Representation of States on MV-algebras by Probabilities on R-generated Boolean Algebras

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Abstract— *Any MV-algebra M can be embedded as a lattice in the Boolean algebra $B(M)$ that is R-generated by M. We relate the study of states on an MV-algebra M to the study of finitely additive probabilities on $B(M)$. In particular, we show that each state on M can be uniquely extended to a finitely additive probability on $B(M)$. In case that M is a PMV-algebra, the conditional state $s(a|b)$ defined for $a, b \in M$ with $s(b) \neq 0$ is extended to the classical conditional probability $p(a \cdot b|b)$ on $B(M)$ of the a-proportion of the event b, given the event b.*

Keywords— MV-algebra, R-generated Boolean algebra, state, probability, conditional probability

1 Introduction

States on MV-algebras were investigated by Mundici in [1] as $[0, 1]$-valued additive functionals on (equivalence classes of) formulas in Łukasiewicz propositional logic with the intention to capture the notion of "average" truth degree of a formula. In [2] a probability theory on MV-algebras is systematically developed. The study of states is frequently enhanced by looking directly at certain probabilities induced by the states: there is a one-to-one correspondence between the states on an MV-algebra and the Borel probability measures on the maximal spectrum of the MV-algebra [3, 4], where the bijection is the Lebesgue integral of a continuous function on the maximal spectrum with respect to a uniquely determined Borel probability measure. In this paper we introduce another way of representing a state within a Boolean probability theory. By Jenča's result [5], each MV-algebra M is embedded as a lattice in a uniquely determined Boolean algebra $B(M)$ that is said to be R-generated by M (see [6], [5]). We will show how to relate the study of states over M to the study of certain finitely additive probabilities on $B(M)$. In particular, our approach sheds a new light on the definition of conditional state appearing in [7].

In Section 2 we repeat the basic notion concerning MV-algebras and R-generated Boolean algebras. Section 3 contains the main results: Proposition 3, in which we show that there is only one probability on $B(M)$ extending a state on M, and Proposition 4, which gives a geometrical and topological description of such extensions in case M is a semisimple MV-algebra. In Section 4 we relate conditional states on M to conditional probabilities on $B(M)$ (Proposition 5).

2 Preliminary Notions

2.1 MV-algebras

MV-algebras [8] were introduced by Chang in 1958 as the algebraic counterpart of propositional Łukasiewicz logic.

Definition 1. An MV-algebra is a structure $\langle A, \oplus, \neg, 0\rangle$ with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0 such that $\langle A, \oplus, 0\rangle$ is an abelian monoid and the following equations hold for every $x, y \in A$:

$$\neg\neg x = x$$
$$x \oplus \neg 0 = \neg 0$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

The most important example of an MV-algebra is the real unit interval $[0, 1]$ equipped with operations $x \oplus y = \min(1, x+y)$ and $\neg x = 1 - x$. Indeed, the class of MV-algebras form a variety that is generated by $[0, 1]$.
On each MV-algebra A we define $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$, $x \ominus y = x \odot \neg y$. Note that in the MV-algebra $[0, 1]$ we have $x \odot y = \max(0, x + y - 1)$ and $x \ominus y = \max(0, x - y)$.

Let A be an MV-algebra. For any two elements x and y of A we write $x \leq y$ if and only if $\neg x \oplus y = 1$. It follows that $\leq$ is a partial order that in the case of the algebra $[0, 1]$ coincides with the natural order. Further, defined connectives

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad (1)$$
$$x \wedge y = \neg(\neg x \vee \neg y) \quad (2)$$

are such that the structure $\langle A, \wedge, \vee, 0, 1\rangle$ is a distributive lattice with bottom element 0 and top element 1. An *MV-chain* is an MV-algebra in which the order relation $\leq$ is total.

For any MV-algebra M, the set $\{x \in M \mid x \oplus x = x\}$ is the largest Boolean algebra contained in M.

Given an MV-algebra A and a set X, the set A^X of all functions $f : X \to A$ becomes an MV-algebra if the operations $\oplus$ and $\neg$ and the element 0 are defined pointwise. MV-algebras of functions taking values in $[0, 1]$ can be characterized by means of their ideals.

An *ideal* of an MV-algebra A is a subset I of A such that $0 \in I$, if $x, y \in I$ then $x \oplus y \in I$, and if $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$. An ideal is *maximal* if it is not contained in any proper ideal. An MV-algebra A is said to be *semisimple* if and only if A is non trivial and the intersection of all maximal ideals of A is $\{0\}$. By the *maximal spectrum* we mean the (nonempty) set of all maximal ideals of A. This set can be made into a compact Hausdorff space.

It can be shown that an MV-algebra A is semisimple if and only if A is isomorphic to a separating MV-algebra of $[0,1]$-valued continuous functions on some nonempty compact Hausdorff space (actually the maximal spectrum of A), with pointwise operations. For other notions related to MV-algebras we refer the reader to [9].

A *product MV-algebra* (or PMV-algebra, for short, see [10], [11]) is a structure $\langle A, \oplus, \neg, \cdot, 0\rangle$, where $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and $\cdot$ is a binary associative and commutative operation on A such that for any $x, y, z \in A$, $x \cdot 1 = x$ and $x \cdot (y \ominus z) = (x \cdot y) \ominus (x \cdot z)$.

Note that $[0,1]$ is a PMV-algebra where the operation $\cdot$ is the usual multiplication of real numbers. Further, a semisimple MV-algebra A is a PMV-algebra if and only if A is closed with respect to the pointwise real product of functions and, in this case, the unique product on A is the pointwise real product of functions.

2.2 Boolean algebra R-generated by an MV-algebra

Let M be an MV-algebra. There exists a unique (up to a Boolean isomorphism) Boolean algebra $B(M)$ such that the lattice reduct of M

(i) is a sublattice of $B(M)$ containing both the elements 0 and 1 of M,

(ii) generates $B(M)$ as a Boolean algebra.

The Boolean algebra $B(M)$ is called an *R-generated Boolean algebra*. See [6, Chapter II.4] for details. For every $a \in B(M)$ there exists $n \in \mathbb{N}$ and a finite chain $a_1 \leq \cdots \leq a_{2n}$ in M such that

$$a = \bigvee_{i=1}^{n} (a_{2i} \setminus a_{2i-1}) \tag{3}$$

holds true in $B(M)$ and $(a_{2i} \setminus a_{2i-1}) \wedge (a_{2j} \setminus a_{2j-1}) = 0$ for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$, where $\setminus$ denote the symmetric difference in the Boolean algebra $B(M)$.

Theorem 1 (Jenča [5], Theorem 2)**.** *Let M be an MV-algebra. Then there exists a surjective mapping $\varphi_M : B(M) \to M$ such that*

$$\varphi_M(a) = \bigoplus_{i=1}^{n} (a_{2i} \ominus a_{2i-1}), \quad a \in B(M), \tag{4}$$

where $a_1, \ldots, a_{2n} \in M$ are as in (3) *and the value of $\varphi_M(a)$ is independent on the choice of the representation* (3)*. Moreover, the mapping φ_M satisfies*

(i) $\varphi_M(1) = 1$,

(ii) $\varphi(a) = a$, *for every* $a \in M$,

(iii) if $a, b \in B(M)$ are such that $a \wedge b = 0$, then $\varphi_M(a) \odot \varphi_M(b) = 0$ *and* $\varphi_M(a \vee b) = \varphi(a) \oplus \varphi(b)$.

The following two examples appear in [5].

Example 1. Let M be an MV-chain. Given two elements $a, b \in M$, put $[a, b) = \{x \in M \mid a \leq x < b\}$. By [6], $B(M)$ is isomorphic to the Boolean algebra of all subsets of M having the form

$$[a_1, b_1) \cup \cdots \cup [a_n, b_n)$$

where $a_i, b_i \in M$ and $[a_i, b_i) \cap [a_j, b_j) = \emptyset$, for every $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Note that each $a \in M$ can be identified with $[0, a) \in B(M)$. Then

$$\varphi_M([a_1, b_1) \cup \cdots \cup [a_n, b_n)) = (b_1 \ominus a_1) \oplus \cdots \oplus (b_n \ominus a_n).$$

Example 2. Suppose now M is a semisimple MV-algebra, then M can be viewed as an MV-algebra of continuous functions from the maximal spectrum X of M to the unit interval $[0,1]$. Let $\mathcal{B} = B([0,1])$ be the Boolean algebra of all subsets of $[0,1]$ of the form $[a_1, b_1) \cup \cdots \cup [a_n, b_n)$ for some $n \in \mathbb{N}$, where $[a_i, b_i) \cap [a_j, b_j) = \emptyset$ with $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Just as in Example 1, each element a of the MV-algebra $[0,1]$ can be identified with the interval $[0, a)$ of $\mathcal{B}$.

Since lattice operations in $[0,1]^X$ (and hence in M) are componentwise, then $B(M)$ is a Boolean subalgebra of $\mathcal{B}^X$. Each element $f \in M \subseteq [0,1]^X$ can be identified with the function $f^* \in B(M)$ such that $f^* : x \in X \mapsto [0, f(x)) \in \mathcal{B}$. Then, according to (3), for every $g \in B(M)$ there is $n \in \mathbb{N}$ and $a_1 \leq b_1 \leq \ldots \leq a_n \leq b_n \in M$ such that

$$g = \bigvee_{i=1}^{n} b_i^* \setminus a_i^*. \tag{5}$$

The mapping φ_M is defined as follows. For every $g \in B(M)$ and any $x \in X$, we have, by (5):

$$g(x) = \bigvee_{i=1}^{n} [0, b_i(x)) \setminus [0, a_i(x)).$$

It suffices to let $\varphi_M(g) : x \in X \mapsto (b_1(x) \ominus a_1(x)) \oplus \cdots \oplus (b_n(x) \ominus a_n(x)) \in [0,1]$.

Observe that the length n of the chain representing g does not depend on the coordinate (maximal ideal) x.

3 States and Probabilities

A *state* on an MV-algebra M is a mapping $s : M \to [0,1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$, for every $a, b \in M$ with $a \odot b = 0$. In case that M is a Boolean algebra, then we denote a state on it by p and call p a *(finitely additive) probability.* Let $\mathscr{S}(M)$ be the *state space* of M, that is, the (nonempty) set of all states on M. Analogously, by $\mathscr{S}(B(M))$ we denote the set of all probabilities on the R-generated Boolean algebra $B(M)$. In the sequel we will introduce and study the relation between the two state spaces.

Proposition 1. *If s is a state on an MV-algebra M, then there is a probability p on $B(M)$ such that $p(a) = s(a)$ for every $a \in M$.*

Proof. Put

$$p(a) = s(\varphi_M(a)), \quad a \in B(M), \tag{6}$$

where φ_M is the mapping from Theorem 1, and observe that this definition is correct since, by the same Theorem 1, it does not depend on the representation of a given in (3). Then $p(1) = s(\varphi_M(1)) = s(1) = 1$. Let $a, b \in B(M)$ be such that $a \wedge b = 0$. Then Theorem 1(iii) yields

$$\begin{aligned} p(a \vee b) &= s(\varphi_M(a \vee b)) = s(\varphi_M(a) \oplus \varphi_M(b)) \\ &= s(\varphi_M(a)) + s(\varphi_M(b)) = p(a) + p(b). \end{aligned}$$

Since every $a \in M$ is a fixed point of φ_M due to Theorem 1(ii), we get that p coincides with s over M. $\square$

The next proposition shows that every probability on $B(M)$ is uniquely determined already on the embedded MV-algebra M.

Proposition 2. *If p and p' are two probabilities on $B(M)$ and $p(b) = p'(b)$ for every $b \in M$, then $p = p'$.*

Proof. Let $a \in B(M)$. By equation (3), there exists a finite chain $a_1 \leq \cdots \leq a_{2n}$ in M such that

$$a = \bigvee_{i=1}^{n} (a_{2i} \setminus a_{2i-1}) \tag{7}$$

in $B(M)$ and $(a_{2i} \setminus a_{2i-1}) \wedge (a_{2j} \setminus a_{2j-1}) = 0$, for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Hence

$$\begin{aligned}
p(a) &= p\Big(\bigvee_{i=1}^{n} (a_{2i} \setminus a_{2i-1})\Big) = \sum_{i=1}^{n} p(a_{2i} \setminus a_{2i-1}) \\
&= \sum_{i=1}^{n} (p(a_{2i}) - p(a_{2i-1})) = \sum_{i=1}^{n} (p'(a_{2i}) - p'(a_{2i-1})) \\
&= \sum_{i=1}^{n} p'(a_{2i} \setminus a_{2i-1}) = p'\Big(\bigvee_{i=1}^{n} (a_{2i} \setminus a_{2i-1})\Big) \\
&= p'(a).
\end{aligned}$$

□

Putting together Proposition 1 with (6) and Proposition 2, we get the following uniqueness result.

Proposition 3. *If s is a state on an MV-algebra M, then the probability $p = s \circ \varphi_M$ on $B(M)$ is the unique probability on $B(M)$ such that $p(a) = s(a)$ for every $a \in M$.*

The above introduced correspondence between the states in $\mathscr{S}(M)$ and the probabilities in $\mathscr{S}(B(M))$ works in general only in one direction. Define a mapping $\Phi : \mathscr{S}(M) \to \mathscr{S}(B(M))$ by

$$\Phi(s)(a) = s(\varphi_M(a)), \tag{8}$$

for every $s \in \mathscr{S}(M)$ and every $a \in B(M)$. Since every MV-algebra M is embedded into $B(M)$, it is natural to expect that the state space of the Boolean algebra $B(M)$ is much larger than that of M. Indeed, already in case that M is the standard MV-algebra $[0, 1]$, the set $\mathscr{S}([0, 1])$ contains only one element (the state s defined by $s(x) = x$, for every $x \in [0, 1]$). So the image of $\Phi(\mathscr{S}([0, 1]))$ is a singleton, but $\mathscr{S}(B([0, 1]))$ contains infinitely-many probabilities: for every $x \in [0, 1]$, the mapping

$$a \in B([0, 1]) \mapsto \begin{cases} 1, & x \in a, \\ 0, & \text{otherwise}, \end{cases}$$

is a two-valued probability and hence it belongs to $\mathscr{S}(B([0, 1]))$.

Properties of the operator Φ are summarized below. In particular, we will show that Φ preserves the geometrical-topological structure of $\mathscr{S}(M)$. In order to show this, the convex sets $\mathscr{S}(M)$ and $\mathscr{S}(B(M))$ are endowed with the subspace topology of the product spaces $[0, 1]^M$ and $[0, 1]^{B(M)}$, respectively, so that both $\mathscr{S}(M)$ and $\mathscr{S}(B(M))$ became compact spaces (see [1]).

Proposition 4. *Let M be a semisimple MV-algebra. Then:*

(i) $\Phi(s)(a) = s(a)$, for every $s \in \mathscr{S}(M)$ and every $a \in M$;
(ii) the mapping Φ is an affine homeomorphism of $\mathscr{S}(M)$ onto the compact convex set $\Phi(\mathscr{S}(M))$;
(iii) the set $\Phi(\mathscr{S}(M))$ is affinely homeomorphic to the set of all Borel probability measures on the maximal spectrum X of M;
(iv) if M is not a Boolean algebra, then $\Phi(\mathscr{S}(M)) \subsetneq \mathscr{S}(B(M))$.

Proof. (i) This follows directly from the definition (8) and Proposition 3.

(ii) First, we will show that the mapping Φ is *affine,* that is, for every $s, s' \in \mathscr{S}(M)$ and $\alpha \in [0, 1]$, we have

$$\Phi(\alpha s + (1 - \alpha)s') = \alpha\Phi(s) + (1 - \alpha)\Phi(s').$$

For every $a \in B(M)$, the definition (8) yields:

$$\begin{aligned}
&\Phi(\alpha s + (1 - \alpha)s')(a) = (\alpha s + (1 - \alpha)s')(\varphi_M(a)) \\
&= \alpha s(\varphi_M(a)) + (1 - \alpha)s'(\varphi_M(a)) \\
&= \alpha\Phi(s)(a) + (1 - \alpha)\Phi(s')(a).
\end{aligned}$$

The mapping Φ is injective. Given $s, s' \in \mathscr{S}(M)$ with $s \neq s'$, find $a \in M$ such that $s(a) \neq s'(a)$. Hence it follows from (i) that $\Phi(s) \neq \Phi(s')$. The mapping Φ is continuous: take a net s_γ of elements of $\mathscr{S}(M)$ such that $s_\gamma \to s$ for some $s \in \mathscr{S}(M)$. This means that $s_\gamma(b) \to s(b)$ for every $b \in M$. Thus for every $a \in B(M)$, we get

$$\Phi(s_\gamma)(a) = s_\gamma(\varphi_M(a)) \to s(\varphi_M(a)) = \Phi(s)(a).$$

The set $\Phi(\mathscr{S}(M))$ is compact convex as an affine continuous image of $\mathscr{S}(M)$. Since Φ is a continuous bijection $\mathscr{S}(M) \to \Phi(\mathscr{S}(M))$, it is actually a homeomorphism by compactness.

(iii) The set $\mathscr{B}(X)$ of all Borel (σ-additive) probability measures on the maximal spectrum X of M is compact and convex [12, Proposition 5.22]. In the topology of $\mathscr{B}(X)$ a net μ_γ converges to μ in $\mathscr{B}(X)$ if for every continuous function $f : X \to \mathbb{R}$ we have $\int f \, \mathrm{d}\mu_\gamma \to \int f \, \mathrm{d}\mu$. By Corollary 29 in [3] or [4, Proposition 1.1], there exists a unique mapping $\Psi : \mathscr{S}(M) \to \mathscr{B}(X)$ such that

$$s(a) = \int_X \hat{a} \, \mathrm{d}\Psi(s) \tag{9}$$

for every $s \in \mathscr{S}(M)$ and every $a \in M$, where $\hat{a}$ is the image of a via the isomorphism identifying the elements of M with the separating MV-algebra of continuous functions $X \to [0, 1]$. We claim that the mapping Ψ is an affine homeomorphism. It is onto since every integral (9) with respect to some Borel probability measure from $\mathscr{B}(X)$ determines a state. It is also injective as every two states $s, s' \in \mathscr{S}(M)$ coincide whenever $\Psi(s) = \Psi(s')$ by the representation (9). The mapping Ψ is affine since the functional $\mu \in \mathscr{B}(X) \mapsto \int_X f \, \mathrm{d}\mu$ is linear for every continuous function $f : X \to [0, 1]$. Finally, we will show that Ψ^{-1} is continuous. Take a net μ_γ and μ with $\mu_\gamma \to \mu$ in $\mathscr{B}(X)$. It results directly from the definition of convergence in $\mathscr{B}(X)$ and in $\mathscr{S}(M)$ that $\Psi^{-1}(\mu_\gamma) \to \Psi^{-1}(\mu)$ in $\mathscr{S}(M)$, so Ψ^{-1} is continuous. Since $\mathscr{S}(M)$ is compact, the mapping Ψ is a homeomorphism. As an inverse of an affine homeomorphism is an affine homeomorphism and a composition of affine homeomorphisms is again an affine homeomorphism, take the mapping $\Psi \circ \Phi^{-1} : \Phi(\mathscr{S}(M)) \to \mathscr{B}(X)$ to finish the proof.

(iv) The MV-algebra M is not a Boolean algebra, so there exists an element $f \in M$ and $x \in X$ with $0 < f(x) < 1$. Put $\beta = \frac{\min(f(x), 1-f(x))}{2}$ and define a mapping $p : B(M) \to [0,1]$ by

$$p(g) = \begin{cases} 1, & \beta \in g(x), \\ 0, & \text{otherwise}, \end{cases} \quad g \in B(M).$$

A routine check shows that p is a probability on $B(M)$. We will show that there is no state $s \in \mathscr{S}(M)$ such that $\Phi(s) = p$. By way of contradiction, assume that such a state s exists. Then Proposition 3 together with (8) imply that $s(g) = p(g)$ for every $g \in M$. Since $s(f) = p(f) = 1$ and $s(\neg f) = p(\neg f) = 1$, we get

$$s(f \oplus \neg f) = s(1) = 1 \neq 2 = s(f) + s(\neg f),$$

which is a contradiction and thus $p \in \mathscr{S}(B(M)) \setminus \Phi(\mathscr{S}(M))$. □

On the one hand, every state on a semisimple MV-algebra can be viewed as the integral (9) with respect to a uniquely determined Borel probability measure that is defined on the maximal spectrum X of M. On the other hand, the operator Φ maps a state to a finitely additive probability on a Boolean subalgebra of the direct sum $\mathcal{B}^X$ (see Example 2). While the transformation Φ enables only to embed $\mathscr{S}(M)$ into a (huge) state space $\mathscr{S}(B(M))$, we will show below that and how probabilities on direct sums of Boolean algebras appear naturally already in classical Kolmogorov model of probability.

Let $B_1, \dots, B_k$ be Boolean algebras. Let p_i be a probability on B_i for $i = 1, \dots, k$. By B we denote a Boolean algebra that is the direct sum of $B_1, \dots, B_k$ (see [13, §16]). Every element of $b \in B$ can be identified with an n-tuple $(b_1, \dots, b_k)$ such that $b_i \in B_i$ for each $i = 1, \dots, k$. The elements b of B are random events whose meaning is "precisely one of all the events $b_1, \dots, b_k$ occurs". These classes of random events capture *two-stage random experiments*, such as the random selection of a ball from a randomly selected box. In this interpretation, each number $i = 1, \dots, k$ denotes a box and the corresponding Boolean algebra B_i models all outcomes of a random selection of a ball from the box i. Precisely, for some nonnegative $\alpha_1, \dots, \alpha_k$ with $\sum_{i=1}^{k} \alpha_i = 1$, the procedure can be described as follows:

(i) select B_i with a probability α_i;
(ii) if B_j was selected in the previous stage, then perform the random experiment with outcomes in B_j and probabilities described by p_j.

A probability p on B should then express the above introduced meaning of the two-stage experiment. This means that for every $b \in B$, we get

$$p(b) = \sum_{i=1}^{k} \alpha_i p(b_i). \tag{10}$$

The interrelationship between states on MV-algebras and probabilities on direct sums as defined above is made possible by the mappings Φ and φ_M.

Example 3. Let M be the direct product of two standard MV-algebras, that is, $M = [0,1]^2$. According to (9) every state s on M is of the form

$$s(a) = s((a_1, a_2)) = \alpha a_1 + (1-\alpha) a_2, \quad a = (a_1, a_2) \in M, \tag{11}$$

for some $\alpha \in [0,1]$. The Boolean algebra $B(M)$ is precisely the direct sum of the two Boolean algebras $\mathcal{B}$, where $\mathcal{B}$ is as in Example 2. The MV-algebra M embeds into $B(M)$ by sending each $a = (a_1, a_2) \in M$ to the function $a^*(i) = [0, a_i)$, $i = 1, 2$. Let λ denotes the restriction of Lebesgue measure to $\mathcal{B}$. Then it follows from (11) that

$$\Phi(s)(g) = \alpha\lambda(g(1)) + (1-\alpha)\lambda(g(2)), \quad g \in B(M). \tag{12}$$

Indeed, as in Example 2, for every $g \in B(M)$ and $i = 1, 2$ find the representation $a_1^i, b_1^i, \dots, a_n^i, b_n^i \in [0,1]$ such that $g(i) = [a_1^i, b_1^i) \cup \dots \cup [a_n^i, b_n^i)$ with all the intervals disjoint. Then

$$\varphi_M(g)(i) = (b_1^i \ominus a_1^i) \oplus \dots \oplus (b_n^i \ominus a_n^i)$$

and $(\varphi_M(g)(1), \varphi_M(g)(2)) \in M$. As a consequence,

$$\begin{aligned} \Phi(s)(g) = s(\varphi_M(g)) &= \alpha\varphi_M(g)(1) + (1-\alpha)\varphi_M(g)(2) \\ &= \alpha(b_1^1 \ominus a_1^1) \oplus \dots \oplus (b_n^1 \ominus a_n^1) \\ &\quad + (1-\alpha)(b_1^2 \ominus a_1^2) \oplus \dots \oplus (b_n^2 \ominus a_n^2) \\ &= \alpha\lambda(g(1)) + (1-\alpha)\lambda(g(2)), \end{aligned}$$

which proves (12).

The model from Example 3 can be given a straightforward probabilistic interpretation. Suppose that some underground platform is accessible from a street either via a stairway or via an escalator. Mr. Smiley chooses the stairway with a probability $\alpha \in [0,1]$ and the escalator with a probability $1-\alpha$. Train arrivals are uniformly distributed over the interval $[0,1]$. What is a probability that Mr. Smiley's waiting time for the train will be smaller or equal to $a_1 \in [0,1]$ (in case he uses the stairway) or smaller or equal to $a_2 \in [0,1]$ (in case he uses the escalator)? The investigated event $a = ([0, a_1), [0, a_2))$ belongs to $B(M)$, while the many-valued event $a = (a_1, a_2)$ is an element of M. If $p(a)$ denotes the probability of a, then (10) together with uniformity of the waiting times yield

$$\begin{aligned} p(a) &= \alpha\lambda([0, a_1]) + (1-\alpha)\lambda([0, a_2]) = \alpha a_1 + (1-\alpha) a_2 \\ &= s(a). \end{aligned}$$

4 Conditional Probability

In this section we use the probabilistic operator Φ to study conditioning in the framework of MV-algebras and states. Conditional probability on MV-algebras was studied from a variety of perspectives. In [7] the conditional probability on MV-algebras is defined in a way that mimicks the classical (Boolean) approach of conditioning "an event by an event" or "an event by a subalgebra". This approach makes use of an additional operation (product) on an MV-algebra and it was employed by Montagna in order to prove de Finetti-style coherence theorem for conditional bets in many-valued logic [14]. A completely different definition of conditional probability (so-called "conditional") was proposed by Mundici in [15].

The following approach to conditioning is taken from [7, Definition 5]. Let s be a state on a PMV-algebra M. Given a many-valued event $b \in M$ with $s(b) > 0$, put

$$s(a|b) = \frac{s(a \cdot b)}{s(b)}, \quad a \in M. \tag{13}$$

The number $s(a|b)$ is called a *conditional state of* a *given* b. If $s(b)$ vanishes, then we leave $s(a|b)$ undetermined. This makes the function $a \in M \mapsto s(a|b)$ defined for "almost all" $b \in M$. Since $(a_1 \oplus a_2) \cdot b = (a_1 \cdot b) \oplus (a_2 \cdot b)$ for every $a_1, a_2 \in M$ such that $a_1 \odot a_2 = 0$, it can be easily verified that the mapping $s(.|b) : M \to [0,1]$ is a state (so-called *conditional state*) on M whenever $b \in M$ is such that $s(b)$ is nonzero. If p is a probability on a Boolean algebra B, then a product operation $\cdot$ on B becomes $\wedge$ and the formula (13) for $p(b) > 0$ reduces to

$$p(a|b) = \frac{p(a \wedge b)}{p(b)}, \quad a \in B. \tag{14}$$

Analogously, the function $p(.|b) : B \to [0,1]$ is a probability on B whenever $p(b) > 0$. In this case we call $p(.|b)$ a *conditional probability*. The major difference between (13) and (14) is the consequence of non-idempotence of the product operation on M. For every $b \in M$ with $s(b) \neq 0$, we have in general only

$$s(b|b) \leq 1 \quad \text{and} \quad s(\neg b|b) \geq 0. \tag{15}$$

In contrast to this, in classical probability $p(b|b) = 1$ and $p(\neg b|b) = 0$ holds true for any $b \in B$ with $p(b) \neq 0$. Surprisingly, this concept of conditioning found a natural application in Montagna's framework [14]: conditional bets are updated in proportion to the truth value of the event in the condition. In that follows we make an effort to justify the definition (13) with the probabilistic interpretation of states via Φ and $B(M)$.

Proposition 5. *Let M be an MV-algebra with product and $s(.|b)$ be a conditional state on it for $b \in M$ with $s(b) \neq 0$. Then there exists a probability p on $B(M)$ such that*

$$\Phi(s(.|b))(a) = p(a \cdot b|b), \quad a \in M. \tag{16}$$

Proof. Put $p = \Phi(s)$. For every $a \in M$, we obtain

$$\begin{aligned}\Phi(s(.|b))(a) = s(a|b) &= \frac{s(a \cdot b)}{s(b)} = \frac{\Phi(s)(a \cdot b)}{\Phi(s)(b)} \\ &= \frac{p(a \cdot b)}{p(b)} = \frac{p((a \cdot b) \wedge b)}{p(b)} \\ &= p(a \cdot b|b),\end{aligned}$$

since $(a \cdot b) \wedge b = a \cdot b$. □

It is worth emphasizing that the probability $\Phi(s(.|b))$ is not a conditional probability on M since $\Phi(s(.|b))(b) = p(b \cdot b|b)$ fails to be equal to 1 in general. So the operator Φ does not map conditional states to conditional probabilities. The formula (16) nevertheless connects the values of conditional states of a given b on M to the values of the classical conditional probability of $a \cdot b$ given b on $B(M)$. Conditioning a by a many-valued event b means in terms of classical conditional probability this: given b, what is a probability $p(a \cdot b|b)$ that an observation of the a-proportion of the many-valued event b will be made? Thus the conditional state defined by the formula (13) leads in accordance with (16) to the conditional probability of $a \cdot b$ given b on the Boolean algebra $B(M)$, which focuses only on the event which is included in b. The use of product in place of infimum makes the difference: the equality $p(a \wedge b|b) = p(a|b)$ holds true in contrast to $p(a \cdot b|b) \neq p(a|b)$. Taking into account the proposed interpretation, the inequalities in (15) look no longer unnatural. The "probabilistic" meaning of $s(b|b)$ is the value of $p(b \cdot b|b)$ that corresponds to the occurrence of the event $b \cdot b$ (given b) rather than b alone as the notation $s(b|b)$ suggests.

Example 4. Consider the situation from Example 3. Assume now that we know the actual waiting time does not exceed $b = (b_1, b_2) \in M = [0,1]^2$. Then the value $s(a|b)$ is according to Proposition 5 the same as $p(a \cdot b|b)$. This probability is precisely the conditional probability that the waiting time will be at most the a-proportion of b given our knowledge that it is at most b.

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