On Two Notions of Independence in Evidence Theory

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Abstract: We study two notions of independence in evidence theory: random set independence and strong independence. We show their relation for special models (Bayesian basic assignments and consonant bodies of evidence) a well as in general case.

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1 Introduction

The complexity of practical problems that are of primary interest in the field of artificial intelligence usually results in the necessity to construct models with the aid of a great number of variables: more precisely, hundreds or thousands rather than tens. Processing distributions of such dimensionality would not be possible without some tools allowing us to reduce demands on computer memory. Independence, which belongs among such tools, allows the expression of these multidimensional distributions by means of low-dimensional ones, and therefore to substantially decrease demands on computer memory.

For three centuries, probability theory has been the only mathematical tool at our disposal for uncertainty quantification and processing. As a result, many important theoretical and practical advances have been achieved in this field. However, during the last forty years some new mathematical tools have emerged as alternatives to probability theory. They are used in situations whose nature of uncertainty does not meet the requirements of probability theory, or those in which probabilistic approaches employ criteria that are too strict. Nevertheless, probability theory has always served as a source of inspiration for the development of these non-probabilistic calculi and these calculi have been continually confronted with probability theory and mathematical statistics from various points of view. Good examples of this fact include the numerous papers studying conditional independence in various calculi [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

With this contribution, we will concentrate ourselves to evidence theory, as it was already done in [12]. After a brief review of basic notation and terminology necessary for understanding the next part of the paper (Section 2) we will present in Section 3 two notions of independence in evidence theory: random set independence and strong independence as well as the independence notions for special cases. In Section 4 we will demonstrate their mutual relationship for both special models of evidence theory and its general case.

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2 Basic Notions

Consider two *frames of discernment* $\mathbf{X}$ and $\mathbf{Y}$ and their Cartesian product

$$\Omega = \mathbf{X} \times \mathbf{Y}.$$

A *projection* of $\omega = (x, y) \in \mathbf{X} \times \mathbf{Y}$ into $\mathbf{X}$ will be denoted $\omega^{\downarrow X}$, i.e. $\omega^{\downarrow X} \in \mathbf{X}$. Analogously, for $A \subset \mathbf{X} \times \mathbf{Y}$, $A^{\downarrow X}$ will denote a *projection* of A into $\mathbf{X}$:

$$A^{\downarrow X} = \{x \in \mathbf{X} | \exists \omega \in A : x = \omega^{\downarrow X}\}.$$

Consider a *basic* (*probability* or *belief*) *assignment* (or just assignment) m on Ω, i.e.

$$m : \mathcal{P}(\Omega) \longrightarrow [0, 1]$$

for which

$$\sum_{A \subseteq \Omega} m(A) = 1. \tag{1}$$

Its *marginal basic assignment* on $\mathbf{X}$ is defined (for each $B \subseteq \mathbf{X}$):

$$m^{\downarrow X}(B) = \sum_{A \subseteq \Omega : A^{\downarrow X} = B} m(A).$$

Given a basic assignment m we can obtain *belief* and *plausibility functions* via the following formulae:

$$\begin{aligned} Bel(A) &= \sum_{B \subseteq A} m(B); \\ Pl(A) &= \sum_{B \cap A \neq \emptyset} m(B). \end{aligned}$$

It is well-known (and evident from these formulae) that for any $A \in \mathcal{P}(\Omega)$

$$Bel(A) \leq Pl(A)$$

and

$$Pl(A) = 1 - Bel(A^C),$$

where A^C is a set complement of $A \in \mathcal{P}(\Omega)$, i.e. plausibility of any set can be obtained from belief of its complement and vice versa. Furthermore, basic assignment can be computed from belief function via Möbius inversion:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(A), \tag{2}$$

i.e. any of these three functions is sufficient to define values of the remaining two.

Now let us concentrate our attention to two special cases of basic assignments.

A basic assignment is called *Bayesian* if all its focal elements* are singletons. In this case $Bel(A) = Pl(A) = P(A)$, called a *probability measure*, and m can be substituted by a point function

$$p : \mathbf{X}_N \longrightarrow [0, 1]$$

called a *probability distribution*.

*A set $A \in \mathcal{P}(X_N)$ is a *focal element* if $m(A) > 0$.

A body of evidence[†] is called *consonant* if its focal elements are nested. In this case

$$Pl(A) = \max_{B\subseteq A} Pl(B),$$

i.e. plausibility function becomes a *possibility measure* Π and its values for any $A \subseteq \mathbf{X}_N$ can be obtained from a *possibility distribution*

$$\pi : \mathbf{X}_N \longrightarrow [0,1]$$

via the formula:

$$\Pi(A) = \max_{x\in A} \pi(x).$$

Connection between basic assignment m and a possibility distribution π is expressed by the following formula:

$$\pi(x) \quad = \sum_{x\in A\in\mathcal{P}(\mathbf{X}_N)} m(A).$$

3 Independence Concepts

In this section we will overview two conditional independence notions called random set independence and strong independence. Then we will present definitions for special cases (Bayesian basic assignments and consonant bodies of evidence). Before doing that let us stress, that we are interested in the independence from the viewpoint of *decomposition* of multidimensional models and not in its behavioral interpretation.

We say that there is *random set independence* between X and Y if their joint basic assignment is of the form

$$m(A) = m^{\downarrow X}(B) \cdot m^{\downarrow Y}(C) \tag{3}$$

for all $A = B \times C, B \subseteq \mathbf{X}, C \subseteq \mathbf{Y}$ and $m(A) = 0$ otherwise.

A *credal set* $\mathcal{M}(X)$ about a variable X is defined as a set of probability measures about the values of this variable. In order to simplify the expression of operations with credal sets, it is often considered [7] that a credal set is the set of probability distributions associated to the probability measures in it. Under such consideration a credal set can be expressed as a *convex hull* of its extreme distributions

$$\mathcal{M}(X) = \mathrm{CH}\{\mathrm{ext}(\mathcal{M}(X))\}.$$

Any belief function (plausibility function, basic assignment) on $\mathbf{X}$ can be associated with a credal set of probability measures

$$\mathcal{M}(Bel) = \{P(A), A \subseteq \mathbf{X} : Bel(A) \leq P(A) \leq Pl(A)\}.$$

Again, there exist numerous definitions of independence for credal sets [4], but we have chosen strong independence, as it seems to be most proper for multidimensional models.

We say that X and Y are *strongly independent* with respect to $\mathcal{M}(XY)$ iff (in terms of probability distributions)

$$\mathcal{M}(XY) = \mathrm{CH}\{p_1 \cdot p_2 : p_1 \in \mathcal{M}(X), p_2 \in \mathcal{M}(Y)\}.$$

Now, let us remind the notion of both stochastic independence and possibilistic T-independence.

[†]A *body of evidence* is a pair $(\mathcal{F}, m)$, where $\mathcal{F}$ is the set of all focal elements.

Variables X and Y are stochastically independent with respect to the joint distribution p if for any pair $(x, y) \in \mathbf{X} \times \mathbf{Y}$ the equality

$$p(x, y) = p^{\downarrow X}(x) \cdot p^{\downarrow Y}(x)$$

holds.

Variables X and Y are T-independent (where T is a continuous t-norm[‡]) with respect to a joint possibility distribution π if for any $x \in \mathbf{X}$ and $y \in \mathbf{Y}$

$$\pi(x, y) = T(\pi^{\downarrow X}(x), \pi^{\downarrow Y}(y)).$$

4 Relations of Independence Concepts

4.1 Bayesian basic assignments

For Bayesian basic assignments both random set independence and strong independence collapse to stochastic independence. Therefore we can conclude:

Lemma 1 *Let X and Y be variables with basic assignments m_X and m_Y, respectively. If both m_X and m_Y are Bayesian, then X and Y are strongly independent if and only in they are independent in the sense of random set independence.*

4.2 Consonant bodies of evidence

The problem of the relation of (conditional) independence concepts for consonant bodies of evidence was thoroughly studied in [12]. I that paper we showed by a simple counterexample, that random set independence is inadequate in this case, as it leads to bodies of evidence which are never more consonant. Furthermore we proved (for the proof see [12]) the following theorem.

Theorem 2 *Let X and Y are strongly conditionally independent in distribution given Z. Then X and Y are conditionally product-independent.*

In case that the condition is empty, we get the relation for unconditional independence concepts. This theorem says that strong independence is stronger than possibilistic independence with respect to product t-norm.

4.3 General basic assignments

Now we will show that strong independence does not imply random set independence and moreover its application to two general basic assignments leads to the resulting model behind the limits of evidence theory.

Example 3 Consider two basic assignments m_X and m_Y on $\mathbf{X} = \{x, \bar{x}\}$ $\mathbf{Y} = \{y, \bar{y}\}$ specified in Table 1 together with their beliefs and plausibilities. From these values we obtain credal sets about variables X and Y:

$$\begin{aligned} \mathcal{M}(X) &= \text{CH}\{(0.3, 0.7), (0.8, 0.2)\}, \\ \mathcal{M}(Y) &= \text{CH}\{(0.6, 0.4), (0.9, 0.1)\}. \end{aligned}$$

[‡] A *triangular norm* (or a *t-norm*) T is an isotonic, associative and commutative binary operator on $[0, 1]$ (i.e. $T : [0, 1]^2 \to [0, 1]$) satisfying the boundary condition: for any $x \in [0, 1]$

$$T(1, x) = x.$$

A t-norm T is called *continuous* if T is a continuous function.

Table 1: Basic assignments m_X and m_Y.

$A \subseteq \mathbf{X}$	$m_X(A)$	$Bel_X(A)$	$Pl_X(A)$
$\{x\}$	0.3	0.3	0.8
$\{\bar{x}\}$	0.2	0.2	0.7
$\mathbf{X}$	0.5	1	1

$A \subseteq \mathbf{Y}$	$m_Y(A)$	$Bel_Y(A)$	$Pl_Y(A)$
$\{y\}$	0.6	0.6	0.9
$\{\bar{y}\}$	0.1	0.1	0.4
$\mathbf{Y}$	0.3	1	1

Table 2: Basic assignment $m_X \times m_Y$.

$C \subseteq \mathbf{X} \times \mathbf{Y}$	$\underline{P}_{XY}(C)$	$\overline{P}_{XY}(C)$	$m_{XY}(C)$
$\{xy\}$	0.18	0.72	0.18
$\{x\bar{y}\}$	0.03	0.32	0.03
$\{\bar{x}y\}$	0.12	0.63	0.12
$\{\bar{x}\bar{y}\}$	0.02	0.28	0.02
$\{x\} \times \mathbf{Y}$	0.3	0.8	0.09
$\{\bar{x}\} \times \mathbf{Y}$	0.2	0.7	0.06
$\mathbf{X} \times \{y\}$	0.6	0.9	0.3
$\mathbf{X} \times \{\bar{y}\}$	0.1	0.4	0.05
$\{xy, \bar{x}\bar{y}\}$	0.34	0.74	0.14
$\{x\bar{y}, \bar{x}y\}$	0.26	0.66	0.11
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}\bar{y}\}$	0.72	0.98	0.55
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}y\}$	0.37	0.88	0.32
$\mathbf{X} \times \mathbf{Y} \setminus \{x\bar{y}\}$	0.68	0.97	0.87
$\mathbf{X} \times \mathbf{Y} \setminus \{xy\}$	0.28	0.82	0.23

Under the assumption of strong independence we get

$$\begin{aligned}\mathcal{M}(XY) \quad = \quad & \mathrm{CH}\{(0.18, 0.12, 0.42, 0.28), (0.27, 0.03, 0.63, 0.07),\\ & (0.48, 0.32, 0.12, 0.08), (0.72, 0.08, 0.18, 0.02)\}.\end{aligned}$$

Let us compute lower and upper probabilities of all nonempty subsets of $\mathbf{X} \times \mathbf{Y}$. Their values can be found in second and third columns of Table 2.

In the last column one can find hypothetical values of basic assignment corresponding the these lower and upper probabilities taken as beliefs and plausibilities computed via formula (2). From this column one can see that X and Y do not satisfy random set independence, as m_{XY} assigns positive values also to subsets which are not of the form $A = B \times C$. Furthermore, if we take into account the equality (1), we get that $m_{XY}(\mathbf{X} \times \mathbf{Y}) = -2.07$, which violates the nonnegativity of basic assignment, i.e. we are behind the limits of evidence theory. $\diamond$

5 Conclusions

We have presented two different notions of independence in evidence theory: random set independence and strong independence. We have shown, that although both of them are generalizations of independence concept in probability theory, it does not hold in other cases. While the first one cannot be applied to consonant bodies of evidence, the second one can. On the other hand for bodies of evidence, which are neither Bayesian nor consonant application of strong independence produces models behind the limits of evidence theory.

These results lead us to the conclusion, that although the mentioned theories can be partially ordered in the following way:

$$\text{Possibility theory} \sqsubseteq \text{Evidence theory} \sqsubseteq \text{Credal sets}$$

and

$$\text{Probability theory} \sqsubseteq \text{Evidence theory} \sqsubseteq \text{Credal sets}$$

their independence concepts are different and it is necessary to use as specific independence concept as possible in order not to get out of the framework in question.

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