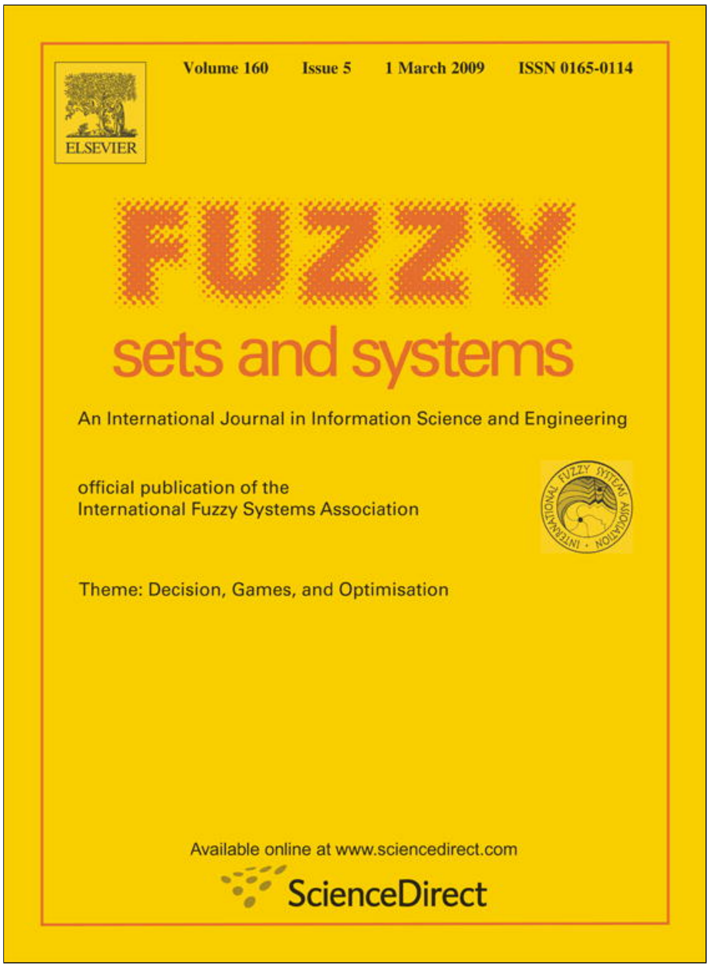

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Fuzzy Sets and Systems 160 (2009) 635–643
FUZZY sets and systems
www.elsevier.com/locate/fss

Enlarged cores and bargaining schemes in games with fuzzy coalitions

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Received 11 January 2008; received in revised form 11 April 2008; accepted 3 June 2008
Available online 6 June 2008

Abstract

In this paper we introduce a new concept of solution for games with fuzzy coalitions, which we call an enlarged core. The enlarged core captures an idea that various groups of fuzzy coalitions can have different bargaining power or influence on the final distribution of wealth resulting from the cooperation process. We study a bargaining scheme for the enlarged core, which is an iterative procedure for generating sequences converging to elements of the enlarged core. It is shown that the enlarged core coincides with Aubin's core for a specific class of games with fuzzy coalitions.
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Keywords: Games with fuzzy coalitions; Core; Bargaining schemes

1. Introduction

Let $N = \{1, \ldots, n\}$ be a set of elements that are called *players*. By 2^N we denote the set of all subsets of N whose elements are called *crisp coalitions*. A vector $a = (a_1, \ldots, a_n) \in [0, 1]^n$ is called a *fuzzy coalition*. The set of all fuzzy coalitions forms the n-dimensional unit cube $[0, 1]^n$. Each crisp coalition is identified with the fuzzy coalition a whose coordinates $a_i \in \{0, 1\}$ for $i \in N$ determine whether the player i belongs (or not) to the crisp coalition a. We put $0 = (0, \ldots, 0)$ and $1 = (1, \ldots, 1)$. The *Łukasiewicz* t-norm $\odot$ and the *Łukasiewicz* t-conorm $\oplus$ are the binary operations on $[0, 1]$ defined by $\alpha \odot \beta = \max(\alpha + \beta - 1, 0)$ and $\alpha \oplus \beta = \min(\alpha + \beta, 1)$ for every $\alpha, \beta \in [0, 1]$, respectively. The Łukasiewicz operations are natural generalizations of the intersection and the union of coalitions [1,6]. Precisely, when a and b are fuzzy coalitions, Łukasiewicz t-norm and Łukasiewicz t-conorm are applied coordinatewise: $a \odot b = (a_1 \odot b_1, \ldots, a_n \odot b_n)$ and $a \oplus b = (a_1 \oplus b_1, \ldots, a_n \oplus b_n)$, respectively. In this way the fuzzy coalitions $a \odot b$ and $a \oplus b$ can be viewed as the intersection and the union of a and b, respectively.

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doi:10.1016/j.fss.2008.06.001

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Definition 1. A (*cooperative*) *game with fuzzy coalitions* is a function $v : [0, 1]^n \to \mathbb{R}$ with $v(0) = 0$. A game with fuzzy coalitions v is called *superadditive* when the following implication holds true:

$$\text{if } a, b \in [0, 1]^n \quad \text{and} \quad a \odot b = 0 \quad \text{then } v(a \oplus b) \geqslant v(a) + v(b).$$

A function $v : 2^N \to \mathbb{R}$ with $v(0) = 0$ is called a *game with crisp coalitions only*. A game with fuzzy coalitions v can be associated with the game v_0 with crisp coalitions only, where v_0 is the restriction of the function v to 2^N. Note that $a \odot b = 0$ is a "disjointness condition'' (see [6, Section 4]), which is equivalent to $a \oplus b = a + b$. Therefore, the notion of superadditivity introduced in Definition 1 is a natural generalization of the classical superadditivity notion. Aubin defined in [1,2] the concept of core for games with fuzzy coalitions which he called "fuzzy games'' (jeux floux).

Definition 2. Let v be a game with fuzzy coalitions. The *core* of v is the (possibly empty) set

$$\mathbf{C}(v) = \{x \in \mathbb{R}^n | \langle 1, x\rangle = v(1) \text{ and } \langle a, x\rangle \geqslant v(a) \text{ for every } a \in [0, 1]^n \setminus \{1\}\}.$$

If $\mathbf{C}(v) \neq \emptyset$, then v is said to be *balanced*.

This definition generalizes the classical concept of core for superadditive games with crisp coalitions only, which is exactly the set of undominated imputations—see [9]. An axiomatic characterization of the core for games with fuzzy coalitions was given by Hwang in [10].

For every $a \in [0, 1]^n$, put

$$C_a(v) = \begin{cases} \{x \in \mathbb{R}^n | \langle a, x\rangle \geqslant v(a)\} & \text{if } a \in [0, 1]^n \setminus \{1\}, \\ \{x \in \mathbb{R}^n | \langle 1, x\rangle = v(1)\} & \text{if } a = 1. \end{cases}$$

Obviously, $C_0(v) = \mathbb{R}^n$ and

$$\mathbf{C}(v) = \bigcap_{a\in[0,1]^n} C_a(v).$$

In general, a game with fuzzy coalitions can be unbalanced as the example below shows.

Example 1. Let $N = \{1, 2\}$. Define

$$u(a_1, a_2) = \begin{cases} 0 & a_1 \odot a_2 = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and observe that the game with fuzzy coalitions u is superadditive. This case captures an economic situation in which two players have no incentive to form a fuzzy coalition a unless $a_1 \odot a_2 > 0$, and every fuzzy coalition satisfying this condition generates precisely the worth $u(1) = 1$ of the "grand'' coalition 1. The game with fuzzy coalitions u is unbalanced. Indeed, the hyperplane $C_1(u)$ misses every halfspace $C_b(u)$ with $b = (b_1, b_2)$ such that $b_1 \in (\frac{1}{2}, 1)$ and $b_2 = b_1$. In other words, while the worth of the "grand'' coalition 1 is generated with a full participation of the two players, the same worth can be equivalently produced only by every fuzzy coalition b. The "smaller'' coalition b has thus no motivation to accept the identical worth of the "grand'' coalition 1.

When the game with fuzzy coalitions v is balanced, then each vector $x \in \mathbf{C}(v)$ describes a "rational'' way of distributing the rewards of cooperation among the players: the vector x is efficient (that is, $\sum_{i=1}^n x_i = v(1)$) and each fuzzy coalition $a \in [0, 1]^n$ distributes to the players at least its cooperative profit $v(a)$ by taking into account their individual membership degrees (that is, $\sum_{i=1}^n a_i x_i \geqslant v(a)$). An example of a balanced game with fuzzy coalitions follows.

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Example 2. Given a nonempty set I, take any family of functions $(f_i)_{i\in I}$, where $f_i : \mathbb{R}^n \to \mathbb{R}$ is a concave[1] and positively homogeneous[2] for every $i \in I$. Put

$$v(a) = \inf\{f_i(a) | i \in I\} \quad \text{for every } a \in \mathbb{R}^n,$$

and observe that the function v is also concave and positively homogeneous on $\mathbb{R}^n$. The function v restricted to $[0, 1]^n$ is a superadditive game with fuzzy coalitions. Indeed, for any pair $a, b \in [0, 1]^n$ with $a \odot b = 0$, we have

$$v(a \oplus b) = v(\tfrac{1}{2}(2a) + \tfrac{1}{2}(2b)) \geqslant \tfrac{1}{2}v(2a) + \tfrac{1}{2}v(2b) = v(a) + v(b),$$

where the last inequality results from concavity of v and the last equality from positive homogeneity of v. It follows from Remark 1 in [2] that the core $\mathbf{C}(v)$ is nonempty and coincides with the (necessarily nonempty) superdifferential

$$\partial v(1) = \{x \in \mathbb{R}^n | \langle x, 1 - c\rangle \leqslant v(1) - v(c) \text{ for every } c \in \mathbb{R}^n\}$$

of the function v at 1.

In general, deciding balancedness of a game with fuzzy coalitions v is difficult because this problem is equivalent to that of the existence of solutions for a system of infinitely many affine inequalities. However, in some circumstances, one can reduce the problem of deciding balancedness of a game with fuzzy coalitions v to deciding the balancedness problem of a game with crisp coalitions only. This aspect was studied by Tijs et al. in [5]. Theorem 7 from [5] says that when a game with fuzzy coalitions v is *supermodular*, that is, if

$$v(\max(a, b)) + v(\min(a, b)) \geqslant v(a) + v(b) \quad \text{for every } a, b \in [0, 1]^n,$$

and when for every $i \in N$ and every $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in [0, 1]^{n-1}$,

$$\text{the function } a_i \in [0, 1] \mapsto v(a_1, \dots, a_i, \dots, a_n) \text{ is convex,} \tag{1}$$

then $\mathbf{C}(v)$ and the set

$$\mathbf{C}(v_0) = \{x \in \mathbb{R}^n | \langle 1, x\rangle = v(1) \text{ and } \langle a, x\rangle \geqslant v(a), \text{ for every } a \in 2^N \setminus \{1\}\}$$

coincide. Hence deciding balancedness of v amounts to deciding balancedness of v_0 under the above conditions. Note that even when $\mathbf{C}(v)$ equals $\mathbf{C}(v_0)$, balancedness of v cannot be easily decided. In practical cases the set 2^N can contain a huge number of crisp coalitions even for relatively small numbers n. For example, deciding balancedness of v with $n = 20$ via balancedness of v_0 amounts to solving the large linear programming problem involving 2^{20} constraints (that is, much more than one million of constraints).

Balanced games with fuzzy coalitions were completely characterized by Azrieli and Lehrer in Theorem 1 from [3]: a game with fuzzy coalitions v is balanced if and only if

$$v(1) = \sup\left\{\sum_{i=1}^{m} \lambda_i v(a^i) | m \in \mathbb{N}, 1 = \sum_{i=1}^{m} \lambda_i a^i, \lambda_j \geqslant 0, a^j \in [0, 1]^n, j \leqslant m\right\}.$$

In this paper we introduce a new concept of solution for games with fuzzy coalitions: the enlarged core. The core of v is the set of common points of all the sets $C_a(v)$ with $a \in [0, 1]^n$. The enlarged core of v is the set of vectors x in $\mathbb{R}^n$, which belong to all but "negligibly many'' sets $C_a(v)$ in the sense that the collection of fuzzy coalitions a such that $x \notin C_a(v)$ is "negligible''. What "negligible'' means depends on the way in which one assesses the relative importance or power of various sets of fuzzy coalitions in the context of the game (see Section 2). The enlarged core contains, but does not necessarily equals, the core of the game. We show (see Theorem 2 below) that under quite mild conditions concerning the game with fuzzy coalitions v, there are "bargaining procedures'' producing sequences of vectors whose convergence behavior is indicative for the nonemptiness of the enlarged core of v. For games with fuzzy coalitions whose core and enlarged core coincide (like those described by Theorem 1) these procedures are tools for observing balancedness of the game and for approximating elements of the core provided that such vectors exist.

[1] A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *concave* if, for every $x, y \in \mathbb{R}^n$ and every $\alpha \in (0, 1)$, we have $f(\alpha x + (1-\alpha)y) \geqslant \alpha f(x) + (1-\alpha) f(y)$.
[2] A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *positively homogeneous* when $f(\alpha x) = \alpha f(x)$ for every $\alpha > 0$ and every $x \in \mathbb{R}^n$.

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2. Enlarged core

A *coalitional assessment* of a game with fuzzy coalitions v is a complete probability measure μ defined on the σ-algebra $\mathfrak{A}$ of Lebesgue measurable subsets of $[0,1]^n$. When v is provided with a coalitional assessment μ, a set of fuzzy coalitions $A \in \mathfrak{A}$ is assigned a certain probability $\mu(A)$ of simultaneous formation in the game with fuzzy coalitions v. The number $\mu(A)$ can also be thought of as a relative assessment of the collective power or influence of the fuzzy coalitions in A on the outcome of the game with fuzzy coalitions v.

For example, if there is no reason to prefer any measurable subsets of fuzzy coalitions against others in a game with fuzzy coalitions v, then the coalitional assessment of v can be the Lebesgue measure λ on $[0,1]^n$. On the other hand, the coalitional assessment can also model the situation in which the collective power of fuzzy coalitions in A reflects the worth of every fuzzy coalition in A. This idea is illustrated by the case of a nonnegative game with fuzzy coalitions v that is also Lebesgue measurable and

$$0 < \int_{[0,1]^n} v \, \mathrm{d}\lambda < \infty.$$

It is clear that the mapping defined by

$$\nu(A) = \frac{1}{\int_{[0,1]^n} v \, \mathrm{d}\lambda} \int_A v \, \mathrm{d}\lambda, \quad A \in \mathfrak{A},$$

is a coalitional assessment because it is a probability measure on $\mathfrak{A}$, which is also absolutely continuous with respect to the Lebesgue measure λ. It is obvious that $\nu(A)$ represents the fraction of the total worth that the set of fuzzy coalitions A controls in the game v.

Any set of fuzzy coalitions $A \in \mathfrak{A}$ with $\mu(A) = 0$ is viewed as "negligible'' in the sense that the conditions $\langle a, x\rangle \geqslant v(a)$ can be discarded for each $a \in A$ in deciding how the outcomes of the cooperative process described by the game with fuzzy coalitions v are distributed. In other words, a collection of fuzzy coalitions $A \in \mathfrak{A}$ with $\mu(A) = 0$ does not have either the power or the influence to rise "objections'', which will be further considered by the players in the decision process (either because the players do not expect that the fuzzy coalitions in A will be effectively constituted or because the collective influence of the fuzzy coalitions in A is null). Balancedness of the game with fuzzy coalitions v depends on satisfying the conditions

$$\langle 1, x\rangle = v(1), \quad \langle a, x\rangle \geqslant v(a) \tag{2}$$

for every fuzzy coalition $a \in [0,1]^n \setminus \{1\}$. Balancing a game with fuzzy coalitions (that is, finding a satisfactory distribution x of the outcomes of the cooperative process) may sometimes be blocked by mutually incompatible demands of some negligible families of fuzzy coalitions. Under these circumstances, neglecting the demands of such families can lead to a "more balanced'' outcome although it is not necessarily an element of the core. Denote

$$\mathbf{C}_\mu(v) = \bigcup_{\substack{A \in \mathfrak{A} | \\ \mu(A)=0}} \bigcap_{a \in [0,1]^n \setminus A} C_a(v), \tag{3}$$

the set of "imputations'' $x \in \mathbb{R}^n$ satisfying all but a negligible set of the conditions (2). Clearly, we have

$$\mathbf{C}(v) \subseteq \mathbf{C}_\mu(v). \tag{4}$$

We call the set $\mathbf{C}_\mu(v)$ the *enlarged core* (with respect to the coalitional assessment μ) of the game with fuzzy coalitions v.

A question of interest in that follows is whether the enlarged core $\mathbf{C}_\mu(v)$ coincides with the set of *μ-almost common points of the sets $C_a(v)$* defined in [7]. This question amounts to verifying whether the equality

$$\mathbf{C}_\mu(v) = \{x \in \mathbb{R}^n | \mu(\{a \in [0,1]^n | x \in C_a(v)\}) = 1\} \tag{5}$$

holds true. In general, given $x \in \mathbb{R}^n$, the set

$$A_x := \{a \in [0,1]^n | x \in C_a(v)\} \tag{6}$$

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may even fail to be measurable (that is, it may not belong to $\mathfrak{A}$), which makes the question above meaningless. The next result shows that measurability of the sets A_x can be ensured and equality (5) holds under reasonably general conditions.

Lemma 1. *Let v be a game with fuzzy coalitions endowed with a coalitional assessment μ. If the function v is Lebesgue measurable, then the set A_x is Lebesgue measurable for every $x \in \mathbb{R}^n$, and the equality (5) holds.*

Proof. Let $\Gamma : [0,1]^n \to 2^{\mathbb{R}^n}$ be the point-to-set mapping defined by $\Gamma(a) = C_a(v)$. The claim that $A_x \in \mathfrak{A}$ is proved if we show that Γ is measurable as a point-to-set mapping, that is, if we show that for any closed set $B \subseteq \mathbb{R}^n$, we have

$$\Gamma^-(B) := \{a \in [0,1]^n | \Gamma(a) \cap B \neq \emptyset\} \in \mathfrak{A}. \tag{7}$$

This is true because A_x is precisely $\Gamma^-(\{x\})$ for every $x \in \mathbb{R}^n$. According to [8, Theorem III.30], in order to prove (7), it suffices to show that the function $\operatorname{dist}(x, \Gamma(\cdot)) : [0,1]^n \to \mathbb{R}$ is $\mathfrak{A}$-measurable (as a real function) for every $x \in \mathbb{R}^n$. Note that

$$\operatorname{dist}(x, \Gamma(a)) = \|x - P_a(x)\| \quad \text{for every } a \in [0,1]^n,$$

where $P_a(x)$ is the metric projection of x onto $C_a(v)$, which is given by

$$P_a x = \begin{cases} x + \dfrac{\max\{0, v(a) - \langle a, x\rangle\}}{\|a\|^2} a & \text{if } a \in [0,1]^n \setminus \{0,1\}, \\ x + \dfrac{v(1) - \langle 1, x\rangle}{n} 1 & \text{if } a = 1, \\ x & \text{if } a = 0. \end{cases}$$

By consequence,

$$\operatorname{dist}(x, \Gamma(a)) = \begin{cases} \dfrac{1}{\|a\|} \max\{0, v(a) - \langle a, x\rangle\} & \text{if } a \in [0,1]^n \setminus \{0,1\}, \\ \dfrac{1}{\sqrt{n}} |v(1) - \langle 1, x\rangle| & \text{if } a = 1, \\ 0 & \text{if } a = 0. \end{cases}$$

The last formula implies that the function $\operatorname{dist}(x, \Gamma(\cdot))$ is $\mathfrak{A}$-measurable due to the measurability of v. Hence $A_x = \Gamma^-(\{x\}) \in \mathfrak{A}$, whenever $x \in \mathbb{R}^n$.

It remains to verify the equality (5). Suppose that $x \in \mathbb{R}^n$ is such that $\mu(A_x) = 1$. Then the set $A := [0,1]^n \setminus A_x$ satisfies $\mu(A) = 0$ and $x \in \bigcap_{a \in [0,1]^n \setminus A} C_a(v)$. Hence $x \in \mathbf{C}_\mu(v)$. Conversely, if $x \in \mathbf{C}_\mu(v)$, then there exists some $A \in \mathfrak{A}$ with $\mu(A) = 0$ and $x \in \bigcap_{a \in [0,1]^n \setminus A} C_a(v)$. Since A_x includes $[0,1]^n \setminus A$, we get

$$\mu(A_x) \geqslant \mu([0,1]^n \setminus A) = \mu([0,1]^n) - \mu(A) = \mu([0,1]^n) = 1,$$

and (5) thus holds true. □

Due to (4), the enlarged core $\mathbf{C}_\mu(v)$ includes the core $\mathbf{C}(v)$. The next example shows that the two sets need not coincide, and that the enlarged core can be in fact much larger than the core.

Example 3. Consider a game with fuzzy coalitions $w : [0,1]^2 \to \mathbb{R}$ given by $w(a_1, a_2) = a_1 \odot a_2$. It can be straightforwardly verified that w is superadditive, supermodular, and satisfies (1). Thus Theorem 7 in [5] by Tijs et al. yields

$$\mathbf{C}(w) = C_1(w) \cap C_{(1,0)}(w) \cap C_{(0,1)}(w) = \{x \in [0,1]^2 | x_1 + x_2 = 1\}. \tag{8}$$

Assume that the coalitional assessment in w is the Lebesgue measure λ. We claim that the enlarged core $\mathbf{C}_\lambda(w)$ contains every payoff $x' = (x'_1, x'_2)$ such that $x'_1, x'_2 \geqslant 0$. Indeed, since $\lambda(\{1\}) = 0$, it follows from (3) that it suffices to show $(x'_1, x'_2) \in \bigcap_{a \in [0,1]^2 \setminus \{1\}} C_a(w)$. However, the first equality in (8) implies $\bigcap_{a \in [0,1]^2 \setminus \{1\}} C_a(w) = C_{(1,0)}(w) \cap C_{(0,1)}(w)$ and since $(x'_1, x'_2) \in C_{(1,0)}(w) \cap C_{(0,1)}(w)$, we have $(x'_1, x'_2) \in \mathbf{C}_\lambda(w)$ in conclusion.

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In Theorem 1 below we show that the core coincides with the enlarged core for a fairly large class of games with fuzzy coalitions and coalitional assessments.

Theorem 1. *Let v be a game with fuzzy coalitions that is a continuous real function on $[0,1]^n$ and let μ be a coalitional assessment such that the following condition holds true*:

$$\text{for every } A \in \mathfrak{A}, \quad \text{if } A \text{ is open or } 1 \in A, \quad \text{then } \mu(A) > 0. \tag{9}$$

Then $\mathbf{C}(v) = \mathbf{C}_\mu(v)$.

Proof. Let $x \in \mathbf{C}_\mu(v)$. Due to Lemma 1, there exists the nonempty set A_x defined by (6) such that $A_x \in \mathfrak{A}$ and

$$\mu([0,1]^n \setminus A_x) = 0. \tag{10}$$

Our aim is to show $A_x = [0,1]^n$ since this implies $x \in \bigcap_{a\in[0,1]^n} C_a(v) = \mathbf{C}(v)$. Assume, by contradiction, that A_x is a nonempty proper subset of $[0,1]^n$.

If $1 \in [0,1]^n \setminus A_x$, then (10) gives the contradiction with the assumption (9). Hence $1 \notin [0,1]^n \setminus A_x$. Therefore, if $a \in [0,1]^n \setminus A_x$, then $x \notin C_a(v)$, which is equivalent to $\langle a, x\rangle - v(a) < 0$ since $a \neq 1$. As the mapping $\langle \cdot, x\rangle - v(\cdot)$ is continuous at a, there exists an open neighborhood $B \subseteq [0,1]^n$ of a with $\langle b, x\rangle - v(b) < 0$ for every $b \in B$. Hence $x \notin C_b(v)$ for every $b \in B$, which implies that $B \subseteq [0,1]^n \setminus A_x$. This inclusion gives a contradiction: while $\mu(B) > 0$ due to the assumption (9), the set $[0,1]^n \setminus A_x$ has measure zero due to (10). As a conclusion, the set $[0,1]^n \setminus A_x$ must be empty and we have $A_x = [0,1]^n$. $\square$

The coalitional assessment μ satisfying (9) guarantees that neither any open set A nor any measurable set of fuzzy coalitions A containing the "grand" coalition 1 is negligible, which means that A cannot be completely disregarded during the negotiations about the distribution of the worth. There exist a host of examples of such coalitional assessments. Let μ' be a coalitional assessment such that $\mu'(A) > 0$ for every open set $A \subseteq [0,1]^n$ and $\alpha \in (0,1)$. Define

$$\mu = \alpha\mu' + (1-\alpha)\delta_1,$$

where δ_1 is the Dirac measure concentrated at the point 1, and note that μ satisfies the conditions from Theorem 1.

3. Bargaining schemes

Let v be a game with fuzzy coalitions. As noted above, deciding balancedness of v and, when the game with fuzzy coalitions v is balanced, computing elements of $\mathbf{C}(v)$, are difficult problems because they involve extremely "large" systems of affine inequalities (in general, those systems involve infinitely many inequalities). In the framework of games with crisp coalitions only, this led Wu [12] to the question whether, and under which conditions, players can "bargain" for an element of the core or of the "enlarged core" (note that Wu's notion of enlarged core is different from ours). Following Wu's idea, a *bargaining scheme* for the core (or for the enlarged core) of v is an iterative procedure which, starting from an arbitrarily chosen initial distribution of wealth $x^0 \in \mathbb{R}^n$ among the players, generates a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to a point of the core $\mathbf{C}(v)$ (or of the enlarged core $\mathbf{C}_\mu(v)$), provided that such a point exists. In this context, each vector x^{k+1} is seen as a redistribution of wealth emerging as the result of a bargaining process in which the terms of the distribution of wealth x^k are renegotiated at each step k according to specific rules. These rules are determined by the procedure generating the sequence $(x^k)_{k\in\mathbb{N}}$.

The core of a game v with crisp coalitions only is the solution set of a finite (although usually large) convex feasibility problem and, therefore, iterative projection methods for solving finite feasibility problems can be naturally interpreted as bargaining schemes for the core or for the enlarged core in the sense of Wu. The bargaining scheme analyzed by Wu in [12] can be seen as one of the many bargaining schemes resulting from the game-theoretical interpretation of projection methods for solving finite convex feasibility problems (see [4] for a survey on this topic). In contrast to the case of a game with crisp coalitions only (for which the number of crisp coalitions in 2^N is 2^n), in games with fuzzy coalitions the set of all fuzzy coalitions $[0,1]^n$ is infinite.

This rises the question whether bargaining schemes for the core (or for the enlarged core in the sense of (3)) can be found even in this setting. We will show in the sequel that this is indeed the case when the game with fuzzy coalitions v

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endowed with a coalitional assessment μ satisfies the following conditions:

(A1) The function v is Lebesgue measurable;

(A2) The function $\xi : [0,1]^n \to \mathbb{R}$ given by

$$\xi(a) = \begin{cases} \dfrac{(\max\{0, v(a)\})^2}{\|a\|^2} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0, \end{cases}$$

is μ-integrable.

Lemma 1 together with the condition (A1) guarantee that the enlarged core of v is precisely the set of μ-almost common points of the sets $C_a(v)$, $a \in [0,1]^n$. Observe that $\xi(a) = \|P_a 0\|^2$, where $P_a 0$ is the minimal norm element of the set $C_a(v)$ and it is given by

$$P_a 0 = \begin{cases} \dfrac{\max\{0, v(a)\}}{\|a\|^2} a & \text{if } a \in [0,1]^n \setminus \{0, 1\}, \\ \dfrac{v(1)}{n} 1 & \text{if } a = 1, \\ 0 & \text{if } a = 0. \end{cases}$$

The condition (A2) implies that the function $a \in [0,1]^n \mapsto P_a 0$ is a square μ-integrable selector of the family of the sets $C_a(v)$, $a \in [0,1]^n$. According to [7], the mapping $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^n$ defined by[3]

$$\mathbf{P}x = \int_{[0,1]^n} (P_a x) \, \mathrm{d}\mu(a),$$

as well as the function $\mathbf{g} : \mathbb{R}^n \to [0, \infty]$ given by

$$\mathbf{g}(x) = \frac{1}{2} \int_{[0,1]^n} \|P_a x - x\|^2 \, \mathrm{d}\mu(a),$$

are then well defined. It can be easily seen that $\mathbf{g}$ is a convex differentiable function with $\nabla \mathbf{g}(x) = \mathbf{P}x - x$, for every $x \in \mathbb{R}^n$. This implies that the set of global minimizers of $\mathbf{g}$ coincides with the set of fixed points of $\mathbf{P}$.

Definition 3. The *Cimmino type bargaining scheme* (in a game with fuzzy coalitions v) is the following rule of generating sequences $(x^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^n$:

$$x^0 \in \mathbb{R}^n \quad \text{and} \quad x^{k+1} = \mathbf{P}x^k \quad \text{for every } k = 0, 1, 2, \ldots .$$

The vector x^0 is called the *initial point* of the bargaining scheme.

It is worth noting that the initial point x^0 completely determines the sequence $(x^k)_{k \in \mathbb{N}}$ generated by the Cimmino type bargaining scheme. The question of convergence of the sequences generated by the Cimmino type bargaining scheme to the enlarged core is the subject of the theorem below. The theorem shows that, whenever $\mathbf{C}_\mu(v) \neq \emptyset$, the Cimmino type bargaining scheme generates approximations for elements in the enlarged core. Otherwise, the sequences generated by the Cimmino type bargaining scheme are unbounded.

Theorem 2. *Let $y^0 \in \mathbb{R}^n$ and let $(y^k)_{k \in \mathbb{N}}$ be the sequence generated by the Cimmino type bargaining scheme starting from the initial point y^0. The next two statements are true*:

(A) *If the sequence $(y^k)_{k \in \mathbb{N}}$ is bounded, then*
  (i) *for any $x^0 \in \mathbb{R}^n$, the sequence $(x^k)_{k \in \mathbb{N}}$ generated by the Cimmino type bargaining scheme starting from the initial point x^0 converges, its limit x^* is a (global) minimizer of the function $\mathbf{g}$, and $\lim_{k \to \infty} \mathbf{g}(x^k) = \mathbf{g}(x^*)$;*
  (ii) *the enlarged core $\mathbf{C}_\mu(v)$ is nonempty if and only if $\lim_{k \to \infty} \mathbf{g}(y^k) = 0$.*

(B) *If the sequence $(y^k)_{k \in \mathbb{N}}$ is unbounded or $\lim_{k \to \infty} \mathbf{g}(y^k) \neq 0$, then $\mathbf{C}_\mu(v)$ is empty.*

[3] The integral below of the vector-valued function $a \mapsto P_a x$ is the vector of integrals of its coordinates.

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Proof. (A)(i) It results from [7, Theorem 2.1(A)] that if the sequence $(y^k)_{k\in\mathbb{N}}$ is bounded, then all the sequences generated by the Cimmino type bargaining scheme are convergent and their limits are global minimizers of $\mathbf{g}$. Since the function $\mathbf{g}$ is also continuous (as being convex and finite on $\mathbb{R}^n$), it follows that $\lim_{k\to\infty}\mathbf{g}(x^k)=\mathbf{g}(x^*)$ whenever $(x^k)_{k\in\mathbb{N}}$ is a sequence generated by the Cimmino type bargaining scheme.

(A)(ii) Suppose that $\mathbf{C}_\mu(v)$ is nonempty. Then, by [7, Theorem 2.1(B)], since $(y^k)_{k\in\mathbb{N}}$ is bounded it is also convergent. By the continuity of $\mathbf{g}$ we have $\lim_{k\to\infty}\mathbf{g}(y^k)=\mathbf{g}(y^*)$, where $y^*=\lim_{k\to\infty}y^k$ and y^* is a global minimizer of $\mathbf{g}$. If z is any point in $\mathbf{C}_\mu(v)$, we have by Lemma 1 that there exists a set $A\in\mathfrak{A}$ with $\mu(A)=0$ such that $z\in C_a$ for every $a\in[0,1]^n\setminus A$. Hence, for any $a\in[0,1]^n\setminus A$, we have $P_a z=z$, that is,

$$\mathbf{g}(z)=\frac{1}{2}\int_{[0,1]^n}\|P_a z-z\|^2\,\mathrm{d}\mu=\frac{1}{2}\int_{[0,1]^n\setminus A}\|P_a z-z\|^2\,\mathrm{d}\mu=0.$$

This and the minimality of $\mathbf{g}(y^*)$ imply $\mathbf{g}(y^*)=0$.

Conversely, suppose that $\lim_{k\to\infty}\mathbf{g}(y^k)=0$. Since $(y^k)_{k\in\mathbb{N}}$ is bounded, it has a convergent subsequence $(y^{i_k})_{k\in\mathbb{N}}$. Let $\bar{y}$ be the limit of this subsequence. Due to the continuity of $\mathbf{g}$ we get

$$\mathbf{g}(\bar{y})=\lim_{k\to\infty}\mathbf{g}(y^{i_k})=\lim_{k\to\infty}\mathbf{g}(y^k)=0.$$

This implies by the definition of $\mathbf{g}$ that $\bar{y}=P_a(\bar{y})\in C_a$ for μ-almost all $a\in[0,1]^n$. In other words, we have that $\bar{y}\in\mathbf{C}_\mu(v)$, that is, $\mathbf{C}_\mu(v)\neq\emptyset$.

(B) If the sequence $(y^k)_{k\in\mathbb{N}}$ is unbounded, then [7, Theorem 2.1(B)] implies that $\mathbf{C}_\mu(v)=\emptyset$. If the sequence $(y^k)_{k\in\mathbb{N}}$ is bounded and $\lim_{k\to\infty}\mathbf{g}(y^k)\neq 0$, then the limit $y^*=\lim_{k\to\infty}y^k$ exists. We have $0\neq\lim_{k\to\infty}\mathbf{g}(y^k)=\mathbf{g}(y^*)$, where y^* is a global minimizer of $\mathbf{g}$ (see (i)). Suppose by contradiction that in this situation $\mathbf{C}_\mu(v)\neq\emptyset$. An argument similar to that involved in the proof of (A)(ii) shows that for some $z\in[0,1]^n$ we have $\mathbf{g}(z)=0$. Since $0\leqslant\mathbf{g}(y^*)\leqslant\mathbf{g}(z)=0$, we deduce that $g(y^*)=0$ and this is a contradiction. □

The next result is a direct consequence of Theorem 2 combined with Theorem 1. It shows that, in special circumstances, Cimmino type bargaining scheme approximates elements of the core of the games with fuzzy coalitions.

Theorem 3. *Let v be a game with fuzzy coalitions that is a continuous function $[0,1]^n\to\mathbb{R}$ and μ be a coalitional assessment satisfying (9). For any initial point x^0, if $(x^k)_{k\in\mathbb{N}}$ is a bounded sequence generated by the Cimmino type bargaining scheme in the game with fuzzy coalitions v and $\lim_{k\to\infty}\mathbf{g}(x^k)=0$, then $\lim_{k\to\infty}x^k\in\mathbf{C}(v)$.*

The following examples show how the Cimmino type bargaining scheme can be used in balancedness analysis of games with fuzzy coalitions. The numerical experiments were carried out in the software package Mathematica. Since the precise calculation of $\mathbf{P}x$ involves rather complicated primitive functions, standard numerical integrations techniques were employed (see [11, Chapter 12], for example).

Example 4. Let u be a game defined in Example 1, which was shown to be unbalanced.

1. Let the coalitional assessment be given by

$$\mu=\tfrac{1}{2}\lambda+\tfrac{1}{2}\delta_1, \tag{11}$$

   where λ is Lebesgue measure and δ_1 is the Dirac measure concentrated at the point 1. The enlarged core $\mathbf{C}_\mu(u)$ is empty. Let us see how the emptiness of $\mathbf{C}_\mu(u)$ (and thus $\mathbf{C}(u)$ too) can be observed from one run of Cimmino type bargaining scheme by using Theorem 2.
   Take the initial point $y^0=(1,2)$, for example. The Cimmino type bargaining scheme generates a sequence $(y^k)_{k\in\mathbb{N}}$ such that the vector $y^{644}=(0.583,0.583)$ is the fixed point of $\mathbf{P}$. As noted above, the fixed points of $\mathbf{P}$ are the global minimizers of $\mathbf{g}$. The value of the function $\mathbf{g}$ in each iteration is depicted in Fig. 1. Since $\mathbf{g}(y^*)=0.022$, it follows from Theorem 2 that $\mathbf{C}_\mu(u)$ is indeed empty.
2. Consider now the same game u as above endowed with the coalitional assessment given by Lebesgue measure λ. Note that the enlarged core $\mathbf{C}_\lambda(u)$ is nonempty because it contains every payoff vector (x_1,x_2) such that $x_1,x_2>0$ and $x_1+x_2\geqslant 1$.

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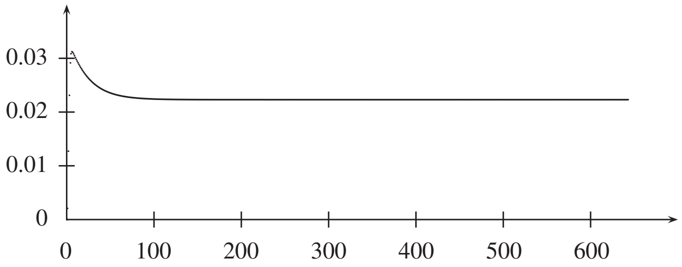

Fig. 1. Function **g** from Example 4.

Starting from the initial point $y^0 = (1, 2)$, the Cimmino type bargaining scheme generates in 1000 iterations only the constant sequence equal to y^0 and thus $y^* = y^0$. The computed value $\mathbf{g}(y^*) \approx 10^{-50}$ can be safely attributed to the error of numerical integration. Hence we can conclude by Theorem 2 that $\mathbf{C}_\lambda(u)$ is nonempty and $y^0 = (1, 2) \in \mathbf{C}_\lambda(u)$.

Example 5. Let w be the game from Example 3. Its core $\mathbf{C}(w)$ is nonempty—see (8).

1. Assume that the coalitional assessment μ is given by (11). The Cimmino type bargaining scheme starting from the initial point $x^0 = (0.05, 0.2)$ converges to $x^* = (0.426, 0.574)$, which is reached in the 20th iteration. This is a fixed point of $\mathbf{P}$ and the almost null value $\mathbf{g}(x^*) \approx 10^{-32}$ together with Theorem 3 suggests that $x^* \in \mathbf{C}(w)$.
   When the initial point is $y^0 = (1.3, 0.1)$, the convergence of the Cimmino type bargaining scheme to a point from $\mathbf{C}(w)$ is much slower: after 1000 iterations, $x^{1000} = (1.010, -0.010)$ with $\mathbf{g}(x^{1000}) \approx 10^{-7}$. Nevertheless, the convergence to a point from the core $\mathbf{C}(w)$ is guaranteed by Theorem 3.

Acknowledgments

The authors are grateful to Prof. Milan Mareš for making their cooperation possible. The work of Tomáš Kroupa was supported by Grant no. 1M0572 of the Ministry of Education, Youth and Sports of the Czech Republic and by Grant GA ČR 102/08/0567.

References

[1] J.-P. Aubin, Coeur et équilibres des jeux flous sans paiements latéraux, C. R. Acad. Sci. Paris Sér. A 279 (1974) 963–966.
[2] J.-P. Aubin, Locally Lipschitz cooperative games, J. Math. Econom. 8 (3) (1981) 241–262.
[3] Y. Azrieli, E. Lehrer, On some families of cooperative fuzzy games, Internat. J. Game Theory 36 (1) (2007) 1–15.
[4] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (3) (1996) 367–426.
[5] R. Brânzei, D. Dimitrov, S. Tijs, Convex fuzzy games and participation monotonic allocation schemes, Fuzzy Sets and Systems 139 (2) (2003) 267–281.
[6] D. Butnariu, E.P. Klement, Triangular Norm Based Measures and Games with Fuzzy Coalitions, Kluwer Academic Publishers, Dordrecht, 1993.
[7] D. Butnariu, B.Z. Shklyar, Existence and approximation of solutions for Fredholm equations of the first kind with applications to a linear moment problem, Optim. Methods Software 23 (1) (2008) 21–37.
[8] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, Vol. 580, Springer, Berlin, 1977.
[9] D.B. Gillies, Solutions to general non-zero-sum games, in: Contributions to the Theory of Games, Vol. IV, Annals of Mathematics Studies, no. 40, Princeton University Press, Princeton, NJ, 1959, pp. 47–85.
[10] Y. Hwang, Fuzzy games: a characterization of the core, Fuzzy Sets and Systems 158 (22) (2007) 2480–2493.
[11] C.W. Ueberhuber, Numerical Computation, 2, Springer, Berlin, 1997 (Methods, software, and analysis, Translated and revised from the 1995 German original).
[12] L.S.Y. Wu, A dynamic theory for the class of games with nonempty cores, SIAM J. Appl. Math. 32 (2) (1977) 328–338.