# RAMSEY GROWTH MODEL UNDRE UNCERTAINTY

Karel Sladký

Department of Econometrics
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
sladky@utia.cas.cz

**Abstract.** We consider an extended version of the Ramsey growth model under stochastic uncertainties modelled by Markov processes. In contrast to the standard model we assume that splitting of production between consumption and capital accumulation is influenced by some random factor, e.g. governed by transition probabilities depending on the current value of the accumulated capital, along with possible additional interventions of the decision maker. Basic properties of the standard formulation are summarized and compared with their counterparts in the extended version. Finding optimal policy of the extended model can be either performed by additional compensation of the (random) disturbances or can be also formulated as finding optimal control of a Markov decision process.
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## 1 Classical Ramsey Growth Model

The heart of the seminal paper of F. Ramsey [6] on mathematical theory of saving is an economy producing output from labour and capital and the task is to decide how to divide production between consumption and capital accumulation to maximize the global utility of the consumption. Ramsey's original results from 1928 were revisited and significantly extended only after almost thirty years and at present the Ramsey model can be considered as one of the three most significant tools for the dynamic general equilibrium model in modern macroeconomics. In [6] the problem was considered in continuous-time setting, Ramsey suggested some variational methods for finding an optimal policy how to divide the production between consumption and capital accumulation. However, in the recent literature on economic growth models (see e.g. Le Van and Dana [2] or Majumdar, Mitra, and Nishimura [4]) the discrete-time formulation is preferred.

The Ramsey growth model in the discrete-time setting can be formulated as follows:

We consider at discrete time points $t = 0, 1, \ldots$, an economy in which at each time $t$ there are $L_t$ (merely identical) consumers with consumption $c_t$ per individual. The number of consumers grow very slowly in time, i.e. $L_t = L_0(1+n)^t$ for $t$ with $\alpha := (1+n) \approx 1$. The economy produces at time $t$ gross output $Y_t$ using only two inputs: capital $K_t$ and labour $L_t = L_0(1+n)^t$. A production function $F(K_t, L_t)$ relates input to output, i.e.

$$Y_t = F(K_t, L_t) \quad \text{with} \quad K_0 > 0,\ L_0 > 0 \ \text{ given.} \tag{1}$$

We assume that that $F(\cdot, \cdot)$ is a homogeneous function of degree one, i.e. $F(\theta K, \theta L) = \theta F(K, L)$ for any $\theta \in \mathbb{R}$.

The output must be split between consumption $C_t = c_t L_t$ and gross investment $I_t$, i.e.

$$C_t + I_t \le Y_t = F(K_t, L_t). \tag{2}$$

Investment $I_t$ is used in whole (along with the depreciated capital $K_t$) for the capital at the next time point $t+1$. In addition, capital is assumed to depreciate at a constant rate $\delta \in (0,1)$, so capital related to gross investment at time $t+1$ is equal to

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{3}$$

Preferences over consumption of a single consumer (resp. the considered $L_t$ consumers) for the discount factor $\beta \in (0,1)$ and the considered time horizon $T$ are expressed by means of utility function $u(\cdot)$ as

$$U^\beta(k_0,T) = \sum_{t=0}^{T} \beta^t u(c_t) \quad (\text{resp. } \bar{U}^\beta(k_0,T) = L_0 \sum_{t=0}^{T} (\alpha\beta)^t u(c_t)). \tag{4}$$

The problem is to find the rule how to split production between consumption and capital accumulation that maximizes global utility $U^\beta(k_0,T)$ of the consumers for a finite or infinite time horizon $T$.

In what follows let $k_t := K_t/L_t$ be the capital per consumer at time $t$, and similarly let $y_t := Y_t/L_t$ be the per capita output at time $t$. Recalling that the production function $F(\cdot,\cdot)$ is assumed to be homogeneous of degree one, then $f(k_t) := F(k_t,1)$ denotes the per capita production per unit time. In virtue of (2), (3) we get

$$c_t + (1+n)k_{t+1} - (1-\delta)k_t \le y_t = f(k_t), \tag{5}$$

and if we set for simplicity $\alpha \equiv (1+n) = 1$ then (5) can be written as

$$c_t + k_{t+1} + (1-\delta)k_t \le y_t = f(k_t). \tag{6}$$

In the above formulation we assume that the per capita production function $f(k)$ and the consumption function $u(c)$ fulfil some standard assumptions on production and consumption functions, in particular, that:

**AS 1.** The function $u(c) : \mathbb{R}^+ \to \mathbb{R}^+$ is twice continuously differentiable and satisfies $u(0) = 0$. Moreover, $u(c)$ is strictly increasing and concave (i.e., its derivatives satisfy $u'(\cdot) > 0$ and $u''(\cdot) < 0$) with $u'(0) = +\infty$ (so-called Inada Condition).

**AS 2.** The function $f(k) : \mathbb{R}^+ \to \mathbb{R}^+$ is twice continuously differentiable and satisfies $f(0) = 0$. Moreover, $f(k)$ is strictly increasing and concave (i.e., its derivatives satisfy $f'(\cdot) > 0$ and $f''(\cdot) < 0$) with $f'(0) = M < +\infty$, $\lim_{k\to\infty} f'(k) < 1$.

Since $u(\cdot)$ is increasing (cf. assumption AS 1) in order to maximize global utility of the consumers is possible to replace (6) by the (nonlinear) difference equation

$$k_{t+1} + (1-\delta)k_t - f(k_t) = -c_t \quad \text{with } k_0 > 0 \text{ given} \tag{7}$$

or equivalently for $\tilde{f}(k) := f(k) - (1-\delta)k$ by

$$k_{t+1} - \tilde{f}(k_t) = -c_t \quad \text{with } k_0 \text{ given}, \tag{8}$$

where $c_t$ $(t = 0,1,\ldots)$ with $c_t \in [0, f(k_{t-1})]$ is selected by the decision maker. Since the considered system is purely deterministic the initial capital $k_0$ along with the control policy $c_t$ fully determines development of $(k_t,c_t)$ over time. In particular, (cf. (4), (7)) for a given initial capital $k_0$ policy $c_t$ is optimal for a finite or infinite time horizon $T$ if the global utility

$$U^\beta(k_0,T) = \sum_{t=0}^{T} \beta^t u(c_t) = \sum_{t=0}^{T} \beta^t u(\tilde{f}(k_t) - k_{t+1}) \tag{9}$$

attains maximum for policy $\hat{c}_t = u(\tilde{f}(\hat{k}_t) - \hat{k}_{t+1})$, i.e. *the value function*

$$\hat{U}^\beta(k_0,T) = \max_{(\boldsymbol{k},\boldsymbol{c})} \sum_{t=0}^{T} \beta^t u(\tilde{f}(k_t) - k_{t+1}) \quad \text{where } (\boldsymbol{k},\boldsymbol{c}) = \{k_0,c_0,k_1,c_1,\ldots,k_T,c_T\}. \tag{10}$$

Moreover, since the performance function is separable (in particular, additive) by the well-known “principle of optimality” of dynamic programming we immediately conclude that for any time point $\tau = 0,1,\ldots$ it holds

$$\hat{U}^\beta(k_0,T) = \sum_{t=0}^{\tau-1} \beta^t u(\hat{c}_t) + \beta^\tau \hat{U}^\beta(k_\tau, T-\tau) \tag{11}$$

$$\hat{U}^\beta(k_\tau, T-\tau) = \max_{k_{\tau+1}} [u(\tilde{f}(\hat{k}_\tau) - k_{\tau+1}) + \beta \hat{U}^\beta(k_{\tau+1}, T-\tau-1)]. \tag{12}$$

The following simple facts may be useful for better understanding the development of the considered economy over time.

**Fact 1.** i) If $f'(0) \le 1$ (and hence $f'(k) < 1$ for all $k > 0$), then by AS 2 every sequence $\{k_0, k_1, \ldots, k_t, \ldots\}$ must be decreasing and $\lim_{t\to\infty} k_t = 0$.

ii) If $f'(0) > 1$ (and hence, since $\lim_{k\to\infty} f'(k) < 1$, there exists some $k'$ such that $f'(k) < 1$ for all $k > k'$), then there exists some $k^* > 0$ such that $f(k^*) = k^*$ and some $k_m \in (0, k^*)$ such that $f(k_m) - k_m = \max_k[f(k) - k]$.

iii) Supposing that $k_0 > k^*$ then elements of any sequence $\{k_0, k_1, \ldots, k_t, \ldots\}$ must be decreasing for all $k_t > k^*$. Furthermore, if for some $t = t_\ell$ it holds $k_{t_\ell} < k^*$ then $k_t < k^*$ for all $t \ge t_\ell$, but $\{k_t, t \ge t_\ell\}$ need not be monotonous. However, in any case $k_t \le k_{\max} = \max(k_0, k^*)$ and $f(k_t) \le f(k_{\max}) =: y_{\max}$ for all $t = 0, 1, \ldots$.

In what follows we summarize some basic properties of value functions and sketch the corresponding proofs (for details see e.g. [2]).

**Result 1.** *Properties of the value function $\hat{U}^\beta(k_0, T)$.*

i) *(Monotonicity of value function at initial condition.)*

In case that $k_0' > k_0 > 0$ then it holds $\hat{U}^\beta(k_0', T) \ge \hat{U}^\beta(k_0, T)$.

ii) *(Continuity and differentiability of value function $\hat{U}^\beta(k_0, T)$.)*

$\hat{U}^\beta(k_0, T)$ is differentiable function at initial condition $k_0$.

iii) *(Concavity and continuity of value function $\hat{U}^\beta(k_0, T)$.)*

$\hat{U}^\beta(k_0, T)$ is concave and continuous with respect to initial condition $k_0$.

iv) *(Continuity and differentiability of value function $\hat{U}^\beta(k_0, T)$ at discount factor $\beta$.)*

$\hat{U}^\beta(k_0, T)$, as well as any feasible $U^\beta(k_0, T)$, is a continuous and differentiable function of discount factor $\beta$.

v) *(Truncation and infinite horizon.)*

There exists limits $\hat{U}^\beta(k_0) := \lim_{T\to\infty} \hat{U}^\beta(k_0, T)$, and $\hat{U}^\beta(k_0)$ converges monotonously to $\hat{U}^\beta(k_0)$ as $T \to \infty$. Note that since the discount factor $\beta < 1$ maximal global utility $\hat{U}^\beta(k_0)$ is finite also if $f'(0) < \beta^{-1}$ (cf. Fact 1i).

To verify parts i), ii), observe that if we start with initial condition $k_0' > k_0$ and follow optimal policy with respect to initial condition $k_0$, except of enlarging initial $c_0$ by $\Delta c_0 = f(k_0') - f(k_0) > 0$ (recall that by AS 1 and AS 2 $f(k_0') > f(k_0)$ and $u(\cdot)$ is increasing). For such policy we have $\tilde{U}^\beta(k_0', T) > \hat{U}^\beta(k_0, T)$, $\hat{U}^\beta(k_0', T) \ge \tilde{U}^\beta(k_0, T)$ and obviously $\hat{U}^\beta(k_0', T) > \hat{U}^\beta(k_0, T)$.
Moreover, $\hat{U}^\beta(k_0, T)$ must be continuous and differentiable function of the initial condition $k_0$. To this end observe that that $\hat{U}^\beta(k_0', T) - \hat{U}^\beta(k_0, T) = u(f(k_0')k_1) - u(f(k_0) - k_1)$ and $u(\cdot)$, $f(\cdot)$ are continuous and differentiable functions by assumptions AS 1 and AS 2. Using the same way of reasoning we can conclude continuity of any feasible function $U^\beta(k_0, T)$ with respect of $k_1, k_2, \ldots, k_t$ and $\hat{U}^\beta(k_0, T)$ are continuous functions of $k_0$.

Part iii) is a consequence of concavity of instantaneous utility function $u(\cdot)$ (cf. AS 1) and per capita production function $f(\cdot)$ (cf. AS 2). Since concave function defined on an open interval must be on this interval continuous, it suffices to verify concavity. For details see [2].

Part iv) follows immediately for any feasible policy and by taking into account part i) and ii) for optimal policies.

To verify part v) observe that if $k \in [k_0, k^*]$ (cf. Fact 1ii)) and $c_t \in [0, f(k_t)]$, hence $u(c_t) > 0$ must be bounded by some $C < \infty$ implying that $\hat{U}^\beta(k_\tau, T-\tau) = \max_{c_t \in [\tau, T)} \sum_{t=\tau}^{T} \beta^t u(c_t) \le \beta^\tau C/(1-\beta)$, and $\hat{U}^\beta(k_\tau, T-\tau)$ is decreasing in $\tau$. Moreover, condition $\sum_{t=0}^{\infty} \beta^t u(c_t) < \infty$ is fulfilled even for unbounded $u(c_t)$ if for some $\bar{\beta} > \beta$ $u(c_t) < \bar{\beta}^t C$ (this holds if $f'(0) < \beta^{-1}$).

## 2 The Growth Model Under Uncertainty

To include random shocks or imprecisions into the model, we shall assume that for a given value of $k_t$ we obtain the output $y_t$ only with certain probability, in particular we assume that $y_t \in [f_{\min}(k_t), f_{\max}(k_t)]$ (i.e. $f_{\min}(\cdot) \le f(\cdot) \le f_{\max}(\cdot)$, AS 2 also hold for $f_{\bullet}(k)$). Obviously, better results can be obtained if we replace the rough estimates of $y_t$ generated by means of $f_{\max}(k_t)$ and $f_{\min}(k_t)$ by a more detailed information on the (random) output $y_t$ generated by the capital $k_t$.

To this end we shall assume that that in (6) $y_t = f(k_t)$ is replaced by $y_t = Z(k_t)$, where $Z(\cdot)$ is a Markov process with state space $\mathcal{I}_1 \subset \mathbb{R}$ and transition probabilities $p(y|k)$ from state $k \in \mathcal{I}_1$ in state $y \in \mathcal{I}_2 \subset \mathbb{R}$ such that $p(y_t|k_t) \gg p(y|k_t)$ for each $y \neq y_t = f(k_t)$ (obviously, $\sum_{y\in\mathcal{I}_2} p(y|k) = 1$ for each $k \in \mathcal{I}_1$). Moreover, we assume that the current value of the total output $y_t$ is known to the decision maker *and then the recourse decision (intervention) may be taken to reach immediately the optimal value of* $k_{t+1}$ for the original model (cf. (10)–(12). Such an extension well corresponds to the models introduced and studied in [8] and also in [3, 4]. Up to now we have assumed that the transition probabilities cannot be influenced by the decision maker. In what follows we extend the model in such a way that $p(y|k)$ will be replaced by $p(y|k,d)$ for $d \in \mathcal{D} = \{1, 2, \ldots, D\}$ and some *cost,* denoted $c(k,d)$, *will be accrued to this decision.* Similarly, additional cost, denoted $\bar{c}(y,\bar{k})$, will be accrued to the recourse decision transferring state $y \in \mathcal{I}_2$ to the desired state $\bar{k} \in \mathcal{I}_1$ (of course, $\bar{c}(\bar{k},\bar{k}) = 0$).

So the development of the considered system over time is given by the following diagram

$$k = k_t \xrightarrow[\;]{\substack{c(k,d)\\ p(y|k,d)}} y = y_t \xrightarrow{\bar{c}(y;\bar{k})} \bar{k} = k_{t+1}$$

The above model can be also treated as a structured controlled Markov reward process $X$ with compact state space $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ (with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$), finite set $\mathcal{D} = \{0, 1, \ldots, D\}$ of possible decisions (actions) in each state $k \in \mathcal{I}_1$ and the following transition and cost structure:

$p(y|k,d)$ : transition probability from $k \in \mathcal{I}_1 \to y \in \mathcal{I}_2$ if decision $d \in \mathcal{D}$ is selected,

$c(k,d)$ : cost of decision $d \in \mathcal{D}$ in state $k \in \mathcal{I}_1$,

$\bar{c}(y,\bar{k})$ : cost for intervention, i.e. immediate transition from state $y \in \mathcal{I}_2$ in $\bar{k} \in \mathcal{I}_1$,

$r(y|k,d)$ : expected value of the one-stage reward obtained in state $k$ if decision $d \in \mathcal{D}$ is selected in state $k$; in particular

$$r(y|k,d) = \int_{y\in\mathcal{I}_2} p(y|k,d)[u(f(y)) - (1-\delta)k - y]\,\mathrm{d}y,$$

$\bar{r}(y|k,d)$ : total expected reward earned by transition (including possible intervention) from state $k$ to state $\bar{k}$, i.e.,

$$\bar{r}(y|k,d) = r(y|k,d) - c(k,d) - \int_{y\in\mathcal{I}_2} p(y|k,d)\bar{c}(y,\bar{k})\,\mathrm{d}y.$$

Now we have two options:

1. Follow optimal policy found for the corresponding deterministic model (cf. (12)) in such a way that we maximize by means of intervations transition rewards between consecutive states, i.e. at time points $t = 0, 1, \ldots$ we maximize $\bar{r}(k_t, k_{t+1}, d)$ with respect to $d \in \mathcal{D}$ and follow the same policy as if $p(y|k_t, d) = 1$ for $y = f(k_t)$.
2. Consider the problem as finding optimal control policy of a $\beta$-discounted Markov decision chain with compact state space and finite action space.

In the latter case for the considered time horizon $T$ let the value function $\hat{U}^\beta(k, T-\tau)$ denote expectation of the maximal (random) global utility received in the remaining $\tau$ next transitions if the considered Markov reward chain $X$ is in state $k \in \mathcal{I}_1$ and optimal policy is followed. Then obviously

$$\hat{U}^\beta(k, T-\tau) = \max_{d\in\mathcal{D}} \int_{y\in\mathcal{I}_2} p(y|k,d)[\bar{r}(y|k,d) + \beta\hat{U}^\beta(y, T-\tau-1)]\,\mathrm{d}y, \qquad k \in \mathcal{I}_1, y \in \mathcal{I}_1 \tag{13}$$

and for $\tau$ tending to infinity, i.e. when $\lim_{T\to\infty} \hat{U}^\beta(k,T) = \hat{U}^\beta(k)$, then

$$\hat{U}^\beta(k) = \max_{d\in\mathcal{D}} \int_{y\in\mathcal{I}_2} p(y|k,d)[\bar{r}(y|k,d) + \beta\hat{U}^\beta(y)]\,\mathrm{d}y \qquad k \in \mathcal{I}_1, y \in \mathcal{I}_1. \tag{14}$$

Equation (14) demonstrates that part v) of Result 1 also holds in the considered extended version. Similarly, also part i) of Result 1 (Monotonicity of value function in initial condition), as a common property of Markov control processes holds for the considered extended version along with continuity and differentiability of value function both at initial condition $k$ and discount factor $\beta$ (cf. Result 1, parts ii) − iv)).

Unfortunately, assuming that $Z$ is a Markov process with compact state space $\mathbb{R}$ then the model given by (13), (14) is not suitable for numerical computation. To make the model computationally tractable we it is necessary to approximate our system governed by (13), (14) by a discretized model with finite state space and estimate the resulting errors caused by such approximation.

To this end, it is necessary to assume that the values of $c_t$, $k_t$, and $y_t$ take on only a finite number of discrete values. In particular, we assume that for sufficiently small $\Delta > 0$ there exist nonnegative integers $\bar{c}_t$, $\bar{k}_t$, and $\bar{y}_t$ such that for every $t = 0, 1, \ldots$ it holds:
$\bar{c}_t\Delta = c_t$, $\bar{k}_t\Delta = k_t$, and $\bar{y}_t\Delta = y_t$ with $\bar{k}_t \le K := k_{\max}/\Delta$ and similarly $\bar{y}_t \le Y := y_{\max}/\Delta$. Such approach was discussed in [7], [9].

*Conclusion.* In this article we indicated possible applications of controlled Markov processes for the analysis of extended versions of the growth models. As it was shown, many properties of the classical model can be extended to the considered more general cases.

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