On Conditional Independence in Evidence Theory

Jiřina Vejnarová
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
&
University of Economics, Prague
vejnar@utia.cas.cz

Abstract

The goal of this paper is to introduce a new concept of conditional independence in evidence theory, to prove its formal properties, and to show in what sense it is superior to the concept introduced previously by other authors.

Keywords. Evidence theory, random sets independence, conditional independence, conditional noninteractivity.

1 Introduction

Any application of artificial intelligence models to practical problems must manage two basic issues: uncertainty and multidimensionality. The models currently most widely used to manage these issues are so-called *probabilistic graphical Markov models.*

In these models, the problem of multidimensionality is solved using the notion of conditional independence, which enables factorisation of a multidimensional probability distribution into small parts, usually marginal or conditional low-dimensional distributions (or generally into low-dimensional factors). Such a factorisation not only decreases the storage requirements for representation of a multidimensional distribution, but it usually induces efficient computational procedures allowing inference from these models as well. Many results analogous to those concerning conditional independence, Markov properties and factorisation from probabilistic framework were also achieved in possibility theory [12, 13].

It is easy to realise that our need of efficient methods for representation of probability and possibility distributions (requiring an exponential number of parameters) logically leads us to greater need of an efficient tool for representation of belief functions, which cannot be represented by a distribution (but only by a set function), and therefore the space requirements for their representation are superexponential.

After a thorough study of relationships among stochastic independence, possibilistic T-independence, random set independence and strong independence [14, 15], we came to the conclusion that the most proper independence concept in evidence theory is random set independence. Therefore, this contribution is fully devoted to two different generalisations of random set independence to conditional independence.

The contribution is organised as follows. After a short overview of necessary terminology and notation (Section 2), in Section 3 we introduce a new concept of conditional independence and show in which sense it is superior to the previously suggested independence notions [10, 1]. In Section 4 we prove its formal properties.

2 Basic Notions

The aim of this section is to introduce as briefly as possible basic notions and notations necessary for understanding the following text.

2.1 Set Projections and Extensions

For an index set $N = \{1, 2, \ldots, n\}$ let $\{X_i\}_{i \in N}$ be a system of variables, each X_i having its values in a finite set $\mathbf{X}_i$. In this paper we will deal with *multidimensional frame of discernment*

$$\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

and its *subframes* (for $K \subseteq N$)

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

When dealing with groups of variables on these subframes, X_K will denote a group of variables $\{X_i\}_{i \in K}$ throughout the paper.

A *projection* of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{X}_N$ into $\mathbf{X}_K$ will be denoted $x^{\downarrow K}$, i.e., for $K = \{i_1, i_2, \ldots, i_k\}$

$$x^{\downarrow K} = (x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \in \mathbf{X}_K.$$

Analogously, for $M \subset K \subseteq N$ and $A \subset \mathbf{X}_K$, $A^{\downarrow M}$ will denote a *projection* of A into $\mathbf{X}_M$:[1]

$$A^{\downarrow M} = \{y \in \mathbf{X}_M \mid \exists x \in A : y = x^{\downarrow M}\}.$$

In addition to the projection, in this text we will also need an opposite operation, which will be called an extension. By an *extension* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ $(K, L \subseteq N)$ we will understand a set

$$A \otimes B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \ \ \& \ \ x^{\downarrow L} \in B\}.$$

Let us note that if K and L are disjoint, then

$$A \otimes B = A \times B.$$

2.2 Set Functions

In evidence theory (or Dempster-Shafer theory) two measures are used to model the uncertainty: belief and plausibility measures. Both of them can be defined with the help of another set function called a *basic (probability* or *belief) assignment* m on $\mathbf{X}_N$, i.e.,

$$m : \mathcal{P}(\mathbf{X}_N) \longrightarrow [0, 1] \tag{1}$$

for which

$$\sum_{A \subseteq \mathbf{X}_N} m(A) = 1. \tag{2}$$

Furthermore, we assume that $m(\emptyset) = 0$.

Belief and *plausibility* measures are defined for any $A \subseteq \mathbf{X}_N$ by the equalities

$$Bel(A) = \sum_{B \subseteq A} m(B),$$
$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B),$$

respectively.

It is well-known (and evident from these formulae) that for any $A \in \mathcal{P}(\mathbf{X}_N)$

$$Pl(A) = 1 - Bel(A^C) \tag{3}$$

holds, where A^C is a set complement of $A \in \mathcal{P}(\mathbf{X}_N)$. Furthermore, basic assignment can be computed from belief function via Möbius inversion:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B), \tag{4}$$

i.e., any of these three functions is sufficient to define values of the remaining two.

[1]Let us remark that we do not exclude situations when $M = \emptyset$. In this case $A^{\downarrow \emptyset} = \emptyset$.

In addition to belief and plausibility measures, *commonality function* can also be obtained from basic assignment m:

$$Q(A) \quad = \quad \sum_{B \supseteq A} m(B).$$

The last notion plays an important role in the definition of so-called (conditional) noninteractivity of variables (cf. Section 3.2) and in Shenoy's valuation-based systems [10]. Similarly to (4), one can obtain basic assignment from commonality function via an analogous formula

$$m(A) = \sum_{B \supseteq A} (-1)^{|B \setminus A|} Q(B). \tag{5}$$

A set $A \in \mathcal{P}(\mathbf{X}_N)$ is a *focal element* if $m(A) > 0$. A pair $(\mathcal{F}, m)$, where $\mathcal{F}$ is the set of all focal elements, is called a *body of evidence*. A basic assignment is called *Bayesian* if all its focal elements are singletons. A body of evidence is called *consonant* if its focal elements are nested.

For a basic assignment m on $\mathbf{X}_K$ and $M \subset K$, a *marginal basic assignment* of m is defined (for each $A \subseteq \mathbf{X}_M$):

$$m^{\downarrow M}(A) = \sum_{B \subseteq \mathbf{X}_K : B^{\downarrow M} = A} m(B).$$

Analogously, $Bel^{\downarrow M}$, $Pl^{\downarrow M}$ and $Q^{\downarrow M}$ will denote the corresponding marginal belief measure, plausibility measure and commonality function, respectively.

Having two basic assignments m_1 and m_2 on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively $(K, L \subseteq N)$, we say that these assignments are *projective* if

$$m_1^{\downarrow K \cap L} = m_2^{\downarrow K \cap L},$$

which occurs if and only if there exists a basic assignment m on $\mathbf{X}_{K \cup L}$ such that both m_1 and m_2 are marginal assignments of m.

3 Random Set Independence and Its Generalisations

3.1 Marginal Case

Let us start this section by recalling the notion of random sets independence [2].[2]

Definition 1 Let m be a basic assignment on $\mathbf{X}_N$ and $K, L \subset N$ be disjoint. We say that groups of

[2]Klir [6] uses the notion *noninteractivity.*

variables X_K and X_L are *independent with respect to basic assignment* m (and denote it by $K \perp\!\!\!\perp L\ [m]$) if

$$m^{\downarrow K\cup L}(A) = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L}) \qquad (6)$$

for all $A \subseteq \mathbf{X}_{K\cup L}$ for which $A = A^{\downarrow K} \times A^{\downarrow L}$, and $m(A) = 0$ otherwise.

It has been shown in [14] that application of Definition 1 to two consonant bodies of evidence leads to a body of evidence which is no longer consonant.

It seemed that this problem could be avoided if we took into account the fact that both evidence and possibility theories could be considered as special kinds of imprecise probabilities. Nevertheless, in [15] we showed that the application of strong independence to two general bodies of evidence (neither Bayesian nor consonant) leads to models beyond the framework of evidence theory.

From these examples one can see that although models based on possibility measures, belief measures and credal sets can be linearly ordered with respect to their generality, nothing similar holds for the corresponding independence concepts.

Therefore, random sets independence presently seems to be the most appropriate independence concept within the framework of evidence theory from the viewpoint of multidimensional models.[3] For this reason in this section we will deal with two generalisations of this concept.

Before doing that, let us present an assertion showing that conditional noninteractivity and conditional independence (presented in the following two subsections) are identical if the condition is empty.

Lemma 1 *Let K, L be disjoint, then $K \perp\!\!\!\perp L\ [m]$ if and only if*

$$Q^{\downarrow K\cup L}(A) = Q^{\downarrow K}(A^{\downarrow K}) \cdot Q^{\downarrow L}(A^{\downarrow L}) \qquad (7)$$

for all $A \subseteq \mathbf{X}_{K\cup L}$.

Proof can be found in [5].

From this lemma one can conjecture why the generalisation of (7) to the conditional case became widely used, while, as far as we know, no direct generalisation of (6) has been suggested up to now.

In the following example we will show that nothing similar holds for beliefs and plausibilities; more exactly, application of formulae analogous to (7) leads to models beyond the theory of evidence.

[3]Let us note that there exist different independence concepts suitable in other situations, for details the reader is referred to [3]

Table 1: Basic assignments m_X and m_Y.

$A \subseteq \mathbf{X}$	$m_X(A)$	$Bel_X(A)$	$Pl_X(A)$
$\{x\}$	0.3	0.3	0.8
$\{\bar{x}\}$	0.2	0.2	0.7
$\mathbf{X}$	0.5	1	1
$A \subseteq \mathbf{Y}$	$m_Y(A)$	$Bel_Y(A)$	$Pl_Y(A)$
$\{y\}$	0.6	0.6	0.9
$\{\bar{y}\}$	0.1	0.1	0.4
$\mathbf{Y}$	0.3	1	1

Table 2: Results of application of formula (8).

$C \subseteq \mathbf{X} \times \mathbf{Y}$	$Bel_{XY}(C)$	$m_{XY}(C)$
$\{xy\}$	0.18	0.18
$\{x\bar{y}\}$	0.03	0.03
$\{\bar{x}y\}$	0.12	0.12
$\{\bar{x}\bar{y}\}$	0.02	0.02
$\{x\} \times \mathbf{Y}$	0.3	0.09
$\{\bar{x}\} \times \mathbf{Y}$	0.2	0.06
$\mathbf{X} \times \{y\}$	0.6	0.3
$\mathbf{X} \times \{\bar{y}\}$	0.1	0.05
$\{xy, \bar{x}\bar{y}\}$	1	0.8
$\{x\bar{y}, \bar{x}y\}$	1	0.85
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}\bar{y}\}$	1	−1.08
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}y\}$	1	−0.43
$\mathbf{X} \times \mathbf{Y} \setminus \{x\bar{y}\}$	1	−0.92
$\mathbf{X} \times \mathbf{Y} \setminus \{xy\}$	1	−0.52
$\mathbf{X} \times \mathbf{Y}$	1	0.45

Example 1 Consider two basic assignments m_X and m_Y on $\mathbf{X} = \{x, \bar{x}\}$ $\mathbf{Y} = \{y, \bar{y}\}$ specified in Table 1 together with their beliefs and plausibilities.

Let us compute joint beliefs and plausibilities via formulae

$$\begin{aligned} Bel^{\downarrow K\cup L}(A) &= Bel^{\downarrow K}(A^{\downarrow K}) \cdot Bel^{\downarrow L}(A^{\downarrow L}), && (8)\\ Pl^{\downarrow K\cup L}(A) &= Pl^{\downarrow K}(A^{\downarrow K}) \cdot Pl^{\downarrow L}(A^{\downarrow L}). && (9) \end{aligned}$$

Their values are contained in Tables 2 and 3, respectively, together with the corresponding values of basic assignments computed via (4) (and also (3), in the latter case). As some values of the "joint basic assignments" are negative, which contradicts to (1) it is evident that these models are beyond the framework of evidence theory. $\diamond$

Therefore, it seems that a definition of independence in terms of beliefs or plausibilities would be much

Table 3: Results of application of formula (9).

$C \subseteq \mathbf{X} \times \mathbf{Y}$	$Pl_{XY}(C)$	$Bel_{XY}(C)$	$m_{XY}(C)$
$\{xy\}$	0.72	0	0
$\{x\bar{y}\}$	0.32	0	0
$\{\bar{x}y\}$	0.63	0	0
$\{\bar{x}\bar{y}\}$	0.28	0	0
$\{x\} \times \mathbf{Y}$	0.8	0.3	0.3
$\{\bar{x}\} \times \mathbf{Y}$	0.7	0.2	0.2
$\mathbf{X} \times \{y\}$	0.9	0.6	0.6
$\mathbf{X} \times \{\bar{y}\}$	0.4	0.1	0.1
$\{xy, \bar{x}\bar{y}\}$	1	0	0
$\{x\bar{y}, \bar{x}y\}$	1	0	0
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}\bar{y}\}$	1	0.72	-0.18
$\mathbf{X} \times \mathbf{Y} \setminus \{\bar{x}y\}$	1	0.37	0.07
$\mathbf{X} \times \mathbf{Y} \setminus \{x\bar{y}\}$	1	0.68	-0.12
$\mathbf{X} \times \mathbf{Y} \setminus \{xy\}$	1	0.28	-0.02
$\mathbf{X} \times \mathbf{Y}$	1	1	0.05

more complicated than Definition 1.

3.2 Conditional Noninteractivity

Ben Yaghlane et al. [1] generalised the notion of noninteractivity in the following way: Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. Groups of variables X_K and X_L are *conditionally noninteractive given* X_M *with respect to* m if and only if the equality

$$Q^{\downarrow K\cup L\cup M}(A) \cdot Q^{\downarrow M}(A^{\downarrow M}) = Q^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot Q^{\downarrow L\cup M}(A^{\downarrow L\cup M}) \quad (10)$$

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$.

Let us note that the definition presented in [1] is based on conjunctive Dempster's rule, but the authors proved its equivalence with (10). Let us also note that (10) is a special case of the definition of conditional independence in valuation-based systems[4] introduced by Shenoy [10].

The cited authors proved in [1] that conditional noninteractivity satisfies the so-called graphoid properties.[5]

Nevertheless, this notion of independence does not seem to be appropriate for construction of multidimensional models. As already mentioned by Studený [11], it is not consistent with marginalisation. What that means can be seen from the following definition and illustrated by a simple example from [1] (originally suggested by Studený).

An independence concept is *consistent with marginalisation* iff for arbitrary projective basic assignments (probability distributions, possibility distributions, etc.) m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ there exists a basic assignment (probability distribution, possibility distribution, etc.) on $\mathbf{X}_{K\cup L}$ satisfying this independence concept and having m_1 and m_2 as its marginals.

Example 2 Let X_1, X_2 and X_3 be three binary variables with values in $\mathbf{X}_1 = \{a_1, \bar{a}_1\}, \mathbf{X}_2 = \{a_2, \bar{a}_2\}$, $\mathbf{X}_3 = \{a_3, \bar{a}_3\}$ and m_1 and m_2 be two basic assignments on $\mathbf{X}_1 \times \mathbf{X}_3$ and $\mathbf{X}_2 \times \mathbf{X}_3$ respectively, both of them having only two focal elements:

$$\begin{aligned} m_1(\{(a_1, \bar{a}_3), (\bar{a}_1, \bar{a}_3)\}) &= .5, \\ m_1(\{(a_1, \bar{a}_3), (\bar{a}_1, a_3)\}) &= .5, \\ m_2(\{(a_2, \bar{a}_3), (\bar{a}_2, \bar{a}_3)\}) &= .5, \\ m_2(\{(a_2, \bar{a}_3), (\bar{a}_2, a_3)\}) &= .5. \end{aligned} \quad (11)$$

Since their marginals are projective

$$\begin{aligned} m_1^{\downarrow 3}(\{\bar{a}_3\}) = m_2^{\downarrow 3}(\{\bar{a}_3\}) &= .5, \\ m_1^{\downarrow 3}(\{a_3, \bar{a}_3\}) = m_2^{\downarrow 3}(\{a_3, \bar{a}_3\}) &= .5, \end{aligned}$$

there exists (at least one) common extension of both of them, but none of them is such that it would imply conditional noninteractivity of X_1 and X_2 given X_3. Namely, the application of equality (10) to basic assignments m_1 and m_2 leads to the following values of the joint "basic assignment":

$$\begin{aligned} &\bar{m}(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{a}_3\}) = .25, \\ &\bar{m}(\mathbf{X}_1 \times \{a_2\} \times \{\bar{a}_3\}) = .25, \\ &\bar{m}(\{a_1\} \times \mathbf{X}_2 \times \{\bar{a}_3\}) = .25, \\ &\bar{m}(\{(a_1, a_2, \bar{a}_3), (\bar{a}_1, \bar{a}_2, a_3)\}) = .5, \\ &\bar{m}(\{(a_1, a_2, \bar{a}_3)\}) = -.25, \end{aligned}$$

which is outside of evidence theory. $\diamondsuit$

Therefore, instead of the conditional noninteractivity, in [5] we proposed to use another notion of conditional independence which will be introduced in the following subsection.

3.3 Conditional Independence

Definition 2 Let m be a basic assignment on $\mathbf{X}_N$ and $K, L, M \subset N$ be disjoint, $K \neq \emptyset \neq L$. We say that groups of variables X_K and X_L are *conditionally independent given* X_M *with respect to* m (and denote it by $K \perp\!\!\!\perp L | M\ [m]$), if the equality

$$\begin{aligned} &m^{\downarrow K\cup L\cup M}(A) \cdot m^{\downarrow M}(A^{\downarrow M}) \\ &= m^{\downarrow K\cup M}(A^{\downarrow K\cup M}) \cdot m^{\downarrow L\cup M}(A^{\downarrow L\cup M}) \end{aligned} \quad (12)$$

[4] Nevertheless, in valuation-based systems commonality function is a primitive concept (and basic assignment is derived by formula (5)).

[5] The reader not familiar with graphoid axioms is referred to the beginning of Section 4.

holds for any $A \subseteq \mathbf{X}_{K\cup L\cup M}$ such that $A = A^{\downarrow K\cup M} \otimes A^{\downarrow L\cup M}$, and $m(A) = 0$ otherwise.

Let us note that for $M = \emptyset$ the concept coincides with Definition 1, which enables us to use the term conditional independence. Let us also note that (12) resembles, from the formal point of view, the definition of stochastic conditional independence [7].

The following assertion expresses the fact (already mentioned above) that this concept of conditional independence is consistent with marginalisation. Moreover, it presents a form expressing the joint basic assignment by means of its marginals.

Theorem 1 *Let m_1 and m_2 be projective basic assignments on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. Let us define a basic assignment m on $\mathbf{X}_{K\cup L}$ by the formula*

$$m(A) = \frac{m_1(A^{\downarrow K}) \cdot m_2(A^{\downarrow L})}{m_2^{\downarrow K\cap L}(A^{\downarrow K\cap L})} \tag{13}$$

for $A = A^{\downarrow K} \otimes A^{\downarrow L}$ such that $m_1^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ and $m(A) = 0$ otherwise. Then

$$\begin{aligned} m^{\downarrow K}(B) &= m_1(B), & (14)\\ m^{\downarrow L}(C) &= m_2(C) & (15) \end{aligned}$$

for any $B \in \mathbf{X}_K$ and $C \in \mathbf{X}_L$, respectively, and $(K \setminus L) \perp\!\!\!\perp (L \setminus K)|(K \cap L)\ [m]$. Furthermore, m is the only basic assignment possessing these properties.

Proof. To prove equality (14) we have to show that for any $B \subseteq \mathbf{X}_K$

$$\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} m(A) = m_1(B). \tag{16}$$

Since, due to the definition of m, $m(A) = 0$ for any $A \subseteq \mathbf{X}_{K\cup L}$ for which $A \neq A^{\downarrow K} \otimes A^{\downarrow L}$, we see that

$$\begin{aligned} &\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} m(A) \\ &= \sum_{\substack{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B \\ A=A^{\downarrow K}\otimes A^{\downarrow L}}} m(A) \\ &= \sum_{\substack{C\subseteq \mathbf{X}_L \\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} m(B\otimes C). \end{aligned}$$

To prove formula (16), we have to distinguish between two situations depending on the value of $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})$. If this value is positive then

$$\begin{aligned} &\sum_{A\subseteq \mathbf{X}_{K\cup L}:A^{\downarrow K}=B} m(A) \\ &= \sum_{\substack{C\subseteq \mathbf{X}_L \\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} \frac{m_1(B)\cdot m_2(C)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} \\ &= \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} \sum_{\substack{C\subseteq \mathbf{X}_L \\ C^{\downarrow K\cap L}=B^{\downarrow K\cap L}}} m_2(C) \\ &= \frac{m_1(B)}{m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L})} m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = m_1(B). \end{aligned}$$

If $m_2^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = 0$ then, according to the definition of m, $m(A) = 0$. But $m_1^{\downarrow K\cap L}(B^{\downarrow K\cap L}) = 0$ also, due to the projectivity of m_1 and m_2, and therefore also $m_1(B) = 0$.

The proof of equality (15) is completely analogous due to the projectivity of m_1 and m_2.

Now, let us prove that $X_{K\setminus L}$ and $X_{L\setminus K}$ are conditionally independent given $X_{K\cap L}$ with respect to a basic assignment m defined via (13) for any $A \subseteq \mathbf{X}_{K\cup L}$, such that $A = A^{\downarrow K} \otimes A^{\downarrow L}$ and $m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$ and $m(A) = 0$ otherwise. First let us show, that

$$\begin{aligned} m^{\downarrow K\cup L}(A) \cdot m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) \\ = m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L}), \end{aligned} \tag{17}$$

holds for all $A = A^{\downarrow K} \otimes A^{\downarrow L}$. If $m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$, then multiplying both sides of the formula (13) by $m^{\downarrow K\cap L}(A^{\downarrow K\cap L})$ we obtain the equality (17), as (14) and (15) are satisfied and $m^{\downarrow K\cup L}(A) = m(A)$ for any $A \subseteq \mathbf{X}_{K\cup L}$. If $m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$ then $m^{\downarrow L}(A^{\downarrow L}) = 0$ also, and therefore both sides of (17) equal 0. If $A \neq A^{\downarrow K} \otimes A^{\downarrow L}$, then $m(A) = 0$ by assumption.

Let $X_{K\setminus L}$ and $X_{L\setminus K}$ be conditionally independent given $X_{K\cap L}$ with respect to a basic assignment m, and $A \subseteq \mathbf{X}_{K\cup L}$ be such that $A = A^{\downarrow K} \otimes A^{\downarrow L}$ and $m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) > 0$. Then (17) holds and therefore

$$m^{\downarrow K\cup L}(A) = \frac{m^{\downarrow K}(A^{\downarrow K}) \cdot m^{\downarrow L}(A^{\downarrow L})}{m^{\downarrow K\cap L}(A^{\downarrow K\cap L})},$$

i.e., (13) holds due to (14) and (15) and the fact that $m^{\downarrow K\cup L}(A) = m(A)$ for any $A \subseteq \mathbf{X}_{K\cup L}$. If $m^{\downarrow K\cap L}(A^{\downarrow K\cap L}) = 0$ then also $m^{\downarrow K}(A^{\downarrow K}) = 0, m^{\downarrow K\cap L}(A^{\downarrow L}) = 0$ and $m(A) = 0$. If $A \neq A^{\downarrow K} \otimes A^{\downarrow L}$ then $m(A) = 0$, which directly follows from Definition 2. $\square$

Let us close this section by demonstrating application of the conditional independence notion (and Theorem 1) to Example 2.

Example 2 *(Continued)* Let us go back to the problem of finding a common extension of basic assignments m_1 and m_2 defined by (11). Theorem 1 says that for basic assignment m defined as follows

$$m(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{a}_3\}) = .5,$$
$$m(\{(a_1, a_2, \bar{a}_3), (\bar{a}_1, \bar{a}_2, a_3)\}) = .5,$$

variables X_1 and X_3 are conditionally independent given X_2. $\diamondsuit$

4 Formal Properties of Conditional Independence

Among the properties satisfied by the ternary relation $K \perp\!\!\!\perp L|M\ [m]$, the following are of principal importance:

(A1) $K \perp\!\!\!\perp L|M\ [m] \ \Rightarrow\ L \perp\!\!\!\perp K|M\ [m]$,

(A2) $K \perp\!\!\!\perp L \cup M|I\ [m] \ \Rightarrow\ K \perp\!\!\!\perp M|I\ [m]$,

(A3) $K \perp\!\!\!\perp L \cup M|I\ [m] \ \Rightarrow\ K \perp\!\!\!\perp L|M \cup I\ [m]$,

(A4) $K \perp\!\!\!\perp L|M \cup I\ [m] \ \wedge\ K \perp\!\!\!\perp M|I\ [m] \implies K \perp\!\!\!\perp L \cup M|I\ [m]$,

(A5) $K \perp\!\!\!\perp L|M \cup I\ [m] \ \wedge\ K \perp\!\!\!\perp M|L \cup I\ [m] \implies K \perp\!\!\!\perp L \cup M|I\ [m]$.

Let us recall that stochastic conditional independence satisfies the so-called *semigraphoid* properties (A1)–(A4) for any probability distribution, while axiom (A5) is satisfied only for strictly positive probability distributions. Conditional noninteractivity referred to in Section 3.2, on the other hand, satisfies axioms (A1)–(A5) for general basic assignment m, as proven in [1].

Before formulating an important theorem justifying the definition of conditional independence, let us formulate and prove an assertion concerning set extensions.

Lemma 2 *Let $K \cap L \subseteq M \subseteq L \subseteq N$. Then, for any $C \subseteq \mathbf{X}_{K\cup L}$, condition* (a) *holds if and only if both conditions* (b) *and* (c) *hold true.*

(a) $C = C^{\downarrow K} \otimes C^{\downarrow L}$;

(b) $C^{\downarrow K\cup M} = C^{\downarrow K} \otimes C^{\downarrow M}$;

(c) $C = C^{\downarrow K\cup M} \otimes C^{\downarrow L}$.

Proof. Before proving the required implications let us note that $C \subseteq C^{\downarrow K} \otimes C^{\downarrow L}$, therefore $C = C^{\downarrow K} \otimes C^{\downarrow L}$ is equivalent to

$$\forall x \in \mathbf{X}_{K\cup L} \left(x^{\downarrow K} \in C^{\downarrow K}\ \&\ \ x^{\downarrow L} \in C^{\downarrow L} \Longrightarrow x \in C\right).$$

(a) $\Longrightarrow$ (b). Consider $x \in \mathbf{X}_{K\cup M}$, such that $x^{\downarrow K} \in C^{\downarrow K}$ and $x^{\downarrow M} \in C^{\downarrow M}$. Since $x^{\downarrow M} \in C^{\downarrow M}$ there must exist (at least one) $y \in C^{\downarrow L}$, for which $y^{\downarrow M} = x^{\downarrow M}$. Now construct $z \in \mathbf{X}_{K\cup L}$ for which $z^{\downarrow K} = x^{\downarrow K}$ and $z^{\downarrow L} = y$ (it is possible because $y^{\downarrow M} = x^{\downarrow M}$). From this construction we see that $z^{\downarrow K\cup M} = x$. Therefore $z^{\downarrow K} = x^{\downarrow K} \in C^{\downarrow K}$ and $z^{\downarrow L} = y \in C^{\downarrow L}$ from which, because we assume that (a) holds, we get that $z \in C$, and therefore also $x = z^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$.

(a) $\Longrightarrow$ (c). Consider now $x \in \mathbf{X}_{K\cup L}$, with projections $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$ and $x^{\downarrow L} \in C^{\downarrow L}$. From $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$ we immediately get that $x^{\downarrow K} \in C^{\downarrow K}$, which in combination with $x^{\downarrow L} \in C^{\downarrow L}$ (due to the assumption (a)) yields that $x \in C$.

(b) & (c) $\Longrightarrow$ (a). Consider $x \in \mathbf{X}_{K\cup L}$ such that $x^{\downarrow K} \in C^{\downarrow K}$ and $x^{\downarrow L} \in C^{\downarrow L}$. From the latter property one also gets $x^{\downarrow M} \in C^{\downarrow M}$, which, in combination with $x^{\downarrow K} \in C^{\downarrow K}$ gives, because (b) holds true, that $x^{\downarrow K\cup M} \in C^{\downarrow K\cup M}$. And the last property in combination with $x^{\downarrow L} \in C^{\downarrow L}$ yields the required $x \in C$. $\square$

Since all I, K, L, M are disjoint, we will omit symbol $\cup$ and use, for example, KLM instead of $K \cup L \cup M$ in the rest of the paper.

Theorem 2 *Conditional independence satisfies (A1)–(A4).*

Proof. **ad (A1)** The validity of the implication immediately follows from the commutativity of multiplication.

ad (A2) The assumption $K \perp\!\!\!\perp LM|I\ [m]$ means that for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI} \otimes A^{\downarrow LMI}$ the equality

$$m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) = m^{\downarrow KI}(A^{\downarrow KI}) \cdot m^{\downarrow LMI}(A^{\downarrow LMI}) \tag{18}$$

holds, and if $A \neq A^{\downarrow KI} \otimes A^{\downarrow LMI}$, then $m(A) = 0$. Let us prove first that also for any $B \subseteq \mathbf{X}_{KMI}$ such that $B = B^{\downarrow KI} \otimes B^{\downarrow MI}$, the equality

$$m^{\downarrow KMI}(B) \cdot m^{\downarrow I}(B^{\downarrow I}) = m^{\downarrow KI}(B^{\downarrow KI}) \cdot m^{\downarrow MI}(B^{\downarrow MI}) \tag{19}$$

is valid. To do so, let us compute

$$\begin{aligned} m^{\downarrow KMI}(B) \cdot m^{\downarrow I}(B^{\downarrow I}) &= \sum_{\substack{A \subseteq \mathbf{X}_{KLMI} \\ A^{\downarrow KMI} = B^{\downarrow KI} \otimes B^{\downarrow MI}}} m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) \\ &= \sum_{\substack{A \subseteq \mathbf{X}_{KLMI} \\ A = A^{\downarrow KI} \otimes A^{\downarrow LMI} \\ A^{\downarrow KMI} = B^{\downarrow KI} \otimes B^{\downarrow MI}}} m^{\downarrow KLMI}(A) \cdot m^{\downarrow I}(A^{\downarrow I}) \end{aligned}$$

$$= \sum_{\substack{A\subseteq \mathbf{X}_{KLMI}\\ A=A^{\downarrow KI}\otimes A^{\downarrow LMI}\\ A^{\downarrow KMI}=B^{\downarrow KI}\otimes B^{\downarrow MI}}} m^{\downarrow KI}(A^{\downarrow KI})\cdot m^{\downarrow LMI}(A^{\downarrow LMI})$$
$$= m^{\downarrow KI}(A^{\downarrow KI})\cdot \sum_{\substack{C\subseteq \mathbf{X}_{LMI}\\ C^{\downarrow MI}=B^{\downarrow MI}}} m^{\downarrow LMI}(C)$$
$$= m^{\downarrow KI}(B^{\downarrow KI})\cdot m^{\downarrow MI}(B^{\downarrow MI}),$$

as

$$\begin{aligned} m^{\downarrow I}(B^{\downarrow I}) &= m^{\downarrow I}(A^{\downarrow I}),\\ m^{\downarrow KI}(B^{\downarrow KI}) &= m^{\downarrow KI}(A^{\downarrow KI}). \end{aligned}$$

So, to finish this step we still must prove that if $B \neq B^{\downarrow KI}\otimes B^{\downarrow MI}$ then $m^{\downarrow KMI}(B)=0$. Also, in this case

$$m^{\downarrow KMI}(B) = \sum_{\substack{A\subseteq \mathbf{X}_{KLMI}\\ A^{\downarrow KMI}=B}} m^{\downarrow KLMI}(A),$$

but since $B = A^{\downarrow KMI} \neq A^{\downarrow KI}\otimes A^{\downarrow MI}$ then, because of Lemma 2, also $A \neq A^{\downarrow KI}\otimes A^{\downarrow LMI}$ for any A such that $A^{\downarrow KMI}=B$. But for these A's, $m^{\downarrow KLMI}(A)=0$ and therefore also $m^{\downarrow KMI}(B)=0$.

ad (A3) Again, let us suppose validity of $K \perp\!\!\!\perp LM|I\ [m]$, i.e., for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI}\otimes A^{\downarrow LMI}$ equality (18) holds, and $m^{\downarrow KLMI}(A) = 0$ otherwise. Our aim is to prove that for any $C \subseteq \mathbf{X}_{KLMI}$ such that $C = C^{\downarrow KMI}\otimes C^{\downarrow LMI}$, the equality

$$\begin{aligned} &m^{\downarrow KLMI}(C)\cdot m^{\downarrow MI}(C^{\downarrow MI})\\ &\quad = m^{\downarrow KMI}(C^{\downarrow KMI})\cdot m^{\downarrow LMI}(C^{\downarrow LMI}) \end{aligned} \tag{20}$$

is satisfied as well, and $m^{\downarrow KLMI}(C)=0$ otherwise. Let C be such that $m^{\downarrow I}(C^{\downarrow I})>0$. Since we assume that $K \perp\!\!\!\perp LM|I\ [m]$ holds, we have for such a C

$$\begin{aligned} &m^{\downarrow KLMI}(C)\cdot m^{\downarrow I}(C^{\downarrow I})\\ &\quad = m^{\downarrow KI}(C^{\downarrow KI})\cdot m^{\downarrow LMI}(C^{\downarrow LMI}), \end{aligned}$$

and therefore we can compute

$$\begin{aligned} &m^{\downarrow KLMI}(C)\cdot m^{\downarrow MI}(C^{\downarrow MI})\\ &= m^{\downarrow KLMI}(C)\cdot m^{\downarrow I}(C^{\downarrow I})\cdot \frac{m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})}\\ &= m^{\downarrow KI}(C^{\downarrow KI})\cdot m^{\downarrow LMI}(C^{\downarrow LMI})\cdot \frac{m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})}\\ &= \frac{m^{\downarrow KI}(C^{\downarrow KI})\cdot m^{\downarrow MI}(C^{\downarrow MI})}{m^{\downarrow I}(C^{\downarrow I})}\cdot m^{\downarrow LMI}(C^{\downarrow LMI})\\ &= m^{\downarrow KMI}(C^{\downarrow KMI})\cdot m^{\downarrow LMI}(C^{\downarrow LMI}), \end{aligned}$$

where the last equality is satisfied due to (A2) and the fact that $m^{\downarrow I}(C^{\downarrow I})>0$. If $m^{\downarrow I}(C^{\downarrow I})=0$ then also $m^{\downarrow KMI}(C^{\downarrow KMI})=0$, $m^{\downarrow LMI}(C^{\downarrow LMI})=0$ and $m^{\downarrow KLMI}(C)=0$ and therefore (20) also holds true.

It remains to be proven that $m(C)=0$ for all $C \neq C^{\downarrow KMI}\otimes C^{\downarrow LMI}$. But in this case, as a consequence of Lemma 2, also $C \neq C^{\downarrow KI}\otimes C^{\downarrow LMI}$, and therefore $m(C)=0$ due to the assumption.

ad (A4) First, supposing $K \perp\!\!\!\perp L|MI\ [m]$ and $K \perp\!\!\!\perp M|I\ [m]$, let us prove that for any $A \subseteq \mathbf{X}_{KLMI}$ such that $A = A^{\downarrow KI}\otimes A^{\downarrow LMI}$, the equality (18) holds. Since from $A = A^{\downarrow KI}\otimes A^{\downarrow LMI}$ it also follows due to Lemma 2 that $A = A^{\downarrow KMI}\otimes A^{\downarrow LMI}$, and therefore (since we assume $K \perp\!\!\!\perp L|MI\ [m]$)

$$\begin{aligned} &m^{\downarrow KLMI}(A)\cdot m^{\downarrow MI}(A^{\downarrow MI})\\ &\quad = m^{\downarrow KMI}(A^{\downarrow KMI})\cdot m^{\downarrow LMI}(A^{\downarrow LMI}). \end{aligned} \tag{21}$$

Now, let us further assume that $m^{\downarrow MI}(A^{\downarrow MI})>0$ (and thus also $m^{\downarrow I}(A^{\downarrow I})>0$). Since from $A = A^{\downarrow KI}\otimes A^{\downarrow LMI}$ Lemma 2 implies $A^{\downarrow KMI} = A^{\downarrow KI}\otimes A^{\downarrow MI}$, one gets from $K \perp\!\!\!\perp M|I\ [m]$ that

$$\begin{aligned} &m^{\downarrow KMI}(A^{\downarrow KMI})\cdot m^{\downarrow I}(A^{\downarrow I})\\ &\quad = m^{\downarrow KI}(A^{\downarrow KI})\cdot m^{\downarrow MI}(A^{\downarrow MI}), \end{aligned}$$

which, in combination with equality (22), yields

$$\begin{aligned} &m^{\downarrow KLMI}(A)\cdot m^{\downarrow MI}(A^{\downarrow MI})\\ &\quad = \frac{m^{\downarrow KI}(A^{\downarrow KI})\cdot m^{\downarrow MI}(A^{\downarrow MI})}{m^{\downarrow I}(A^{\downarrow I})}\cdot m^{\downarrow LMI}(A^{\downarrow LMI}), \end{aligned}$$

which is (for positive $m^{\downarrow MI}(A^{\downarrow MI})$) evidently equivalent to (18). If, on the other hand, $m^{\downarrow MI}(A^{\downarrow MI})=0$, then also $m^{\downarrow LMI}(A^{\downarrow LMI})=0$ and $m^{\downarrow KLMI}(A)=0$ and both sides of (18) equal 0.

It remains to prove that $m^{\downarrow KLMI}(A) = 0$ for all $A \neq A^{\downarrow KI}\otimes A^{\downarrow LMI}$. But $m^{\downarrow KLMI}(A)=0$ because Lemma 2 says that either $A \neq A^{\downarrow KMI}\otimes A^{\downarrow LMI}$ (and therefore $m^{\downarrow KLMI}(A)=0$ from the assumption that $K \perp\!\!\!\perp L|MI\ [m]$) or $A^{\downarrow KMI} \neq A^{\downarrow KI}\otimes A^{\downarrow MI}$ (and then $m^{\downarrow KMI}(A^{\downarrow KMI}) = 0$ due to the assumption $K \perp\!\!\!\perp M|I\ [m]$), and therefore also $m^{\downarrow KLMI}(A)=0$). □

Analogous to a probabilistic case, conditional independence $K \perp\!\!\!\perp L|MI\ [m]$ does not generally satisfy (A5), as can be seen from the following simple example.

Example 3 Let X_1, X_2 and X_3 be three variables with values in $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ respectively, $\mathbf{X}_i = \{a_i, \bar{a}_i\}, i = 1,2,3$, and their joint basic assignment is defined as follows:

$$\begin{aligned} &m(\{(x_1,x_2,x_3\}) = \tfrac{1}{16},\\ &m(\mathbf{X}_1\times\mathbf{X}_2\times\mathbf{X}_3) = \tfrac{1}{2}, \end{aligned}$$

for $x_i = a_i, \bar{a}_i$, values of m on the remaining sets being 0. Its marginal basic assignments on $\mathbf{X}_1\times\mathbf{X}_2, \mathbf{X}_1\times$

$\mathbf{X}_3, \mathbf{X}_2 \times \mathbf{X}_3$ and $\mathbf{X}_i, i = 1, 2, 3$ are

$$m^{\downarrow 12}(\{x_1, x_2\}) = \tfrac{1}{8},$$
$$m^{\downarrow 12}(\mathbf{X}_1 \times \mathbf{X}_2) = \tfrac{1}{2},$$

$$m^{\downarrow 13}(\{x_1, x_3\}) = \tfrac{1}{8},$$
$$m^{\downarrow 13}(\mathbf{X}_1 \times \mathbf{X}_3) = \tfrac{1}{2},$$
$$m^{\downarrow 23}(\{x_2, x_3\}) = \tfrac{1}{8},$$
$$m^{\downarrow 23}(\mathbf{X}_2 \times \mathbf{X}_3) = \tfrac{1}{2},$$

and

$$m^{\downarrow i}(x_i) = \tfrac{1}{4},$$
$$m^{\downarrow i}(\mathbf{X}_i) = \tfrac{1}{2},$$

respectively. It is easy (but somewhat time-consuming) to check that

$$m(A^{\downarrow 13} \otimes A^{\downarrow 23}) \cdot m^{\downarrow 3}(A^{\downarrow 3}) = m^{\downarrow 13}(A^{\downarrow 13}) \cdot m^{\downarrow 23}(A^{\downarrow 23})$$

and

$$m(A^{\downarrow 12} \otimes A^{\downarrow 23}) \cdot m^{\downarrow 2}(A^{\downarrow 2}) = m^{\downarrow 12}(A^{\downarrow 12}) \cdot m^{\downarrow 23}(A^{\downarrow 23}),$$

the values of remaining sets being zero, while e.g.

$$m(\{(a_1, \bar{a}_2, \bar{a}_3)\}) = \frac{1}{16} \neq \frac{1}{4} \cdot \frac{1}{8} = m^{\downarrow 1}(\{a_1\}) \cdot m^{\downarrow 23}(\{(\bar{a}_2, \bar{a}_3)\}),$$

i.e., $\{1\} \perp\!\!\!\perp \{2\}|\{3\}\ [m]$ and $\{1\} \perp\!\!\!\perp \{3\}|\{2\}\ [m]$ hold, but $\{1\} \perp\!\!\!\perp \{2,3\}|\emptyset\ [m]$ does not. $\diamondsuit$

This fact perfectly corresponds to the properties of stochastic conditional independence. In probability theory (A5) need not be satisfied if the joint probability distribution is not strictly positive. But the counterpart of strict positivity of probability distribution for basic assignments is not straightforward. It is evident that it does not mean strict positivity on all subsets of the frame of discernment in question — in this case variables are not (conditionally) independent (cf. Definitions 1 and 2). On the other hand, it can be seen from Example 3 that strict positivity on singletons is not sufficient (and, surprisingly, as we shall see later, also not necessary). At present we are able to formulate Theorem 3. To prove it, we need the following lemma.

Lemma 3 *Let K, L, M be disjoint subsets of N, $K, L \neq \emptyset$ and m be a joint basic assignment on $\mathbf{X}_N$. Then the following statements are equivalent:*

(i) $K \perp\!\!\!\perp L|M\ [m]$.

(ii) *The basic assignment $m^{\downarrow KLM}$ on $\mathbf{X}_{KLM}$ has for $A = A^{\downarrow KM} \otimes A^{\downarrow LM}$ the form*

$$m^{\downarrow KLM}(A) = f_1(A^{\downarrow KM}) \cdot f_2(A^{\downarrow LM}), \qquad (22)$$

where f_1 and f_2 are set functions on $\mathbf{X}_{KM}$ and $\mathbf{X}_{LM}$, respectively, and $m(A) = 0$ otherwise.

Proof. Let (i) be satisfied. Then for any $A = A^{\downarrow KM} \otimes A^{\downarrow LM}$ we have

$$m^{\downarrow KLM}(A) \cdot m^{\downarrow M}(A^{\downarrow M}) = m^{\downarrow KM}(A^{\downarrow KM}) \cdot m^{\downarrow LM}(A^{\downarrow LM}).$$

If $m^{\downarrow M}(A^{\downarrow M}) > 0$, we may divide both sides of the above equality by it and we obtain

$$m^{\downarrow KLM}(A) = \frac{m^{\downarrow KM}(A^{\downarrow KM}) \cdot m^{\downarrow LM}(A^{\downarrow LM})}{m^{\downarrow M}(A^{\downarrow M})}.$$

Therefore (ii) is obviously fulfilled, e.g. for

$$f_1(A^{\downarrow KM}) = m^{\downarrow K\cup M}(A^{\downarrow K\cup M})$$

and

$$f_2(A^{\downarrow LM}) = \frac{m^{\downarrow L\cup M}(A^{\downarrow L\cup M})}{m^{\downarrow M}(A^{\downarrow M})}.$$

If, on the other hand, $m^{\downarrow M}(A^{\downarrow M}) = 0$, then also $m^{\downarrow KM}(A^{\downarrow KM}) = 0$, $m^{\downarrow LM}(A^{\downarrow LM}) = 0$ and $m^{\downarrow KLM}(A^{\downarrow KLM}) = 0$, and therefore (22) trivially holds. To finish the proof of this implication we must prove that $m(A) = 0$ if $A \neq A^{\downarrow KM} \otimes A^{\downarrow LM}$, but it follows directly from the definition.

Let (ii) be satisfied. Then denoting

$$f_1^{\downarrow M}(A^{\downarrow M}) = \sum_{\substack{C \subseteq X_{KM} \\ C^{\downarrow M} = A^{\downarrow M}}} f_1(C)$$

and

$$f_2^{\downarrow M}(A^{\downarrow M}) = \sum_{\substack{C \subseteq X_{LM} \\ C^{\downarrow M} = A^{\downarrow M}}} f_2(C),$$

we have

$$\begin{aligned} m^{\downarrow KM}(A^{\downarrow KM}) &= \sum_{\substack{C \subseteq X_{KLM} \\ C^{\downarrow KM} = A^{\downarrow KM}}} m^{\downarrow KLM}(C) \\ &= \sum_{\substack{C \subseteq X_{KLM} \\ C^{\downarrow KM} = A^{\downarrow KM}}} f_1(C^{\downarrow KM}) \cdot f_2(C^{\downarrow LM}) \\ &= f_1(A^{\downarrow KM}) \cdot \sum_{\substack{D \subseteq X_{LM} \\ D^{\downarrow M} = A^{\downarrow M}}} f_2(D) \\ &= f_1(A^{\downarrow KM}) \cdot f_2^{\downarrow M}(A^{\downarrow M}) \end{aligned}$$

and similarly

$$m^{\downarrow LM}(A^{\downarrow LM}) \qquad = f_2(A^{\downarrow LM}) \cdot f_1^{\downarrow M}(A^{\downarrow M}).$$

Therefore

$$\begin{aligned}
&m^{\downarrow M}(A^{\downarrow M}) \\
&\quad = \sum_{\substack{C \subseteq X_{KLM} \\ C^{\downarrow M} = A^{\downarrow M}}} m^{\downarrow KLM}(C) = \sum_{\substack{D \subseteq X_{KM} \\ D^{\downarrow M} = A^{\downarrow M}}} m^{\downarrow KM}(D) \\
&\quad = \sum_{\substack{D \subseteq X_{KM} \\ D^{\downarrow M} = A^{\downarrow M}}} f_1(D^{\downarrow KM}) \cdot f_2^{\downarrow M}(D^{\downarrow M}) \\
&\quad = f_2^{\downarrow M}(A^{\downarrow M}) \cdot \sum_{\substack{D \subseteq X_{KM} \\ D^{\downarrow M} = A^{\downarrow M}}} f_1(D^{\downarrow KM}) \\
&\quad = f_2^{\downarrow M}(A^{\downarrow M}) \cdot f_1^{\downarrow M}(A^{\downarrow M}).
\end{aligned}$$

Hence, multiplying both sides of (22) by $m^{\downarrow M}(A^{\downarrow M})$ one has

$$\begin{aligned}
&m(A) \cdot m^{\downarrow M}(A^{\downarrow M}) \\
&\quad = f_1(A^{\downarrow KM}) \cdot f_2(A^{\downarrow LM}) \cdot f_1^{\downarrow M}(A^{\downarrow M}) \cdot f_2^{\downarrow M}(A^{\downarrow M}) \\
&\quad = f_1(A^{\downarrow KM}) \cdot f_2^{\downarrow M}(A^{\downarrow M}) \cdot f_2(A^{\downarrow LM}) \cdot f_1^{\downarrow M}(A^{\downarrow M}) \\
&\quad = m^{\downarrow KM}(A^{\downarrow KM}) \cdot m^{\downarrow LM}(A^{\downarrow LM}),
\end{aligned}$$

i.e., (i) holds (as $m(A) = 0$ if $A \neq A^{\downarrow KM} \otimes A^{\downarrow LM}$ by assumption). □

Theorem 3 *Let m be a basic assignment on $\mathbf{X}_N$ such that $m(A) > 0$ if and only if $A = \times_{i \in N} A_i$, where A_i is a focal element on $\mathbf{X}_i$. Then (A5) is satisfied.*

Proof. Let $K \perp\!\!\!\perp L|MI\ [m]$ and $K \perp\!\!\!\perp M|LI\ [m]$. Then by Lemma 3 there exist functions f_1, f_2, g_1 and g_2 such that

$$\begin{aligned}
m^{\downarrow KLMI}(A) &= f_1(A^{\downarrow KMI}) \cdot f_2(A^{\downarrow LMI}) \\
m^{\downarrow KLMI}(A) &= g_1(A^{\downarrow KLI}) \cdot g_2(A^{\downarrow LMI})
\end{aligned}$$

for any $A = A^{\downarrow KMI} \otimes A^{\downarrow LMI}$ and any $A = A^{\downarrow KLI} \otimes A^{\downarrow LMI}$, respectively.

If $m(A) > 0$ we can write

$$f_1(A^{\downarrow KMI}) = \frac{g_1(A^{\downarrow KLI}) \cdot g_2(A^{\downarrow LMI})}{f_2(A^{\downarrow LMI})}. \tag{23}$$

Let us note, that if $m(A) > 0$, then by assumption $A = \times_{i \in N} A_i$ and therefore it can be written as $A = A^{\downarrow K} \times A^{\downarrow L} \times A^{\downarrow M} \times A^{\downarrow I}$. Hence (23) may be rewritten into the form

$$\begin{aligned}
&f_1(A^{\downarrow K} \times A^{\downarrow M} \times A^{\downarrow I}) \qquad (24) \\
&\quad = \frac{g_1(A^{\downarrow K} \times A^{\downarrow L} \times A^{\downarrow I}) \cdot g_2(A^{\downarrow L} \times A^{\downarrow M} \times A^{\downarrow I})}{f_2(A^{\downarrow L} \times A^{\downarrow M} \times A^{\downarrow I})}.
\end{aligned}$$

Let us choose $B \subseteq \mathbf{X}_L$ such that $B = A^{\downarrow L}$. Then (24) can be written in the form

$$f_1(A^{\downarrow K} \times A^{\downarrow M} \times A^{\downarrow I}) = h_1(A^{\downarrow KI}) \cdot h_2(A^{\downarrow MI}),$$

where

$$\begin{aligned}
h_1(A^{\downarrow KI}) &= g_1(A^{\downarrow K} \times B \times A^{\downarrow I}), \\
h_2(A^{\downarrow MI}) &= \frac{g_2(B \times A^{\downarrow M} \times A^{\downarrow I})}{f_2(B \times A^{\downarrow M} \times A^{\downarrow I})}.
\end{aligned}$$

Therefore

$$\begin{aligned}
m^{\downarrow KLMI}(A) &= h_1(A^{\downarrow KI}) \cdot h_2(A^{\downarrow MI}) \cdot f_2(A^{\downarrow LMI}) \\
&= h_1(A^{\downarrow KI}) \cdot h_2'(A^{\downarrow LMI}). \qquad (25)
\end{aligned}$$

Now, we shall prove that (25) is valid also for $A = A^{\downarrow KI} \otimes A^{\downarrow LMI}$ such that $m(A) = 0$. The validity of

$$\begin{aligned}
&m^{\downarrow KLMI}(A) \cdot m^{\downarrow M}(A^{\downarrow MI}) \\
&\qquad = m^{\downarrow KMI}(A^{\downarrow KMI}) \cdot m^{\downarrow LMI}(A^{\downarrow LMI})
\end{aligned}$$

for $A = A^{\downarrow KMI} \otimes A^{\downarrow LMI}$ implies that at least one of $m^{\downarrow LMI}(A^{\downarrow LMI})$ and $m^{\downarrow KMI}(A^{\downarrow KMI})$ must also equal zero. In the first case, (25) holds for $h_2'(A^{\downarrow LMI}) = m^{\downarrow LMI}(A^{\downarrow LMI})$ and h_1 arbitrary.

If, on the other hand, $m^{\downarrow LMI}(A^{\downarrow LMI}) > 0$, then $m^{\downarrow KMI}(A^{\downarrow KMI})$ must equal zero. We also must prove that in this case $m^{\downarrow KI}(A^{\downarrow KI}) = 0$, from which (25) immediately follows. To prove it, let us suppose the contrary. Since $A^{\downarrow KMI} = \times_{i \in KMI} A_i$, there must exist at least one $j \in M$ such that A_j is not a focal element on $\mathbf{X}_j$. From this fact it follows that also $m^{\downarrow LMI}(A^{\downarrow LMI}) = 0$, as $m^{\downarrow j}(A_j)$ is marginal to $m^{\downarrow LMI}(A^{\downarrow LMI})$, and it contradicts the assumption that $m^{\downarrow LMI}(A^{\downarrow LMI}) > 0$.

It remains to be proven that $m(A) = 0$ if $A \neq A^{\downarrow KI} \otimes A^{\downarrow LMI}$. But it follows directly from the assumption, as $m(A) > 0$ only for $A = \times_{i \in N} A_i$. □

Example 3 suggests that the assumption of positivity of $m(A)$ on any $A = \times_{i \in N} A_i$, where A_i is a focal element on $\mathbf{X}_i$, is substantial. On the other hand, the assumption that $m(A) = 0$ otherwise may not be so substantial and (A5) may hold for more general bodies of evidence than those characterised by the assumption of Theorem 3 (at present we are not able to find a counterexample).

Let us note that, for Bayesian basic assignments, assumption of Theorem 3 seems to be more general than that of strict positivity of the probability distribution. But the generalisation is of no practical consequence — if probability of a marginal value is equal to zero, than this value may be omitted.

5 Summary and Conclusions

This paper started with a brief discussion, based on recently published results, why random sets independence is the most appropriate independence concept (from the viewpoint of multidimensional models) in evidence theory. Then we compared two generalisations of random sets independence — conditional noninteractivity and the new concept of conditional independence. We showed that, although from the viewpoint of formal properties satisfied by these concepts, conditional noninteractivity seems to be slightly better than conditional independence, from the viewpoint of multidimensional models the latter is superior to the former, as it is consistent with marginalisation.

There is still a problem to be solved, namely: can the sufficient condition be weakened while keeping the validity of (A5)?

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