ADJOINT PROCESSES OF THE PORTFOLIO OPTIMIZATION PROBLEM AND EQUILIBRIUM ASSET PRICES

ALEXIS DERVIZ[1]
Institute of Information Theory and Automation
Pod vodárenskou věží 4
CZ-182 08 Prague 8 Czech Republic,

E-mail: aderviz@utia.cas.cz .

The paper studies dynamic asset market equilibria in economies with optimally consuming and investing agents who face positive portofolio adjustment costs and have preferences for liquidity and covered positions, defined as increasing disutilities of short positions in available assets. In the constructed model, the adjustment rates of asset holdings in a dynamically optimized portfolio serve as control variables, while the asset holdings play the role of the states. The arising stochastic dymanic optimization problem contains state variable constraints in the form of the state vector dependence of the utitlity function, for which the traditional Consumption-based Capital Asset Pricing Model techniques do not work. Instead, the investor's problem is analyzed by means of the stochastic maximum principle including the adjoint equation for the co-state process. The proposed adjoint equation method for the shadow asset prices (co-states of the optimization problem) provides new insights in the equilibrium dynamics of asset prices. One of the applications of the latter is a generalized uncovered asset reutrn parity condition on the exchange rate of an open economy.

KEYWORDS: Asset pricing, portfolio optimization, stochastic maximum principle, co-state variables, adjoint equation

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1 Introduction

The present paper studies asset prices in a dynamic equilibrium model with portfolio-optimizing agents. The main result is a collection of equilibrium asset prices in terms of the underlying shadow values of these assets. The latter emerge as co-state variables alias stochastic adjoint processes of the optimization problems of representative investors. The result is obtained by means of a stochastic maximum principle for a suitable class of optimal control problems with Brownian uncertainty. Namely, the adjoint/co-state process is introduced, for which the differential adjoint equation is derived. The used statement of the adjoint equation is a synthesis of several known approaches. It is general enough to cover possible nonlinearities in the transition equation.

The shadow asset pricing technique is applied to the following three examples. First, I study the basic model of consumption and portfolio choice under cash constraints. Second, this basic model is expanded to the case of an open economy with both domestic and international consumption goods, cash balances and securities. Third, I formulate an optimizing model of a productive firm which issues its own securities. I costruct a solution to this problem using the same principles as in the basic model of consumption and investment.

The obtained solution of the optimal portfolio problem leads to a revision of two fundamental results of finance theory.

First, I am faced with the assumption of the traditional portfolio theory in frictionless and complete markets (due to Merton, Karatzas and collegues, Huang and others), which requires the diffusion matrix of the price vector process to be nonsingular. Consequently, one can change the probability measure to make the properly deflated cum-consumption wealth process a martingale. On the other hand, one of the conventional equilibrium conditions on the asset markets often utilized in the Consumption-based Capital Asset Pricing Model (CCAPM) literature is a constant net supply of all basic assets. I show, that, under this condition, the individual wealth process contains only the risk factors which are non-diversifiable by the available assets. In particular, "complete" markets would in such a setting mean a risk-free wealth. Accordingly, one must conclude that the basic asset price processes have linearly dependent risk parameter vectors in equilibrium, contrary to the assumption used to obtain the classical results on optimal portfolios. In this sense, the traditional CCAPM is “wrong” unless it gives up the constant net supply assumption and allows for a non-trivial supply side of the capital market.

Second, I identify the difference in the results of the present model and the traditional CCAPM in the treatment of instantaneous returns of individual securities. I show, among other things, that ther exists a “generalized” total return rate parity between any pair of assets, regardless of the existence of a risk-free interest rate. This total return parity generalizes the corresponding CCAPM relationship. In the special case of a two-country model with assets denominated in different currencies, the return parity provides a generalization of the classical uncovered interest rate parity for the exchange rate, which is, differently from the latter, free of Siegel's paradox (Siegel, 1972).

The no-bancruptcy constraint on wealth used in the traditional solutions of the portfolio problem is replaced in my approach by “soft” liquidity constraints or, more generally, limitations on borrowing. The latter are defined as increasing disutilities of short positions in available assets, i.e. agents are penalized by reduced utility for excessive debt. The utility function becomes dependent on the vector of positions in the basic assets. The adjustment rates of asset holdings in a dynamically optimized portfolio become the control variables, while the asset holdings themselves play the role of the states. The arising stochastic dymanic optimization problem contains smooth state variable constraints in the form of the state vector dependence of the utitlity function. In addition, there are non-zero portfolio reshuffling costs. Under these circumstances, it is impossible either to solve the Bellman equation of the agent's optimization problem explicitly or apply the so-called “martingale technique” (as described by Duffie, 1992, Ch. 8). On the other hand, useful results can be obtained by applying the stochastic maximum principle for optimal control of diffusions, involving the adjoint equation for the co-state process.

The adjoint equation statement in Itô form establishes an analogy between the Hamiltonian equations of deterministic optimal control theory and a pair of mutually adjoint Itô equations for the state and co-state processes in stochastic control problems. Consequently, one is able, as in deterministic theory, to study gradual changes of the controlled system starting at an arbitrary initial state.

The international extension of the basic model can be compared to the monetary two-country asset pricing model of Lucas, 1982. Several results of

Section 3 highlight similarities and differences of asset pricing obtained with my method of shadow prices on the one hand and Lucas' pricing formulae for basic securities on the other. The latter - after redefinition in continuous time - turn out to be special cases of the valuation by means of shadow prices in my model. All one needs is the assumption that there is zero aggregate external supply of additional asset quantities, and that preferences and budget constraints of agents in both modeled nations are identical. Alternatively, "almost Lucas' results" emerge in my model when transaction costs in one part of the world are negligible compared to the other.

Specifically when the properties of the shadow prices are used to calculate the international capital market equilibrium, the so obtained information about the evolution of exchange rates turns out to be richer than what is offered in earlier literature.

Explicit solutions to the consumption and investment problem under uncertainty both in a closed economy (e.g. Merton, 1971, Karatzas et al., 1986), and an open economy setting (Lucas, 1982, Stulz, 1984, Penati and Pennacchi, 1989, Claessens, 1991, and many others) have been obtained for a restrictive case when the agent's utility does not depend on the state variable. This is also the pre-condition for a direct application of martingale techniques to CCAPM, as in Cox and Huang, 1989, Karatzas et al., 1987, Zapatero, 1995. However, this reduction of the utility class limits the power and informativeness of the results. First, accomodating the natural cash-in-advance and other short position constraints on the porfolio as constraints on control variables entails considerable technical complications (see, e.g. Cvitanic and Karatzas, 1992, or Xu and Shreve, 1992). Second, the very choice of the state and control variables in the optimization problem must be adapted to the solution method: one must solve directly for *optimal asset holdings*, not the values by which these holdings are being optimally *adjusted* over time. Third, what is obtained as a solution is, in fact, that control trajectory which makes a properly chosen state process a martingale. Put differently, a stochastic analogue of the steady state is calculated. Outside the steady state, a part of the control vector (namely, asset holdings) can be indeterminate. Therefore, what in the theory under discussion is sometimes presented as an optimal portfolio derived by means of stochastic calculus of variations, also called Malliavin calculus, see Karatzas and Ocone, 1991, would be better called a martingale-preserving control of the state process.

In short, the presently existing martingale technique in optimal control supplies *stock* values in a "steady-up-to-random-walk" state for a limited class of utility functions. Financial applications, on the other hand, would evidently benefit from solutions in terms of *flow* values of portfolio adjustment rates, starting anywhere outside steady state equilibria, possibly for utility functions which are allowed to depend on state variables (e.g. cash holdings for a liquidity-valuing individual, which is known to be equivalent to the Clower constraint, see Feenstra, 1986).

Formally, the above objective may seem to be met by the dynamic programming approach to optimal control. Unfortunately, closed form solutions to the Bellman equation are known only for a small number of special utility function cases (HARA, logarithmic). In other cases, results in terms of the value function (provided it is differentiable) are often hard to interpret. Besides, the indeterminacy of portfolio selection reappears in the dynamic programming setting when one tries to simplify the problem by resorting to "stock" control variables (see above), as the example of the "mutual fund theorem" in Karatzas et al., 1986, manifests.

The difficulty can be overcome by using the stochastic maximum principle which can be stated, analogously to deterministic problems, in terms of the adjoint differential equation for the co-state variable(cf. Hausmann, 1981, or Peng, 1990).

The finance theory has so far limited its scope to problems with state-independent utilities and controls associated with stocks of wealth allocated to different assets, as discussed above. However, the necessity of the adjoint equation for applications in finance and investment theory was recognized long ago. Thus, Lucas, 1978, works with a "stochastic Euler equation" for an optimization problem in a class of Markov processes, and so do Duffie and Zame, 1989, Sec. 5. Their formulations, however, skip any maximum principle as such; instead, they directly link the equilibrium asset prices and the marginal utilities of consumption. The co-state in the portfolio problem is explicitly used by Cadenillas and Karatzas, 1995, to derive a result already known from Karatzas' earlier work.[2]

The approach presented here, on the contrary, pursues the analogy with

[2]In fact, the application of the adjoint process by Cadenillas and Karatzas appears to be vacuous. Their current value Hamiltonian happens to be linear in portfolio holdings. A subsequent assumption then eliminates the portfolio holdings from the Hamiltonian altogether. The claimed "optimal" investment strategy is then the same martingale-preserving one known from Karatzas et al., 1987.

the deterministic control theory as far as is possible. I take advantage of the existence of a stochastic maximum principle in "Hamiltonian form", as stated in Elliott, 1982, Theorem 16.25, for a fairly general class of objective functions and transition equations in Itô form. What has so far been missing in the literature is a proper link between the co-state process appearing there, which is the natural second argument in the Hamiltonian function, and a differential equation with coefficients depending on utility and the right hand side of the transition equation, that would generalize the second (adjoint) Hamiltonian equation of deterministic problems.

The paper is structured as follows. Section 2 presents the basic model of consumption and investment with state-dependent utility, culminating with a statement of the adjoint equation system for the vector of shadow asset price processes (proved in Appendix A). Section 3 analyses the international consumption and investment. It also presents an application to the international capital market equilibrium in a Walrasian setting of instantaneously clearing markets. Section 4 studies the differences of the proposed model from the traditional CCAPM and derives the general uncovered asset return parity condition free of Siegel's asymmetry paradox. The conclusion summarizes the main results.

2 The Consumption and Investment Model

This section defines a version of consumption and portfolio choice under uncertainty and its solution with the help of the stochastic maximum principle. The Euler equation for the adjoint process of the problem is formulated, with the proof relegated to Appendix A.

2.1 The model basics

Consider an economy with the uncertainty factors represented by a vector Z of d mutually independent Brownian motions w.r.t. an information filter $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the "*usual conditions*" (Karatzas and Shreve, 1991, p. 10). The diffusion parts of all random processes appearing below are spanned by Z. No uncertainty is assumed at time 0: $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Let there be n securities indexed by $k = 1, ..., n$. Each one is characterized by the following three $\mathcal{F}$-adapted processes: the spot ex-dividend price P^k, cumulative divdend γ^k (so that the dividend rate is $d\gamma^k = g^k dt + v^k dZ$), the internal growth π^k (that is, $d\pi^k$ is the growth rate if the drift of process π^k is positive and the attrition rate if it is negative).

There is a single numéraire - the consumption good. Denote by x^k the number of shares of security k held by the representative agent.

Let c be the consumption rate and $d\Phi^k$ - the purchase (or sale if negative) of security k during time dt in exchange for the numéraire. In the absence of other sources of income, the extant tradition in the stochastic consumption/investment literature, started probably by Merton, 1971, implies the binding budget constraint

$$cdt + \sum_{k=1}^{n} x^k d\gamma^k + \sum_{k=1}^{n} P^k d\Phi^k = 0. \tag{1}$$

for every agent[3]. This is used to derive the dynamic equation for the agent's total wealth, defined as $W = \sum_{k=1}^{n} P^k x^k$. Given the evolution rules of price processes P^k ($1 \leq k \leq n$), the evolution of wealth can be stated without any mention of investment rates $d\Phi^k$. Therefore, the optimal consumption path $t \mapsto c_t$ and the optimal portfolio composition should be calculable in wealth terms, $\{P^k x^k\}_{k=1}^{n}$. Clearly, the latter can be a well-defined vector of control variables only if the agent is allowed to adjust asset holdings instantaneously. The obtained result is a kind of steady state for the wealth process adjusted for growth, disturbed solely by random fluctuations.

Short position constraints and the utility function The assumption of instantaneous portfolio choice is unrealistic in most macroeconomic applications of finance theory, where the dynamic of investment processes is the central issue of the whole analysis (e.g. the international financial flows, see Section 3). In these cases, one needs to assume that $d\Phi^k = \varphi^k dt$ and analyze the portfolio adjustment rates φ^k explicitly instead of hiding them inside the total financial wealth variable. The consequence is the agent's optimization problem whith state variables x^k ($1 \leq k \leq n$) and control variables c, $\varphi = [\varphi^1, ..., \varphi^n]^\mathsf{T}$. It remains to formulate a suitable budget constraint for this new setup.

[3]Note that (1) implies that the volatility terms in $d\gamma$ and $d\Phi$ are linearly dependent and cancel out after summation.

To do this, observe that the constraint (1) tacitly pre-supposes the existence of some sort of "liquid" asset, very similar to the real money balances of macroeconomic theory, measured in the same units as the consumption good and serving as a medium of exchange in a market-place where revenues from asset sales and dividends are traded for consumption. In other words, "money" naturally emerges in an investment model with explicitly defined (dis)investment rates. This liquid asset will denoted by $\mathcal{M}$ in the sequel.

In the absence of transaction costs, the agent's current holdings x^0 of $\mathcal{M}$ would evolve according to

$$dx^0 = \sum_{k=1}^{n} x^k d\gamma^k - \sum_{k=1}^{n} P^k \varphi^k dt - cdt. \tag{2}$$

Condition (1) would mean the requirement of instantaneous spending of all liquidity at every given moment.[4] It is, of course, possible to impose a much less restrictive liquidity constraint, e.g. in the form $x^0 \geq \overline{x^0}$ for some critical level $\overline{x^0}$ of disposable liquid assets.

Restrictions on short positions of other securities are also imaginable, for example, $x^k \geq \overline{x^k}$, $k = 1, ..., n$, meaning that the maximum allowed short position size in security k is $-\overline{x^k}$. In the investor's optimization problem, such restrictions would be equivalent to the existence of $n+1$ instantaneous (i.e. time-dependent and stochastic) Lagrange multipliers ω_k, $(k = 0, 1, ..., n)$ that should be added to the integrand in the objective function. Specifically, if $U(c)$ is the investor's utility function, these constraints dictate the replacement of U by

$$\tilde{U}(x, c) = U(c) + \sum_{k=0}^{n} \omega_k (\bar{x^k} - x^k).$$

Obviously, the liquidity and other short position constraints force the state variable x into the utility function even if the latter was originally independent of it. This circumstance, and the poor analytical tractability of intertemporal optimization problems with static Lagrange multipliers, suggests the desirability of adopting restrictions on x in the form of x-argument

[4] The treatment of liquidity constraints in the present model can be compared to that of Lucas, 1982. The cash-in-advance requirement utilized by Lucas is more restrictive compared to a possible discrete-time analogue of my condition (1): the securities in his model must be traded separately in advance of the consumption goods trade, the latter resulting in exhaustive spending of available money balances. In continous time models, more loose constraints on liquidity seem to be more natural.

in the utility function from the outset. For instance, a natural condition prohibiting infinite debt (negative $\mathcal{M}$-holdings) can be given by

$$\lim_{x^0 \to -\infty} U_{x^0}(x^0, x', c) = -\infty, \ \lim_{x^0 \to -\infty} U'_{x^0}(x^0, x', c) = +\infty$$

for all admissible c and $x' = [x^1, ..., x^n]^\mathsf{T}$.

For simplicity, the present model concentrates on $\mathcal{M}$-debt constraints alone. Therefore, I shall work with the utility as a function of two variables, the liquidity and the consumption rate: $(x^0, c) \mapsto u(x^0, c)$. Increasing disutility of negative liquidity holdings as well as diminishing marginal utility of positive liquidity is assumed. This means different valuation of consumption under different levels of liquidity holdings, specifically, a highly negative valuation of big debt (i.e. this is a "smooth" Clower constraint on cash holdings). The dependence of u on consumption satisfies the standard Inada conditions.

Transaction costs Agents in the model are allowed to transmit funds from liquidity to other assets and back, at a cost.

Specifically, for a given security k ($k = 1, ..., n$), let $\varphi^k > 0$ be the intended volume of k-shares purchase. Then, the amount $P^k \psi^0 = P^k j^k(x^0, \varphi^k)$ must be paid out of the $\mathcal{M}$-account for this transaction ($\psi^0 > \varphi^k$).

For $\varphi^k < 0$, $-P^k \psi^0 = -P^k j^k(x^0, \varphi^k) > 0$ is the increment in the $\mathcal{M}$-account in exchange for $-\varphi^k$ sold units of k ($-\psi^0 < -\varphi^k$). In both cases, the difference $j^k(x^0, \varphi^k) - \varphi^k$, a smooth increasing convex function of the transaction volume $\mid \varphi^k \mid$, constitutes the transaction fee portion of the security price paid to an intermediary. Zero transaction costs are impossible unless there is no trade at all: $j^k(x, \varphi) = 0$ if and only if $\varphi = 0$.

The transaction fee is smoothly negatively dependent on the agent's solvency x^0: the wealthier the agent is, the easier it is for him to operate in the security market. In aggregate (if the existence of a representative agent is assumed), one can also interprete x^0 in the transition equation as the measure of depth, or liquidity, of the financial market. Thus, the deeper the market the lower the transaction costs.

An example of transaction cost satisfying the above properties is the quadratic function similar to the physical capital installation cost often exploited by "Tobin's q" models in macroeconomics:

$$\psi = j^k(x, \varphi) = \varphi - \frac{b_k}{2x}(\varphi)^2, \ \varphi = h^k(x, \psi) = \left[\left(\frac{x}{b_k} \right)^2 + \frac{2x}{b_k}\psi \right]^{\frac{1}{2}} - \frac{x}{b_k}, \quad (3)$$

where b_k is a positive constant. The formula is applicable in the range of values of ψ such that $2b_k\psi^0/x^0 > -1$. This means that, if security k is being sold out, the sale rate shall not be too high compared to the current liquidity holdings.

In the equilibrium security price calculations, I will need the *marginal transaction function* $v^k = 1/j^k_\varphi = h^k_\psi$ as a function of x and $\psi = j^k(x,\varphi)$ (subscripts denote partial derivatives). For example, the transaction cost function (3) results in

$$v^k(x,\psi) = j^k_\varphi(x, h^k(x,\psi))^{-1} = \left(1 + \frac{b_k\varphi}{x}\right)^{-1} = \left[1 + \frac{2b_k\psi}{x}\right]^{-\frac{1}{2}}. \tag{4}$$

Taking into account the transaction costs, one must modify the law of motion (2) of x^0 by replacing $\sum_k P^k\varphi^k dt$ with $\sum_k P^k j^k(x^0,\varphi^k)dt$.

To make $\mathcal{M}$ more close to conventional macroeconomic representations of real cash balances, I introduce its internal growth (or deterioration) rate $d\pi^0 = \mu^0 dt + \sigma^0 dZ$. Then, collecting the dynamics of individual state processes, I get the following system of state-transition equations:

$$dx^0 = x^0 d\pi^0 + \sum_k \left(x^k d\gamma^k - P^k j^k(x^0,\varphi^k)dt\right) - cdt, \tag{5a}$$

$$dx^k = -x^k d\pi^k + \varphi^k dt, \ k = 1, ..., n. \tag{5b}$$

The optimization problem Let T be the individual decision horizon. Unlike many portfolio optimization models of the literature, the present one admits a non-zero smooth final bequest function $(T,x) \mapsto B(T,x)$ of the final time horizon T and the state vector value as of that time. As usual, B is assumed increasing and strictly concave in x. In addition, for the infinite time horizon case I assume that there is a limit x_∞ of the state process and a limit $B_\infty(x_\infty)$ of the bequest function, defined in the proper probabilistic sense. The role of B is important, for instance, in the applications of the model to American option pricing.

Allowing for variable time preferences, I assume that for any $s > t$, the agents discount their time s-utilities at time t by means of the same locally riskless discount factor $\Theta^s_t = [\vartheta]^s_t = \int_t^s \vartheta(\tau)d\tau$, with the same ϑ for all agents.

(For an arbitrary process g of finite variation, symbol $[g]_t^s$ denotes $\int_t^s g(\tau)d\tau$ $(t < s)$).

The generic agent solves

$$\max_{\ell} \mathbf{E}\left[\int_0^T e^{-[\vartheta]_0^t} u(x_t^0, c_t)dt + e^{-[\vartheta]_0^T} B(T, x_T) \mid \mathcal{F}_t\right], \tag{6}$$

with respect to the decision path $t \mapsto \ell_t = [c_t, \varphi_t]^\mathsf{T}$, subject to the state-transition equation (5), the initial asset values given. In other words, the agent maximizes the expected present value of future utilities from consumption plus the expected present value of the final bequest B.

The above consumption and portfolio optimization problem has a state-dependent utility. In addition, the right hand side of the transition equation (5) contains transaction cost functions and is, therefore, nonlinear in both state and control variables.

The dynamic programming solution to a class of consumption/investment optimization problems containing (5), (6) by means of the value function (Fleming and Soner, 1993), is but indirect. Unlike traditional consumption and portfolio optimization (Merton, 1991), problems where the utility function contains the state process cannot be treated with the so-called martingale technique (Cox and Huang, 1987). Instead, I base my approach on the stochastic maximum principle and the adjoint Itô equation for the co-state process (an early example is in Bismut, 1976). The present version of the stochastic maximum principle synthesizes a number of earlier formulations found in stochastic control literature (Hausmann, 1981, Elliott, 1982, Peng, 1990, Cadenillas and Karatzas, 1995). The exact formulation of the maximum principle, including the adjoint equation, is given in the next subsection. Technical details of the proof are relegated to Appendix A.

2.2 The maximum principle and the adjoint process

The stochastic maximum principle discussed below covers the class of optimization problems which is slightly more general than (5), (6). Particularly, the diffusion part of the transition equation can be control-dependent. The usefulness of this generalization will become clear in Section 3 dedicated to applications and generalizations of the basic model.

Only a couple of initial steps of the "martingale technique" (see, e.g. Duffie, 1992, Ch. 8) can be imitated in the present model, before the state-dependence of the utility function begins to matter. After that, one must calculate the Fréchet derivative of the maximand to obtain the stochastic adjoint equation in integral form.

Fix a time horizon $0 < T \leq \infty$, a locally riskless discount rate process $(\vartheta_t)_{t\in[0,T)}$, and consider a maximization problem for the functional J (*performance index*) defined for $0 \leq t < T$ on the elements $\ell = t \mapsto \ell_t \in \mathbf{R}^{d'}$ of a properly constructed functional space (specified in Appendix A), as the expected sum of the present values of future utilities:

$$(\ell, t) \mapsto J(\ell, t, x) = \mathbf{E}\left[\int_t^T e^{-\Theta_t^\tau} U(x_\tau, \ell_\tau) d\tau + e^{-\Theta_t^T} B(T, x_T) \mid \mathcal{F}_t, x_t = x\right] \tag{7}$$

(with omitted bequest term if $T = \infty$), subject to the state-transition equation

$$dx_t = \mu(x_t, \ell_t)dt + \sigma(x_t, \ell_t)dZ_t \tag{8}$$

for the n-dimensional state process $(x_t)_{t\geq 0}$. The final bequest function B must possess a limit if the time horizon of the optimization problem is infinite.

The saught optimal control $\hat{\ell}$ must be *time-consistent*. Namely, if $V(t, x) = sup_\ell J(\ell, t, x)$ is the *value function* of the problem (7), (8), then the value of performance index J at $\hat{\ell}$ must equal V for all times: $V(t, x_t) = J(\hat{\ell}, t, x_t)$, $0 \leq t < T$, provided that process x satisfies (8) with $\ell = \hat{\ell}$. That is, the policy picked as optimal initially must still be optimal when viewed at a later time t.

The standard martingale technique utilizes the fact that the transition equation in the traditional portfolio problem is linear in the state variable (wealth), and solves it by means of a change of the original probability measure to the equivalent martingale measure. The name points at the fact that a linear transformation of the state variable becomes a martingale under this change.

In the present model, a change of probability in transition equation (8) would be possible if the diffusion matrix $\sigma(x, \ell)$ of process x in (8) were uniformly invertible. However, I do not impose such an invertability property. Since components of vector x are holdings of different securities, not imposing any invertibility conditions on matrix σ means that I do not require asset markets to be complete.

In general, as is known from incomplete financial market theory (Duffie, 1987, Geankoplos, 1990), a consequence can be that the price of a specific contingent claim is indeterminate in equilibrium. In my approach, the analysis of the price formation mechanism of a specific security will not be precluded by market incompleteness. Proposition 1, to be stated below, allows one to define at least an instantaneous demand function for any contingent claim in terms of its shadow price (which is a linear function of the co-state process ξ, see Proposition 2 and Sections 3, 4).

The optimization problem solution method presented in Appendix A transforms equation (8) by linearizing it, as explained below.[5]

Throughout the text, the following notations will be used : if X and Y are semimartingales, their predictable - and also optional, if both processes are continuous - quadratic covariation, $[X, Y]$, is a *variation finite*-semimartingale (see Elliott, 1982, Karatzas and Shreve, 1991, or any other standard text on stochastic calculus). The following convention will be used for its stochastic differential: $d[X, Y] = \langle X, Y \rangle dt$, $d[X, X] = \langle X \rangle dt$. If σ^X, σ^Y are diffusion matrices of these two processes w.r.t. basis dZ, I shall sometimes abuse notations by writing $d[X, Y] = \langle \sigma^X, \sigma^Y \rangle dt$.

Define the $n \times n$-matrix valued process A by

$$dA = D_x \mu dt + D_x \sigma dZ.$$

Put $a^0 = D_x \mu$, an $n \times n$-matrix, and $a^1 = D_x \sigma$, a 3-tensor of dimensions $n \times n \times d$, with elements $(a^1)^k_{jm}$, $j, k = 1, ..., n$, $m = 1, ..., d$. Then, the predictable quadratic variation $[A]_t = \int_0^t \langle A \rangle_s ds$ of A is an $n \times n$-matrix valued process with elements

$$\int_0^t \sum_{l,p} (a^1)^k_{pl}(s)(a^1)^p_{jl}(s) ds, \; j, k = 1, ..., n,$$

or, according to the convention introduced earlier,

$$\langle D_x \sigma \rangle^k_j = \langle A^1 \rangle^k_j = \sum_{l,p} (a^1)^k_{pl} (a^1)^p_{jl}$$

for all j and k.

Similarly, if $b^1 = D_\ell \sigma$, then

$$\langle D_x \sigma, D_\ell \sigma \rangle^k_i = \langle a^1, b^1 \rangle^k_i = \sum_{j,m} (a^1)^k_{jm} (b^1)^j_{im}$$

[5]For linear s.d.e. this procedure renders a solution formula, cf. Karatzas and Shreve, 1991, subsection 5.6.C, p. 361.

for all i and k.

To formulate the result to be applied in the sequel, I invoke the notion of stochastic matrix-valued left- and right-exponentials (see, e.g. Protter, 1990, p. 264). For a square n-dimensional matrix-valued semimartingale λ and fixed t, let $s \mapsto e^R(\lambda)_t^s$, $s \mapsto e^L(\lambda)_t^s$ be the solutions of linear s.d.e.

$$d_s e^R(\lambda)_t^s = d_s\lambda \cdot e^R(\lambda)_t^s,$$
$$d_s e^L(\lambda)_t^s = e^L(\lambda)_t^s \cdot d_s\lambda,$$

with initial conditions $e^R(\lambda)_t^t = e^L(\lambda)_t^t = \mathbf{1}_n$.

The problem (7), (8) can be solved by forming the *current value Hamiltonian*

$$\mathcal{H}(t, x, \ell, \xi, \Xi) = U(x, \ell) + \xi \cdot \mu(x, \ell) - tr(\Xi \cdot \sigma(x, \ell)),$$

which must be maximized with respect to $\ell = \ell_t$ for all t. Here, ξ and Ξ are the so called first- and second-order adjoint processes (ξ is of the same dimention n as x and Ξ is a $n \times d$-matrix), with $\Xi = \xi \cdot D_x\sigma$ along the optimal path.

Proposition 2.1 *Let the optimal control $t \mapsto \ell_t$ of the optimization problem (7), (8) be characterized by the first order conditions. Then*

$$D_\ell\mathcal{H}(t, x_t, \ell_t, \xi_t, \Xi_t) = 0, \;\; t \in [0, T),$$

where the co-state process $(\xi_t)_{t\in[0,T)}$ is given by

$$\xi_t = E\left[\int_t^T D_xU_s \cdot \left(e^L(\Theta\mathbf{1}_n - A + [A])_t^s\right)^{-1} ds \right. \\ \left. + D_xB(T, x_T) \cdot \left(e^L(\Theta\mathbf{1}_n - A + [A])_t^T\right)^{-1} \mid \mathcal{F}_t\right], \tag{9}$$

and $\Xi_t = \xi_t \cdot D_x\sigma(x_t, \ell_t)$ for $t \in [0, T)$.

If the Hamiltionian is strictly concave in controls, its maximization with respect to ℓ_t for every t is sufficient for the optimality of control path $t \mapsto \ell_t$.[6]

For proof, see Appendix A.

[6]The adjoint process ξ can be also described as the x-gradient of the value function of the problem (7), (8), provided it exists.

Corollary 2.1 *The (first-order) adjoint process ξ of the preceding proposition satisfies the stochastic Euler differential equation*

$$d\xi = \xi \cdot (\vartheta \mathbf{1}_n dt - dA + \langle A \rangle dt) - D_x U dt, \tag{10}$$

with the final condition $\xi_T = D_x B(T, x_T)$ if $T < +\infty$. If $T = +\infty$, the **bequest-at-infinity** *condition $\lim_{T \to +\infty} \xi_T = D_x B_\infty(x_\infty)$ or the transversality condition*

$$\lim_{T \to \infty} \mathbf{E} \left[\xi(T) \cdot \left(e^L (\Theta \mathbf{1}_n - A + [A])_0^T \right)^{-1} \right] = 0,$$

must be added to (19).

Note that (19) becomes the well-known f.o.c. of the deterministic maximum principle (see Fleming and Rishel, 1975) as soon as the diffusion matrix σ in transition equation (8) approaches zero.

Since the state x is the vector of asset holdings, the adjoint process ξ can be called the *shadow price* vector of the corresponding group of assets.

2.3 First order conditions and the equilibrium asset prices

Resuming the analysis of problem (5), (6), let us calculate the matrix-valued process A which appears in the linearized transition equation (5). It is equal to

$$dA = \begin{bmatrix} d\pi^0 - \sum_k P^k j_x^k & d\gamma^K \\ 0 & d\pi^K \end{bmatrix}, \tag{11}$$

and its diffusion "matrix" (in fact, a 3-tensor) is described by

$$A^1 = \begin{bmatrix} \sigma^0 & \upsilon^K \\ 0 & \sigma^K \end{bmatrix}, \ \langle A \rangle = \begin{bmatrix} \mid \sigma^0 \mid^2 & \upsilon^K \cdot (\sigma^0 + \sigma^K)^\intercal \\ 0 & \mid \sigma^K \mid^2 \end{bmatrix}. \tag{12}$$

Here, uppercase index K must be understood as an indicator of vector variables and operations, with the exact meaning obvious from the context. Equation (12) indicates that, as follows from the formulae of Subsection 2, instantaneous covariance matrix $\langle A \rangle$ is obtained by multiplying matrix A^1 by itself, multiplication of the matrix elements understood in the scalar product sense.

Proposition 2.2 *The shadow asset price vector $\xi = [\xi_0, \xi_K]$ of the agent's consumption and portfolio optimization problem satisfies the s.d.e.*

$$d\xi_0 = \xi_0 \left(\vartheta dt - d\pi^0 + \sum_k P^k j_x^k dt + \mid \sigma^0 \mid^2 dt \right) - u_x dt, \tag{13a}$$

$$d\xi_k = \xi_0 \left((\upsilon^k \cdot (\sigma^0 + \sigma^k)^\mathsf{T} dt - d\gamma^k \right) + \xi_k \left(\vartheta dt - d\pi^k + \mid \sigma^k \mid^2 dt \right), \tag{13b}$$

The optimal consumption and investment rates of the agents satisfy the conditions

$$\frac{\partial u}{\partial c} = u_c = \xi_0, \tag{14}$$

$$\xi_k = \xi_0 P^k j_\varphi^k(x^0, \varphi^k) = \frac{\xi_0 P^k}{\upsilon^k}, k = 1, ..., n. \tag{15}$$

PROOF Equations (13)-(15) follow directly from Proposition 2.!. Specifically, (14), (15) result from the current-value Hamiltonian maximization, while (13) - from applying (19) of Corollary 2.1 to the problem (5), (6) •

A long-lasting tradition in the general equilibrium models of asset pricing (cf. Lucas, 1982, Cox et al., 1985, Claessens, 1991, Zapatero, 1995, and other general equilibrium models of international asset markets) requires that simple market-clearing conditions in Walrasian equilibrium are imposed, equating for each asset its total endowment, which is given exogenously, with the sum of holdings of this asset by individual investors, which are their choice, and, hence, endogenous variables.

In the present model, the investors' choice is not associated with asset holdings but with their adjustment rates. Therefore, the simplest way to state the Walrasian equilibrium conditions is to equate these adjustment values in aggregate with the known additional exogenous supplies. In this subsection, I assume zero net exogenous supply rate of all securities, which means that there is no supply to the asset market that would not be reflected by the growth/attrition rates $d\pi^K$. Accordingly, under the representative investor assumption, clearing of the market for security k means $\varphi^k = 0$.

Corollary 2.2 *Assume that there is a representative agent and that the security markets clear at all times, i.e. $\varphi_t^k = 0$, $k = 1, ..., n$, $t \in [0, T)$. Then* *the equilibrium price of security k can be represented by the integral equality*

$$P_t^k = \mathbf{E}\left[\int_t^T e^{-[\vartheta]_t^\tau + \int_t^\tau \left\{d\pi_s^k - \frac{|\sigma^k|^2}{2} ds\right\}} \frac{u_c(\tau)}{u_c(t)} \left\{d\gamma_\tau^k - \upsilon^k \cdot (\sigma^0)^\intercal d\tau\right\} \right.$$
$$\left. + e^{-[\vartheta]_t^T + \int_t^T \left\{d\pi_s^k - \frac{|\sigma^k|^2}{2} ds\right\}} \frac{D_{x^k} B(T, x_T)}{u_c(t)} \mid \mathcal{F}_t \right] \tag{16}$$

PROOF From (15) follows that the price P^k can be written as $\xi_k/\xi_0 \upsilon^k$, where υ^k is the marginal transaction function introduced in Subsection 2.1. Under the assumption of the representative investor and instantaneous market clearing $\upsilon^k = 1$. Therefore, P^k is simply the ratio ξ_k/ξ_0.

Now observe that s.d.e. (13b) can be solved backwards with respect to ξ_k as

$$\xi_k(t) = \mathbf{E}\left[\int_t^T e^{-[\vartheta]_t^\tau + \int_t^\tau \left\{d\pi_s^k - \frac{|\sigma^k|^2}{2} ds\right\}} \xi_0(\tau) \left\{d\gamma_\tau^k - \upsilon^k \cdot (\sigma^0)^\intercal d\tau\right\} \right.$$
$$\left. + e^{-[\vartheta]_t^T + \int_t^T \left\{d\pi_s^k - \frac{|\sigma^k|^2}{2} ds\right\}} D_{x^k} B(T, x_T) \mid \mathcal{F}_t \right]$$

(see Karatsas and Shreve, 1991, subsection 5.6.C, or Protter, 1990, section V.9, p. 266). It remains to use (14) to complete the proof •

Equation (16) for the generic security price can be compared to the continuous time CCAPM security pricing formula (6), Theorem 1 of Duffie and Zame, 1989, of which it bears a strong resemblance. The result of the present model is, however, stronger in that it does not require either the existence of a riskless short rate or the asset market completeness. Clearly, there are formal differences due to the "institutional" deviations of my approach from CCAPM.

First, the dividends in (16) are risk-adjusted by their covariance with the cash growth rate. This feature could not be present in CCAPM where the securities positions are taken directly and not exchanged through a randomly growing liquid asset.

Second, in my model, the "term structure of interest rates" component of the asset pricing is linked to the final bequest function, absent form standard CCAPM. Indeed, let k be a zero-coupon bond with maturity T, instantaneous yield to maturity $d\pi^k$ and the principal at maturity added to the cash account x^0_{T-0}. The remaining securities are indexed by the set K'. Logically, the bequest function B does not depend on x^k, since that security ceases to exist after time T. On the other hand, it must be x^0-dependent (a soft liquidity constraint at the end of the modeled period, so that the agent can continue to consume and invest after time T). Accordingly, the special case of (16) for this security looks like

$$P_t^k = \mathbf{E}\left[e^{-[\vartheta]_t^T + \int_t^T \left\{ d\pi_s^k - \frac{|\sigma^k|^2}{2} ds \right\}} \frac{D_{x^0} B(T, x_{T-0}^0 + 1, x_T^{K'})}{u_c(t)} \mid \mathcal{F}_t \right]$$

and reflects the usual properties of discount bonds in a qualitatively correct way.

3 Extensions and examples

The model of Section 2 contains only one liquid asset. I now extend it to a consumption and portfolio selection problem of an international investor who can maintain one liquid asset at home and another abroad. Such an agent also consumes both domestically and internationally, covering the consumption expenses in each part of the world by the respective cash. This model would naturally incorporate the exchange rate between the chosen country and the composite currency of the rest of the world as the relative price of one liquid asset in terms of another. This two-country model and its solution in terms of shadow prices is discussed first.[7]

Another application of the shadow price technique, given in Subsection 2, is a model of a productive company that raises capital by offering newly issued own assets to the market. The presence of production uncertainty leads to a significant analytic difference compared to the consumption/investment problem. The diffusion coefficient in the state-transition equation becomes dependent on a control variable. This means the need to utilize the full power of the adjoint equation technique including the second order adjoint process. The consequence of this fact will be traced down to the equilibrium prices of the firm's assets.

[7]One can then compare the results of the present model regarding asset pricing and equilibrium exchange rate with the classical formulae of Lucas, 1982, after an heuristic restatement of the latter in continuous time.

3.1 International portfolio optimization

Consider an economy where the agents consume one homogenous domestic and another homogenous foreign consumption good. The existence of a representative agent is assumed.

The economy offers four investment possibilities: cash and other domestic liquid assets $\mathcal{M}$ (real balances), international cash and liquid assets $\mathcal{I}$, domestic financial securities $\mathcal{K}$, numbered by $k = 1, ..., K$, foreign financial securities $\mathcal{F}$, numbered by $f = 1, ..., F$. $\mathcal{I}$ is understood as a synthetic international security aggregating those currencies which are held by domestic individuals. Therefore, when talking about the nominal exchange rate S, I will have in mind the price in nominal domestic terms of this currency basket. Accordingly, $Q = S/P$ (the "real exchange rate") will be the price of this basket in terms of *real domestic balances* $\mathcal{M}$, P being the current domestic price level.

The agents trade in the markets for the four named assets as well as $\mathcal{C}^d$ - domestic consumption goods, and $\mathcal{C}^i$ - the imported good. $\mathcal{M}$ can be traded against $\mathcal{C}^d$, $\mathcal{I}$ and $\mathcal{K}$, while foreign goods and securities are bought and sold in exchange for $\mathcal{I}$. Transfers between $\mathcal{M}$ and $\mathcal{I}$ is the only way to switch from domestic to foreign goods and securities and back.

There is a stochastic instantaneous rate of return $d\pi^0$ on $\mathcal{M}$ (meaning the nominal rate of return on the sight deposit component of $\mathcal{M}$ less the inflation rate), and the $\mathcal{M}$-dividend rates $d\Gamma^k$ $(k = 1, ..., K)$ on $\mathcal{K}$. The foreign liquidity $\mathcal{I}$ grows at the rate $d\pi^i$. Domestic security $k \in \mathcal{K}$ has the internal deterioration rate $d\pi^k$ (one can think of a random default rate, stock dilution, etc. Naturally, there are securities, such as government bonds, for which this rate is zero.) International asset $f \in \mathcal{F}$ has the $\mathcal{I}$-dividend $d\gamma^f$ and the internal deterioration rate $d\pi^f$. Cumulative growth and dividend variables π^0, π^i, Γ^k, γ^f, π^k and π^f are Itô processes.

The diffusion parts of the above global risk factors are spanned by a d-dimensional vector Z of standard mutually independent Brownian motions, generating the filtration $\mathcal{J} = (\mathcal{J}_t)_{t \in \mathbf{R}^+}$ satisfying the usual conditions, whose tth element is the time-t publicly available information. All the processes appearing in this example are diffusions w.r.t. basis Z. When dealing with their stochastic differentials, I will use the following notations. For any Itô process x, define its drift by μ^x and its diffusion - by σ^x. Also, for a strictly positive processes x, put $\hat{dx} = dx/x$ and define δ^x and ρ^x by the equality

$$dx = \mu^x dt + \sigma^x dZ = x\hat{dx} = x(\delta^x dt + \rho^x dZ). \quad (1)$$

The $\mathcal{M}$-prices in $\mathcal{K}$-markets are denoted by P^k, the $\mathcal{I}$-prices in $\mathcal{F}$-markets - by P^{*f}. They are $\mathcal{J}$-adapted processes. The representative agent is assumed to be a price (Q, P^k and P^{*f}) and asset return (π^0, π^i, π^k, π^f, Γ^k and γ^f) taker.

The agent's state variables are defined as follows: x^0 - number of held $\mathcal{M}$-units, x^i - number of held $\mathcal{I}$-units, x^k - number of held shares of security $k \in \mathcal{K}$, x^f - number of held shares of security $f \in \mathcal{F}$. Symbols x^K and x^F denote the vectors formed by components x^k, $k = 1, ..., K$, and x^f, $f = 1, ..., F$, respectively.

The agent's control variables are: c^d - domestic consumption good purchases per period (paid out of $\mathcal{M}$-balances), c^i - imported consumption good per period (paid out of $\mathcal{I}$-balances), φ^i - foreign currency purchases (sales if negative) rate, φ^k - number of purchased/sold shares of k per period, φ^f - number of purchased/sold shares of f per period.

The transaction costs in the asset markets were defined in Section 2. The same definition will be used for the transaction costs in the forext market $\mathcal{M}/\mathcal{I}$. The agent's liquidity will be measured by $x^l = x^0 + Qx^i$. For $\varphi^i > 0$, put $\psi^i = j(x^l, Q\varphi^i)$ to be the amount subtracted from the $\mathcal{M}$-account in exchange for φ^i purchased units of $\mathcal{I}$ ($\psi^i > Q\varphi^i$). For $\varphi^i < 0$, $-\psi^i = -j(x^l, Q\varphi^i) > 0$ is the increment in the $\mathcal{M}$-account in exchange for $-\varphi^i$ sold units of $\mathcal{I}$ ($-\psi^i < -Q\varphi^i$). Function j is assumed increasing and strictly convex in $\phi = Q\varphi^i$. One can think of the same linear-quadratic functional form as was introduced in (3) in Subsection 2.1 as the main example:

$$\psi = j(x, \phi) = \phi - \frac{b}{2x}(\phi)^2, \ \phi = h(x, \psi) = \left[\left(\frac{x}{b}\right)^2 + \frac{2x}{b}\psi\right]^{\frac{1}{2}} - \frac{x}{b}, \quad (2)$$

with b a positive constant. The range of values of ψ is such that $2b\psi^i/x^l > -1$ (the purchase rate of domestic currency shall not be too high compared to the current liquid wealth). The marginal transaction function is

$$v(x^l, \psi^i) = j_\psi(x^l, j(x^l, Q\varphi^i))^{-1} = \left(1 + \frac{bQ\varphi^i}{x^l}\right)^{-1} = \left[1 + \frac{2b\psi^i}{x^l}\right]^{-\frac{1}{2}}. \quad (3)$$

Under these notations, the laws of motion of the components of the state vector $x = [x^0, x^i, x^K, x^F]^\top$ look like

$$dx^0 = x^0 d\pi^0 + x^K \cdot d\Gamma - c^d dt - j(x^l, Q\varphi^i)dt - \sum_k P^k j^k(x^0, \varphi^k)dt, \quad \text{(4a)}$$

$$dx^i = x^i d\pi^i + x^F \cdot d\gamma - c^i dt + \varphi^i dt - \sum_f P^{*f} j^f(x^i, \varphi^f)dt, \quad \text{(4b)}$$

$$dx^k = -x^k d\pi^k + \varphi^k dt, \ k = 1, ..., K, \quad \text{(4c)}$$

$$dx^f = -x^f d\pi^f + \varphi^f dt, \ f = 1, ..., F. \quad \text{(4d)}$$

The limitations as regards debt in either domestic or foreign liquidity are expressed in the "soft" form through the presence of variables x^0 and x^i in the period utility function. The partial utility derivatives w.r.t. both x^0 and x^i are assumed decreasing, and too small cash holdings - penalized by the utility diminishing quickly to zero. In this way, I prohibit negative positions in $\mathcal{M}$ and $\mathcal{I}$ and ensure diminishing marginal utility effects of increased cash hoarding. Specifically, I denote by x^* and c^* the $\mathcal{M}$-values of foreign cash and foreign consumption: $x^* = Qx^i$, $c^* = Qc^i$. Let us define the CES liquidity index

$$A(x^0, x^*) = \left[\beta_x^{\frac{1}{\theta}} (x^0)^{\frac{\theta-1}{\theta}} + (1-\beta_x)^{\frac{1}{\theta}} (x^*)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}$$

and the CES consumption index

$$C(c^d, c^*) = \left[\beta_c^{\frac{1}{\eta}} (c^d)^{\frac{\eta-1}{\eta}} + (1-\beta_c)^{\frac{1}{\eta}} (c^*)^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}},$$

where $\theta > 0$ and $\eta > 0$ are constant substitution elasticities. The period utility is posited to be of the form

$$u(x^0, x^i; c^d, c^i) = \frac{1}{1-\frac{1}{\rho}} \left[A(x^0, Qx^i)^\alpha C(c^d, Qc^i)^{1-\alpha}\right]^{1-\frac{1}{\rho}}, \quad (5)$$

for some $\alpha \in (0,1)$, where $\rho > 0$ is the intertemporal substitution elasticity. Thus, the liquidity index A and the consumption index C are imperfect complements in the Cobb-Douglas sense, while the domestic and foreign liquidity/consumption are imperfect substitutes in the CES sense.

The agent's optimization problem is

$$\max_{\ell} \mathbf{E}\left[\int_0^\infty e^{-[\vartheta]_0^t} u(x_t^0, x_t^i; c_t^d, c_t^i)dt\right] \tag{6}$$

w.r.t. controls $\ell = [c^d, c^i, \varphi^i, \varphi^K, \varphi^F]^\mathsf{T}$, subject to the state-transition equation (4), the value x_0 of the state process at time $t = 0$ given. It is, therefore, a memeber of the problem class (2.7), (2.8).

For the problem (4), (6), the adjoint process ξ is a vector of four shadow prices, ξ_0, ξ_i, ξ_K and ξ_F of $\mathcal{M}$, $\mathcal{I}$, $\mathcal{K}$ and $\mathcal{F}$ (the last two are vectors). The transversality condition in this case is

$$\lim_{T\to\infty} e^{-\vartheta T}[\xi_0(T), \xi_i(T), \xi_K(T), \xi_F(T)] = [0, 0, \mathbf{0}_K, \mathbf{0}_F].$$

Let $d\pi^0 = \mu^0 dt + \sigma^0 dZ$, $d\pi^i = \mu^i dt + \sigma^i dZ$, $d\pi^k = \mu^k dt + \sigma^k dZ$, $d\pi^f = \mu^f dt + \sigma^f dZ$, $d\Gamma^k = \Delta^k dt + \epsilon^k dZ$, $k = 1, ..., K$, $d\gamma^f = \delta^f dt + \varepsilon^f dZ$, $f = 1, ..., F$. Interpreting (2.19) for the problem (4), (6), one derives the following s.d.e. of the shadow prices:

$$\begin{aligned} d\xi_0 &= \xi_0(\vartheta dt - d\pi^0 + \mid \sigma^0 \mid^2 dt + j_x(x^l, Q\varphi^0)dt \\ &\quad + P^K \cdot j_x^K(x^0, \varphi^K)dt) - u_{x^0}dt, \end{aligned} \tag{7a}$$

$$\begin{aligned} d\xi_i &= \xi_0 Q j_x(x^l, Q\varphi^i)dt + \xi_i(\vartheta dt - d\pi^i + \mid \sigma^i \mid^2 dt \\ &\quad + P^{*F} \cdot j_x^F(x^i, \varphi^f)dt) - u_{x^i}dt, \end{aligned} \tag{7b}$$

$$\begin{aligned} d\xi_k &= \xi_0\left((\sigma^0 - \sigma^k)\cdot(\epsilon^k)^\mathsf{T} dt - d\Gamma^k\right) \\ &\quad + \xi_k(\vartheta dt + d\pi^k + \mid \sigma^k \mid^2 dt),\ k = 1, ..., K, \end{aligned} \tag{7c}$$

$$\begin{aligned} d\xi_f &= \xi_i\left((\sigma^i - \sigma^f)\cdot(\varepsilon^f)^\mathsf{T} dt - d\gamma^f\right) \\ &\quad + \xi_f(\vartheta dt + d\pi^f + \mid \sigma^f \mid^2 dt),\ f = 1, ..., F. \end{aligned} \tag{7d}$$

These Euler equations apply to both domestic and foreign agents and are symmetric w.r.t. the country of residence, except for the national liquidity preferences u_{x^0}, u_{x^i} that require an obvious formal correction. I will return to this symmetry property when discussing Siegel's paradox in Subsection 4.

Now I apply the maximum principle to the forex market $\mathcal{M}/\mathcal{I}$. Optimal transactions there are described by the current value Hamiltonian optimization w.r.t. φ^i. The first order condition is $-\xi_0 Q j_\phi(x^l, Q\varphi^i) + \xi_i = 0$. This leads to the following expression for the real exchange rate Q as the optimal reservation price in the forex market:

$$Q = \frac{\xi_i}{\xi_0 j_\phi(x^l, Q\varphi^i)} = \frac{\xi_i v(x^l, \varphi^0)}{\xi_0}. \tag{8}$$

Using the posited functional form (3) of the marginal transaction function v, one can rewrite this as

$$Q = \frac{\xi_i}{\xi_0}\left(1 + \frac{b\varphi^i}{x^l/Q}\right)^{-1} = \frac{\xi_i}{\xi_0}\left(1 + \frac{2b\psi^0}{x^l}\right)^{-\frac{1}{2}}. \tag{9}$$

Note that $x^d = x^l/Q$ is the liquidity measure for the international investor (for him, $\mathcal{I}$ is the "domestic" currency). For domestic investors, (9) is the inverse $\mathcal{I}$-demand function if $\varphi^i > 0$ and the inverse $\mathcal{I}$-supply function if $\varphi^i < 0$. If written in the form

$$\frac{1}{Q} = \frac{\xi_0}{\xi_i}\left(1 + \frac{2b\psi^0}{x^0}\right)^{\frac{1}{2}} = \frac{\xi_0}{\xi_i}\left(1 - \frac{2b(-\psi^0)}{x^0}\right)^{\frac{1}{2}},$$

it can be seen as the inverse $\mathcal{M}$-supply for $\psi^0 = j(x^l, Q\varphi^i) > 0$ and the inverse $\mathcal{M}$-demand in the opposite case.

Optimization w.r.t. transactions φ^k and φ^f in the asset markets are described by the first order conditions similar to (9):

$$P^k = \frac{\xi_k}{\xi_0} v^k(x^0, \varphi^k),\ k \in \mathcal{K};\ P^{*f} = \frac{\xi_f}{\xi_i} v^f(x^i, \varphi^f),\ f \in \mathcal{F}. \tag{10}$$

The Hamiltonian maximization w.r.t. c^d and c^i is characterized by the first order conditions

$$\xi_0 = u_{c^d},\ \xi_i = u_{c^i}. \tag{11}$$

In equilibrium, the market clearing conditions $\varphi^k = 0$, $\varphi^f = 0$ for all k and f are assumed. According to our assumptions, this means $v^k \equiv v^f \equiv 1$ for all k and f. As a result, the following characterization of the asset market equilibrium obtains as a direct consequence of (10):

Proposition 3.1 *In an equilibrium with identical investors and zero net exogenous supply rate of all securities, the security prices are given by*

$$P^k = \frac{\xi_k}{\xi_0},\ k = 1, ..., K, \tag{12}$$

$$P^{*f} = \frac{\xi_f}{\xi_i},\ f = 1, ..., F. \tag{13}$$

Equations (12), (13) allow one to derive an "international" CCAPM related to a generalized uncovered yield parity condition (see the next section). As it turns out, the potential of (7), (9), (12) and (13) is much stronger compared to Merton's and Ross' original results. Thanks to the explicit laws of motion of the shadow prices, one can decompose the risk premia existing in the relation between any pair of assets. The "parity violation" becomes a source of valuable information about the properties of the studied economy (Derviz a Klacek, 1998).

3.2 The producer optimization problem

The second application of the shadow price technique deals with a productive firm characterized by four state variables: y^0 - liquidity, i.e. number of cash ($\mathcal{M}$-) units on the current account; y^a - amount of the capital input ($\mathcal{A}$); y^b - amount of currently outstanding bonds issued by the company; y^e - amount of currently outstanding shares issued by the company.

The amount h^c of output $\mathcal{C}$ is produced per period at a variable (labor) cost of $L(y^0, y^a; h^a, h^c)$, where h^a is the newly installed capital. The dependence on y^0 means that the firm needs free liquidity to run the production, with decreasing marginal benefits, i.e. L goes to $-\infty$ when y^0 falls to zero, while the partial derivative L_{y^0} decreases to zero as y^0 increases. Dependence on y^a means that the production requires inputs of the intermediate good (with the intuitive properties of a positive derivative L_{y^a} and decreasing marginal effect). The positive dependence of production costs on h^a means a disruption of the production process caused by the new capital installation, analogous to Tobin's q models. The dependence of L on h^c is assumed to possess the usual properties, so that it is a smooth increasing marginal cost function $h^c \mapsto L_c(y^0, y^a; h^a, h^c)$. L_c generates the traditional component of the $\mathcal{C}$-market inverse supply function of the producer. There is another component of this supply function resulting from intertemporal optimality considerations, as will be shown below.

The labor costs will have one more, stochastic, component, $\zeta(h^c)dZ$, with ζ an increasing (in the vector norm) function of output h^c. Altogether, the variable production costs of the firm are $dw^c = Ldt + \zeta dZ$.

Bond coupons are repaid at rate dj, while bonds themselves mature at rate $d\pi^b$. The amount $h^b dt$ of new bonds is issued per period.

The didvidends are paid in cash at rate h^d per share, so that, in total, the company pays $y^e h^d dt$ per period. The amount of new equity issued per period is $-y^e d\pi^e + h^e dt$, of which h^e is the decision variable and $d\pi^e$ is the new issue noise rate reflecting the technical conditions in the market beyond the company's control.

I define the market price P^a of one unit of capital good $\mathcal{A}$, P^b of a newly issued bond and P^e of the newly issued equity share. The latter two give rise to issue costs $j^b(y^0, h^b)$, $j^e(y^0, h^e)$, with properties analogous to those of the transaction costs defined in Section 2.

Altogether, the vector of controls in the company's decision-making problem is $h = [h^a, h^b, h^c, h^d, h^e]^\top$.

From the given definitions follows the transition equation for the state process $y = [y^0, y^a, y^b, y^e]^\top$:

$$\begin{aligned} dy^0 &= y^0 d\pi^0 - y^b(dj + P^b d\pi^b) - y^e h^d dt + \left\{ph^c - L(y^0, y^a; h^a, h^c)\right\} dt \\ &\quad - \zeta(h^c)dZ - P^a h^a dt + P^b j^b(y^0, h^b)dt + P^e j^e(y^e, h^e)dt, \qquad (14a) \\ dy^a &= -y^a d\pi^a + h^a dt, \qquad (14b) \\ dy^b &= -y^b d\pi^b + h^b dt, \qquad (14c) \\ dy^e &= -y^e d\pi^e + h^e dt. \qquad (14d) \end{aligned}$$

Next, I define the aggregator function for instantaneous valuation of the firm's assets and operations in a given moment of time. It is assumed to have the form

$$F(y; h) = f(y^0, y^e; h^d) - \frac{\beta_c}{2} \mid \zeta(h^c) \mid^2 . \qquad (15)$$

Here, function f measures the firm's ability to pay dividends and finance operations out of current cash holdings y^0, relative to the number of outstanding shares y^e. The second term on the right hand side of (15) expresses the negative valuation in the production process risk given by ζ.

For simplicity, assume that the firm discounts the future by means of the same rate ϑ as the consumers in Section 2. Then the producer's optimization problem is

$$\max_h \mathbf{E}\left[\int_0^T e^{-[\vartheta]_0^t} F(y_t, h_t)dt + e^{-[\vartheta]_0^T} B(T, y_T)\right] \tag{16}$$

subject to the state-transition equation

$$dy = \mu(y,h)dt + \sigma(y,h)dZ \tag{17}$$

which is the symbolic abbreviation of (14), the holdings of assets y_0 at time $t = 0$ given. The final bequest function B is equal to zero by definition if the time horizon of the optimization problem is infinite ($T = \infty$).

When $T < \infty$, there exists a natural final bequest function,which can be useful in many applications. Define by V_t the time t-value function of the problem (16) with infinite horizon:

$$V_t(y) = \max_h \mathbf{E}\left[\int_t^\infty e^{-[\vartheta]_t^s} F(y_s, h_s)ds \mid \mathcal{F}_t\right], \tag{18}$$

where $y_t = y$ is an $\mathcal{F}_t$-measurable initial condition for the transition equation (17). Assume that this value function exists and depends smoothly on all variables. Then, by taking $B(T, y_T) = V_T(y_T)$, the optimization problem can be split into two periods. Within the planning horizon $0 \leq t < T$, there may exist an additional state variable y^*. Typically, y_t^* is the time t-volume of additional security (e.g. derivative) with maturity T. At $t = T$, the final cash flow generated by this security is included into y_T^0 in accordance with its definition. After that, the firm is faced by the optimization problem (17), (18) with the originally defined set of state variables. This approach was used in an American currency option pricing model of Derviz, 1999.

The solution to (16), (17) can be obtained by forming the current value Hamiltonian

$$\mathcal{H}(t, y, h, \lambda, \Lambda) = F(y,h) + \lambda \cdot \mu(y,h) - tr(\Lambda \cdot \sigma(y,h)),$$

where λ and Λ are the first- and second-order adjoint processes (λ is of the same dimention as y and Λ is a matrix), with $\Lambda = \lambda \cdot D_y \sigma^y$. The Hamiltonian must be maximized with respect to h_t. The costate process λ, which appears in the above formulae, can be also described as the y-gradient of the value function of the problem (16), (17).

According to Proposition 2.1, the (first-order) adjoint process λ satisfies the s.d.e.

$$d\lambda = \lambda \cdot (\vartheta \mathbf{1} dt - dA^c + \langle A^c \rangle dt) - D_y F dt, \tag{19}$$

with the y-linearization matrix-valued stochastic process A^cdefined by

$$\begin{aligned} dA^c &= \begin{bmatrix} d\pi^0 - L_{y^0} dt + P^b j_y^b dt + P^e j_y^e dt & -L_{y^a} dt & -dj - P^b d\pi^b & -h^d dt \\ 0 & -d\pi^a & 0 & 0 \\ 0 & 0 & -d\pi^b & 0 \\ 0 & 0 & 0 & -d\pi^e \end{bmatrix} \\ &= A^{c0} dt + A^{c1} dZ. \end{aligned}$$

Another consequence of Proposition 2.1 is the Hamiltonian optimization result. Skipping the f.o.c., I write it out in terms of the relevant inverse supply and demand functions. The inverse supply of $\mathcal{C}$ is

$$p = L_c(y^0, y^a; h^a, h^c) + \left[\frac{\beta_c}{\lambda_0} \zeta(h^c) - \sigma^0 \right] \cdot [\zeta'(h^c)]^{\mathsf{T}}. \tag{20}$$

Observe the ζ-dependence of the right hand side of this formula - the expression of the production uncertainty effect in the output price. The inverse demand for $\mathcal{A}$ and the inverse supplies of $\mathcal{B}$ and $\mathcal{E}$ are given by

$$P^a = \frac{\lambda_a}{\lambda_0} + L_a, \; P^b = -\frac{\lambda_b}{\lambda_0} \frac{1}{j_h^b}, \; P^e = -\frac{\lambda_e}{\lambda_0} \frac{1}{j_h^e} \tag{21}$$

(cf. (10) in Subsection 1).

4 Shadow Prices and Asset Parity

This section returns to the international portfolio optimization model of Subsection 3.1 and shows the consequences of applying Proposition 2.1 to the Walrasian equilibrium in asset markets. The obtained integral formulae are compared to the ones available in the standard continuous time versions of CCAPM. It is shown that the results generalize the ones of Duffie and Zame, 1989, in what concerns the "correct" discounting of expected future returns on the evaluated security.

Analogously to Subsection 2.3, I impose $\mathcal{M}/\mathcal{I}$-, $\mathcal{K}$- and $\mathcal{F}$-market clearing, which is, under the representative investor assumption, equivalent to $\varphi^i = \varphi^K = \varphi^F = 0$.

4.1 Aggregate wealth in equilibrium

The riskiness of the private wealth process can be characterized in terms of the domestic state-price deflator $p = u_{c^d}$ (the shadow price ξ_0 of liquidity in the present model, see (3.11)). It turns out that all other potential risk factors of the wealth process are diversified away in any equilibrium with zero net exogenous asset suuply rate.

Moreover, the riskiness of the private wealth vanishes completely in the case when the domestic liquidity growth rate $d\pi^0$ is zero. This happens in the economies satisfying the following three conditions. First, it is possible to fully hedge the inflation by the existing assets (the inflation risk of the real money holdings is diversifiable). Second, there are no transfers of liquidity proportional to its the actual holdings.[8] Instead, the supply of nominal money happens through the open market operations alone, i.e. there is an asset in the set $\mathcal{K}$ (government bond) whose sales and purchases allow the monetary authority to infuse the money into the economy and withdraw it back. Third, the nominal domestic liquidity pays no uncertain interest.

The above observations are formalized by the following result.

Proposition 4.1 *In an equilibrium with identical investors and zero net exogenous supply of all securities including foreign liquidity, the only diffusion term in the stochastic differential of the private wealth process* $W = x^0 + Qx^i + P^K \cdot x^K + QP^{*F} \cdot x^F$ *is* $-W\hat{d}\xi_0$.

If the domestic liquidity pays no uncertain interest, there are no direct monetary transfers and there is no inflation uncertainty undiversifiable by available assets, in short, if $\pi^0 \equiv 0$, *then the private wealth process is riskfree (contains no diffusion terms).*

Calculation of the diffusion part of individual wealth is carried out in Appendix B.

[8]This particular shortcut of many monetary models, e.g. Lucas, 1982, proves to be very far from innocuous.

The above proposition arouses certain doubts regarding the universal applicability of traditional CCAPM. Indeed, the latter usually works in equilibria with fixed exogenous supplies of all securities, which also implies their zero net exogenous supply rates (leading to $v \equiv 1$), just as was assumed in Proposition 4.1. On the other hand, CCAPM makes use of individually optimal portfolio choice on condition that the basic security price processes generate a non-singular matrix of diffusion coefficients. This makes the wealth process a martingale after the appropriate change of probability measure (transition to risk-neutral probability). Consequently, private wealth is a process whose diffusion part never vanishes. However, as Proposition 1 shows, equilibrium price processes under fixed supply conditions cannot generate a non-singular diffusion matrix, since under these prices, the wealth process is risk-free.

Thus, at least in the models with a representative investor, only equilibria with non-zero average rates of security infusion/withdrawal can be compatible with Merton's portfolio optimization behavior. As the above confrontation with CCAPM suggests, withdrawals and infusion of non-zero asset quantities should be considered a norm rather than anomaly in the economies with rational investors.

4.2 Shadow asset prices and CCAPM

In this subsection I am examining the generalized asset return parity mechanism. In the context of international portfolio optimization, this means connecting the shadow asset prices with information about the exchange rate expectations.

Untill the end of the section, the real exchange rate Q is replaced by the nominal one, S. Understanding of all variables and equations shall be adjusted accordingly. Justification for this replacement is the focus on international investors who do not care about domestic inflation. From a purely technical point of view, domestic inflation parameters can be eliminated at the expense of additional covariance terms in the state and co-state equations (going over from real to nominal values of return rates and relative price changes leads only to changes in the diffusion parameters).

Take an arbitrary domestic asset $k \in \mathcal{K}$ and an arbitrary foreign asset $f \in \mathcal{F}$. Recall (3.12) and (3.13) which describe their equilibrium prices in terms of the shadow prices, and unify those equations as follows:

$$\xi_0 = \frac{\xi_i v}{S} = \frac{\xi_k}{P^k} = \frac{\xi_f v}{SP^{*f}}. \tag{1}$$

To be able to differentiate this formula, one needs the expressions for relative changes in the shadow prices ξ_k, ξ_f. As follows from the adjoint equations (3.7c), (3.7d) and the equilibrium conditions (1),

$$\hat{d}\xi_k = -\frac{d\Gamma^k}{P^k} + (\sigma^0 - \sigma^k) \cdot \left(\frac{\epsilon^k}{P^k}\right)^{\mathsf{T}} dt + \vartheta dt + d\pi^k + \mid \sigma^k \mid^2 dt, \tag{2a}$$

$$\hat{d}\xi_f = -\frac{d\gamma^f}{P^{*f}} + (\sigma^i - \sigma^f) \cdot \left(\frac{\varepsilon^f}{P^{*f}}\right)^{\mathsf{T}} dt + \vartheta dt + d\pi^f + \mid \sigma^f \mid^2 dt. \tag{2b}$$

Rewriting (1) as $\xi_k SP^{*f} = \xi_f P^k v$ and applying Itô's lemma together with (2a), (2b), obtain

$$\frac{d\Gamma^k}{P^k} + \hat{d}P^k - d\pi^k = \frac{d\gamma^f}{P^{*f}} + \hat{d}P^{*f} - d\pi^f + \hat{d}S - \hat{d}v + h_0 dt \tag{3}$$

(h_0 is a shorthand for the term containing diffusion vector norms and scalar products that are not analyzed in detail).

The left hand side of this equation can be understood as the total return rate (instantaneous yield) dY^k on asset k: the dividend/coupon rate relative to the current price, plus the relative capital gain/loss, minus the internal value loss/attrition rate. In the same way, the first three terms on the right hand side form the total return rate dY^f on asset f. The last two terms, which I shall denote $d\Lambda^{kf}$, are the forex market marginal transaction cost differential and a risk-adjustment term. Collectively, these terms can be dubbed *disparity rate* between assets k and f. The name points at the fact that without $d\Lambda^{kf}$, (3) would reduce to the *uncovered total return parity* condition on assets k and f.

To illustrate the potential of the generalization of the uncovered asset return parity condition (3), I want to establish its connection to CCAPM as well as another empirically controversial topic in international finance: Siegel's paradox (Siegel, 1972).

Take the totality of domestic securities $\mathcal{K}$ and its arbitrary element, k. Following the continuous time formulation of Duffie and Zame, 1989, (later restated in Duffie, 1992), I want to relate the total average return y^k on asset k, the short rate r and the state-price deflator p. If ρ^k is the volatility of the total return on k, then, according to Duffie, 1992, Section 9, equation (37),

$$y^k - r = -\rho^k \cdot (\rho^p)^\top. \tag{4}$$

Going over to our notations, y^k is the drift of process Y^k appearing on the left hand side of (3). The state-price deflator is equal to the marginal utility of consumption in the traditional CCAPM, therefore, in view of (11), $p = \xi_0$. Finally, the short rate r can be identified with the volatility-adjusted drift of the $\mathcal{M}$-return process π^0: $r = \mu^0 - \mid \sigma^0 \mid^2$. It is in this combination that parameters μ^0 and σ^0 appear in the shadow cash price differential. Also, it is in this form that the process π^0 appears in the discount rates of integral asset pricing analogues of Lucas, 1982, which can be derived from (3.7).

Rewrite (3.12) as $P^k = \xi_k/\xi_0$ and differentiate using Itô's lemma and (3.7), to get

$$\begin{aligned} \hat{d}P^k + \frac{d\Gamma^k}{P^k} - d\pi^k - d\pi^0 + \mid \sigma^0 \mid^2 dt = \mid \sigma^k \mid^2 dt + (\sigma^0 - \sigma^k) \cdot \left(\frac{\epsilon^k}{P^k}\right)^\top dt \\ - \rho^{P^k} \cdot (\rho^{\xi_0})^\top dt + \frac{u_{x^0} - j_x}{u_{c^d}} dt. \end{aligned} \tag{5}$$

Taking the drift parts of both sides, observing that $\rho^{\xi_0} = -\sigma^0$ and transforming, obtain

$$y^k - r = -\rho^k \cdot (\rho^p)^\top + (\sigma^0 - \frac{\epsilon^k}{P^k} + \sigma^k) \cdot (\sigma^k)^\top + \frac{u_{x^0} - j_x}{u_{c^d}}. \tag{6}$$

The first term on the right hand side here is the same as the right hand side of (4). The second term is present only when the internal attrition rate of security k is stochastic and non-zero. It can be neglected for many default-free instruments, bringing (6) closer to the classical result. However, the remaining terms are not so easy to discard. They originate in the transaction technologies of the $\mathcal{M}/\mathcal{I}$-market (term $-j_x/u_{c^d}$) and the liquidity preferences of agents (term u_{x^0}/u_{c^d}). Evidently, in our model with soft liquidity constraints, the traditional CCAPM must be always adjusted by the terms given in (6).

Equation (3.12), from which the above formula was derived, is itself a much richer result than (4) of the traditional CCAPM. For example, its diffusion part renders the asset price volatility formula $\rho^{P^k} = \rho^{\xi_k} - \rho^{\xi_0} = \sigma^k - \epsilon^k/P^k + \sigma^0$. This is quite natural, since the results based upon the stochastic maximum principle should contain more information than those relying on the *ad hoc* solutions of Merton or martingale techniques of Cox and Huang.

4.3 Non-existence of Siegel's paradox

Siegel's, 1972, observation that the uncovered interest rate parity, if understood in a naive non-arbitrage way, cannot be valid simultaneously for international and domestic investors, has to do with convexity of the function $x \mapsto 1/x$ which transforms the exchange rate of the domestic agents into that of the international ones. In that sense, the analytical background of the paradox is common with Itô's lemma for the same function.

I am going to show that Siegel's paradox does not arise in the model defined in Subsection 3.1.

Proposition 4.2 *For any pair of assets $k \in \mathcal{K}$ and $f \in \mathcal{F}$, let the $k-f$ disparity Λ^{kf} and the $f-k$ disparity Λ^{fk} be defined by (3). Then they satisfy the necessary symmetry, or consistency condition $d\Lambda^{fk} = -d\Lambda^{kf} - \mid \rho^S \mid^2 dt$, so that*

$$dY^k = dY^f + \hat{d}S + d\Lambda^{kf},\ dY^f = dY^k + \hat{d}\left(\frac{1}{S}\right) + d\Lambda^{fk}$$

hold simultaneously.

Proof The argument is trivial and serves as a testimony of the power of the shadow asset price method. The relation between dY^k and dY^f was obtained from (1). The latter is the fundamental asset pricing formula of the model when "seen with the eyes of" domestic investors. The same equation can be restated in the form corresponding to the foreign perspective:

$$\xi_i = \frac{\xi_f}{P^{*f}} = \frac{\xi_k S}{P^k v}.$$

Now observe that not only S becomes $1/S$, but also v becomes $1/v$ if expressed from the point of view of international investors. Indeed, the marginal transaction process v has a symmetrical definition. It is the derivative of the other country cash w.r.t. the own country cash involved in the forex market transaction: $v = d\psi/d\varphi$.

I conclude that (3) must be just as symmetric and incensitive to the change of perspective from domestic to foreign as the equation from which it was derived, i.e. (1) •

5 Conclusion

This paper has presented an approach to portfolio optimization and asset market equilibrium under diffusion uncertainty, based on the adjoint equation for the problems of stochastic optimal control. The adjoint equation is an s.d.e. for the co-state process of the representative agent's optimization problem. Shadow asset prices, which are co-state processes of the representative agent, become the determinants of equilibrium asset prices. It is argued that the approach based on s.d.e. for the shadow asset prices improves the technical possibilities of analyzing security price and supply/demand formation outside long-run steady states, as compared to the traditional techniques involving total wealth maximizing individuals. I have demonstrated the role played by the co-state (shadow price) process in stochastic general equilibrium models with securities markets. The derived stochastic Euler equation in Itô form enables one to characterize solutions of problems with state-dependent utilities by a law of motion for the co-states, which is analogous to the one appearing in deterministic environments. This proves to be useful, among other things, in problems with borrowing constraints.

In Section 2, the basic model of consumption and portfolio adjustment is analyzed and the principle of applying the stochastic maximum principle to this problem and the adjoint equation is stated. I propose a theorem servicing a more general situation than the one defined by the consumer's portfolio problem. Namely, I formulate an optimization problem with the transition equation where the diffiusion parameter can depend on controls. This statement is later applied to the problem of a firm with production costs uncertainty (Subsection 3.2). The derivation of the adjoint equation is given in Appendix A.

In Section 3, I discuss an application to international portfolio diversification and the equilibrium exchange rate, plus the already mentioned productive firm problem. In Section 4, I compare the shadow price technique with the ones used so far in the CCAPM literature and demonstrate its power in situations where the martingale technique of the extant finance theory does not work. One of the unexpected results obtained with my method is risklessness of the private wealth process in a class of asset market equilibria with zero external infusion or withdrawal of asset quantities (proof is given in Appendix B). This finding would be impossible in the standard theory where the wealth can always be transformed into a martingale by an appropriate growth-adjustment and a change of probability measure. One can conclude that the validity of classical partial equilibrium results in portfolio theory (Merton-type) implies the presence of certain exogenous supply risks in the asset markets.

I further analyze the obtained equilibrium asset prices from the point of view of CCAPM and the uncovered parity of total returns. The shadow price method allows one to obtain a generalization of CCAPM for the problems with transaction costs. Also, I prove that the uncovered total return parity in the proposed model does not give rise to Siegel's papradox, i.e. this generalized parity condition is symmetric with respect to the nationality of the investor.

Appendix

A Proof of Proposition 2.1

In this appendix, Proposition 3.1 will be derived from a more general result having to do with the maximum principle for the case of optimal control of a vector-valued Itô process with control-dependent diffusion coefficient. I start by introducing a number of notations and auxiliary statements.

Lemma A.1 *Let y_t be an $\mathbf{R}^n$-valued (column-vector) semimartingale satisfying the linear s.d.e.*

$$dy = d\alpha \cdot y + d\beta$$

for some quadratic matrix-valued process α and a column-vector process β. If the initial value y_0 is given, then y is a unique strong solution of this equation and for all $0 \leq t \leq T$, the formula

$$y_T = e^R(\alpha)_t^T \cdot y_t + \int_t^T e^R(\alpha)_s^T \cdot \{d\beta_s - d[\alpha, \beta]_s\} \tag{1}$$

is valid.

Similarly, for the unique row-vector process η satisfying the linear s.d.e.

$$d\eta = \eta \cdot dh + dj$$

with the endpoint condition η_T, the "backward-integration" formula

$$\eta_t = E\left[\eta_T \cdot \left(e^L(h)_t^T\right)^{-1} + \int_t^T \{d[j,h]_\tau - dj_\tau\} \cdot \left(e^L(h)_t^\tau\right)^{-1} \mid \mathcal{F}_t\right] \tag{2}$$

is valid for all $0 \le t \le T$.

The proof is a simple consequence of Karatzas and Shreve, 1991, subsection 5.6.C, p. 361, or Protter, 1991, Sec. V.9, Theorem 52, p. 266.

Consider a problem

$$\max_\ell E\left[\int_0^T e^{-\Theta_0^t} U(t, x_t, \ell_t) dt + e^{-\Theta_0^T} B(T, x_T)\right], \tag{3}$$

with maximum taken over $\mathcal{F}$-adapted m-dimensional processes $t \mapsto \ell_t$ from the space $\mathbf{H}_m$ with norm $\| \ell \|^2 = E\left[\int_0^T \mid \ell_t \mid^2 dt\right]$, subject to the dynamic constraint (state-transition equation) for the n-dimensional state process x:

$$dx_t = \mu(t, x_t, \ell_t)dt + \sigma(t, x_t, \ell_t)dZ_t, \tag{4}$$

x_0 given. Here, Z is a d-dimensional standard Wiener process generating the information filter $\mathcal{F} = (\mathcal{F}_t)_{t \ge 0}$.

Let U, B, μ and σ be continuously differentiable functions with Lipschitz first derivatives and polynomial growth in x. In addition, let function U be strictly increasing and concave in ℓ.

For notational simplicity, the function of time $t \mapsto f(t, x_t, \ell_t)$ will be often shortened to f_t for any function f of time, state and control, when x_t, ℓ_t are defined by (3), (4).

Put

$$a_t = \int_0^t \left(D_x\mu_s ds + D_x\sigma_s dZ_s\right).$$

The principal result of this appendix is

Theorem A.1 *Define the current value Hamiltonian of the problem (3), (4) by*

$$\mathcal{H}(t, x, \ell, \xi, \Xi) = U(t, x, \ell) + \xi \cdot \mu - \Xi \bullet \sigma^\mathsf{T}.$$

Here, $\xi \in \mathbf{R}^{n*}$ (a row vector), and $\Xi \in \mathbf{R}^{n*} \otimes \mathbf{R}^d$ (an $n \times d$-matrix), the bullet being a shorthand for taking the trace of the matrix product.

Suppose that there exists a process $\hat{\ell}_t \in \mathbf{H}_m$ giving the internal solution of the problem (3), (4), i.e. the Fréchet differential of the functional defined by (3), calculated at $\hat{\ell}$, is zero. Then this control process must satisfy the first order condition

$$D_\ell \mathcal{H}(t, x_t, \hat{\ell}_t, \xi_t, \Xi_t) = 0,\ 0 \le t \le T, \tag{5}$$

where the **second-order adjoint process** Ξ *is given by* $\Xi_t = \xi_t \cdot D_x\sigma(t, x_t, \hat{\ell}_t)$, *and the* **(first-order) adjoint process** ξ *- by*

$$\begin{aligned}\xi_t = E\Bigg[\int_t^T D_x\hat{U}_s \cdot \left(e^L(\Theta\mathbf{1}_n - a + [a])_t^s\right)^{-1} ds \\ + D_xB(T, x_T) \cdot \left(e^L(\Theta\mathbf{1}_n - a + [a])_t^T\right)^{-1} \mid \mathcal{F}_t\Bigg], \quad (6)\\ \xi_T = D_xB(T, x_T). \quad (7)\end{aligned}$$

For strictly concave U, the Hamiltonian is strictly concave as well, so that (5) is also sufficient for optimality.

The result of the theorem (equation (6)) combined with Lemma A.1 (equation (2)) gives the following

Corollary A.1 *The adjoint process ξ satisfies the linear s.d.e.*

$$d\xi = \xi \cdot (\theta\mathbf{1}_n - da + d[a]) - D_xU dt \tag{8}$$

with the endpoint condition $\xi_T = D_xB(T, x_T)$.

Proposition 3.1 follows from the above theorem as a special case.

PROOF OF THE THEOREM The following proof is a synthesis of results found in existing literature on the stochastic maximum principle and adjoint processes, such as Cadenillas and Karatzas, 1995, Peng, 1990, and some others, written out here on the level of generality which corresponds to the problem under discussion.

For the arbitrary admissible control $t \mapsto \ell_t$, let

$$V_0 = E\left[\int_0^T e^{-\Theta_0^t} U(t, x_t, \ell_t) dt + e^{-\Theta_0^T} B(T, x_T)\right]$$

be the performance index (at $t = 0$) of this control trajectory. The optimal performance index corresponding to $\hat{\ell}$ will be shortened to $\hat{V}_0$. Our aim is to calculate the Fréchet derivative of the functional $\ell \mapsto V_0(\ell)$ acting from $\mathbf{H}_m$ to $\mathbf{R}$, at point $\hat{\ell}$.

Denote by $t \mapsto \hat{x}_t$ the state process controlled by $\hat{\ell}$ (in general, the hat symbol always denotes a function or expression where $\hat{x}$, $\hat{\ell}$ are substituted for x, ℓ), put $\Delta\ell_t = \ell_t - \hat{\ell}_t$, $\Delta x_t = x_t - \hat{x}_t$ and observe that

$$\begin{aligned} V_0 - \hat{V}_0 = E\Bigg[\int_0^T e^{-\Theta_0^t}\Big(&D_\ell U(t,\hat{x}_t,\hat{\ell}_t)\cdot\Delta\ell_t + D_x U(t,\hat{x}_t,\hat{\ell}_t)\cdot\Delta x_t \\ &+ R^1(t,\Delta\ell_t) + R^2(t,\Delta x_t)\Big)\,dt \\ &+ +e^{-\Theta_0^T}\Big(D_x B(T,\hat{x}_T)\cdot\Delta x_T + R^3(T,\Delta x_T)\Big)\Bigg], \end{aligned} \tag{9}$$

where $R^1 = \mathcal{O}(\mid \Delta\ell \mid^2)$, $R^2, R^3 = \mathcal{O}(\mid \Delta x \mid^2)$, as follows from the first order Taylor decomposition. We apply this decomposition once again to the integral version of (4) to get

$$\begin{aligned} \Delta x_t = \int_0^t \{&[D_\ell\hat{\mu}_\tau d\tau + D_\ell\hat{\sigma}_\tau dZ_\tau]\cdot\Delta\ell_\tau + [D_x\hat{\mu}_\tau d\tau + D_x\hat{\sigma}_\tau dZ_\tau]\cdot\Delta x_\tau \\ &+ Q^1(\tau,\Delta x_\tau)d\tau + Q^2(\tau,\Delta x_\tau)dZ_\tau \\ &+ Q^3(\tau,\Delta\ell_\tau)d\tau + Q^4(\tau,\Delta\ell_\tau)dZ_\tau\Big\}, \end{aligned} \tag{10}$$

with $Q^1, Q^2 = \mathcal{O}(\mid \Delta x \mid^2)$, $Q^3, Q^4 = \mathcal{O}(\mid \Delta\ell \mid^2)$.

Consider the higher order terms Q^1, Q^2, Q^3, Q^4 as given processes. Then, according to Lemma A.1 (equation (1)), relation (10), regarded as a linear s.d.e. for Δx, can be "integrated" as follows:

$$\begin{aligned} \Delta x_t = \int_0^t e^R(a)_\tau^t\cdot\{&[(D_\ell\hat{\mu}_\tau - \langle D_x\hat{\sigma}_\tau, D_\ell\hat{\sigma}_\tau\rangle)\,d\tau + D_\ell\hat{\sigma}_\tau dZ_\tau]\cdot\Delta\ell_\tau \\ &+ \Big(Q^5(\tau,\Delta x_\tau) + Q^6(\tau,\Delta\ell_\tau)\Big)\,d\tau \\ &+ \Big(Q^2(\tau,\Delta x_\tau) + Q^4(\tau,\Delta\ell_\tau)\Big)\,dZ_\tau\Big\}, \end{aligned} \tag{11}$$

with $Q^5 = \mathcal{O}(\mid \Delta x \mid^2)$, $Q^6 = \mathcal{O}(\mid \Delta\ell \mid^2)$.

From the uniqueness of the strong s.d.e. solutions and the definitions of stochastic exponentials it follows that

$$e^R(a)_\tau^t = e^R(a)_0^t\cdot\Big(e^R(a)_0^\tau\Big)^{-1}.$$

Therefore,

$$E\left[\int_0^T e^{-\Theta_0^T} D_x\hat{U}_t\cdot\int_0^t e^R(a)_\tau^t\cdot\Big(D_\ell\hat{\sigma}_\tau\cdot\Delta\ell_\tau + Q_\tau^2 + Q_\tau^4\Big)\,dZ_\tau dt\right]$$

$$= E\left[\int_0^T e^{-\Theta_0^T} D_x\hat{U}_t \cdot e^R(a)_0^t \cdot \int_0^t \left[e^R(a)_0^\tau\right]^{-1} \cdot \left(D_\ell\hat{\sigma}_\tau \cdot \Delta\ell_\tau + Q_\tau^2 + Q_\tau^4\right) dZ_\tau dt\right]$$
$$= E\left[\int_0^T f_\tau^1 dZ_\tau\right] = 0 \tag{12}$$

for some process f^1.

Similarly,

$$E\left[e^{-\Theta_0^T} D_x B(T,\hat{x}_T) \cdot e^R(a)_0^T \cdot \int_0^T \left(e^R(a)_0^\tau\right)^{-1}\right.$$
$$\left.\cdot \left(D_\ell\hat{\sigma}_\tau \cdot \Delta\ell_\tau + Q_\tau^2 + Q_\tau^4\right) dZ_\tau\right]$$
$$= E\left[\int_0^T E\left[D_x B_T \cdot e^R(-\Theta\mathbf{1}_n + a)_0^T \mid \mathcal{F}_\tau\right] \left(e^R(a)_0^\tau\right)^{-1}\right.$$
$$\left.\cdot \left(D_\ell\hat{\sigma}_\tau \cdot \Delta\ell_\tau + Q_\tau^2 + Q_\tau^4\right) dZ_\tau\right]$$
$$= E\left[\int_0^T f_\tau^2 dZ_\tau\right] = 0 \tag{13}$$

for some process f^2. From (12) and (13) follows that when (11) is substituted into (9), all terms containing dZ disappear. Collecting the remaining terms, we conclude that

$$V_0 - \hat{V}_0 = E\left[\int_0^T e^{-\Theta_0^t}\left(D_\ell\hat{U}_t \cdot \Delta\ell_t + D_x\hat{U}_t \cdot \int_0^T e^R(a)_\tau^t\right.\right.$$
$$\cdot \ \left\{(D_\ell\hat{\mu}_\tau - \langle D_x\hat{\sigma}_\tau, D_\ell\hat{\sigma}_\tau\rangle) \cdot \Delta\ell_\tau\right.$$
$$\left.\left. + \ Q^5(\tau,\Delta x_\tau) + Q^6(\tau,\Delta\ell_\tau)\right\} + R^1(t,\Delta\ell_t) + R^2(t,\Delta x_t)\right) dt$$
$$+ \ e^{-\Theta_0^T} D_x\hat{B}_T \cdot \int_0^T e^R(a)_\tau^T \cdot \{(D_\ell\hat{\mu}_\tau - \langle D_x\hat{\sigma}_\tau, D_\ell\hat{\sigma}_\tau\rangle) \cdot \Delta\ell_\tau$$
$$\left.+ \ Q^5(\tau,\Delta x_\tau) + Q^6(\tau,\Delta\ell_\tau)\right\} d\tau\Bigg]. \tag{14}$$

It is necessary to establish the order of the terms R^2, Q^5 in (14) as regrards the power of $\mid \Delta\ell \mid$. From the estimates obtained in Bensoussan, 1988, or Peng, 1990, it can be derived that $\| \Delta x \|^2 = \mathcal{O}(\| \Delta\ell \|)$. Since this is not enough to separate the higher order terms in problems with control-dependent diffusion coefficient in the transition equation, the cited authors have to invoke the second order Taylor decompositions w.r.t. state and control explicitly in their arguments. In view of the representation (11) (which, in turn, follows from the explicit solution of linear s.d.e.), we can circumvent this difficulty by separating the trend and diffusion parts of R^2 and Q^5.

The trend parts of R^2, Q^5 are of the second order w.r.t. $\mid \Delta\ell \mid$, as follows from Itô's lemma and the possibility to iterate (11). Therefore, their time integrals are of the second order w.r.t. $\parallel \Delta\ell \parallel$. The diffusion parts of R^2, Q^5 disappear in expectation, which can be shown in the same way as in (12), (13). Henceforth, we can introduce a common notation $\mathbf{Q}$ for the higher order terms w.r.t. $\parallel \Delta\ell \parallel$ in (14) and write

$$\begin{aligned} V_0 - \hat{V}_0 = E\Bigg[\int_0^T e^{-\Theta_0^\tau}\Bigg\{D_\ell \hat{U}_\tau + E\Bigg[\int_\tau^T D_x\hat{U}_t \cdot e^R(-\Theta\mathbf{1}_n + a)_\tau^t dt \\ + \; D_x\hat{B}_T \cdot e^R(-\Theta\mathbf{1}_n + a)_\tau^T \mid \mathcal{F}_\tau\Bigg] \cdot (D_\ell\hat{\mu}_\tau - \langle D_x\hat{\sigma}_\tau, D_\ell\hat{\sigma}_\tau\rangle)\Bigg\} \cdot \Delta\ell_\tau d\tau\Bigg] \\ + \mathbf{Q}(\parallel \Delta\ell \parallel), \end{aligned} \tag{15}$$

meaning that the Fréchet derivative of V is given by

$$D_{\hat{\ell}}^F V(h) = E\left[\int_0^T e^{-\Theta_0^t}\left\{D_\ell\hat{U}_t + \xi_t \cdot (D_\ell\hat{\mu}_t - \langle D_x\hat{\sigma}_t, D_\ell\hat{\sigma}_t\rangle)\right\} \cdot h_t dt\right], \tag{16}$$

for all $h \in \mathbf{H}_m$, with

$$\begin{aligned} \xi_t &= E\left[\int_0^T D_x\hat{U}_s \cdot e^R(-\Theta\mathbf{1}_n + a)_t^s ds + D_x\hat{B}_T \cdot e^R(-\Theta\mathbf{1}_n + a)_t^T \mid \mathcal{F}_t\right], \\ \xi_T &= D_x\hat{B}_T. \end{aligned} \tag{17}$$

In view of the generally valid relation

$$e^R(\pi)_t^s = \left(e^L(-\pi + [\pi])_t^s\right)^{-1}, \; t \le s,$$

for all admissible semimartingales π, one obtains

$$e^R(-\Theta\mathbf{1}_n + a)_t^s = \left(e^L(\Theta\mathbf{1}_n - a + [a])_t^s\right)^{-1}, \; 0 \le t \le s \le T,$$

so that ξ can be expressed as

$$\begin{aligned} \xi_t = E\Bigg[\int_0^T D_x\hat{U}_s \cdot \left(e^L(\Theta\mathbf{1}_n - a + [a])_t^s\right)^{-1} ds \\ + D_x\hat{B}_T \cdot \left(e^L(\Theta\mathbf{1}_n - a + [a])_t^T\right)^{-1} \mid \mathcal{F}_t\Bigg], \end{aligned}$$

with terminal condition (17) •

B Proof of Proposition 4.1

From the adjoint equation of the investor's problem follow the s.d.e. for individual shadow prices. We are interested in their diffusion paprts.

According to (3.8), (3.11), (3.12),

$$Q = \frac{\xi_i}{\xi_0},\ P^k = \frac{\xi_k}{\xi_0},\ P^{*f} = \frac{\xi_f Q}{\xi_i} = \frac{\xi_f}{\xi_0}. \tag{1}$$

Therefore, the wealth is equal to

$$W = \frac{\xi \cdot x}{\xi_0}.$$

For every y, let the shorthand $\hat{d}y$ stand for the stochastic differential expression dy/y. Applying Itô's lemma, get

$$dW = \frac{d(\xi \cdot x)}{\xi_0} - \frac{\xi \cdot x}{\xi_0}\hat{d}\xi_0 + R_1 dt \tag{2}$$

for some process R_1 which is not necessary to specify nearer, since we are interested only in the diffusion part of dW.

The first two terms on the right hand side of (2) have the same diffusion part as the expression

$$\begin{aligned}
&\frac{1}{\xi_0}\left\{d\xi \cdot x + \xi \cdot dx - \xi \cdot x\hat{d}\xi_0\right\} \\
&= \frac{1}{\xi_0}\{\xi \cdot (\theta \mathbf{1}_n dt - dA + \langle A \rangle dt) \cdot x - D_x u \cdot x dt + \xi \cdot (dA \cdot x + R_2(x,\ell)dt) \\
&\quad -\xi \cdot x\hat{d}\xi_0\Big\} = -W\hat{d}\xi_0 + R_3 dt.
\end{aligned}$$

The first equality above follows from Corollary 2.1. Indeed, one can replace dx by $x \cdot dA + R_2(x,\ell)dt$ for some R_2, since the part of the right hand side of the transition equation which is non-linear in x does not contain diffusions.

Now we see that the diffusion part of dW can only originate from that of $d\xi_0$ in the equilibrium we are concerned about.

As follows from the adjoint s.d.e. (3.7a) for ξ_0, the diffusion part of $\hat{d}\xi_0$ is equal to $-d\pi^0$. That is, in the absence of diffusion uncertainties in π^0, ξ_0 has zero diffusion part, and the same is true for W •

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