DISJOINTNESS OF FUZZY COALITIONS

(Discussion)*

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Abstract. The cooperative games with fuzzy coalitions in which some players act in a coalition only with a fraction of their total "power" (endeavor, investments, material, etc.) or in which they can distribute their "power" in more coalitions, are connected with some formal or interpretational problems. Some of these problems can be avoided if we interpret each fuzzy coalition as a fuzzy class of crisp coalitions, as shown in [9, 10, 11]. In this paper, the relation between this model of fuzziness and the original one (in which a fuzzy coalition is a fuzzy set of players) is elucidated, and properties of the model are analyzed and briefly interpreted. The analysis is focused on the concept of disjointness of fuzzy coalitions and its role in the study of superadditivity of games with fuzzy coalitions. In particular, three variants of disjointness are introduced and their consistency is discussed. The derived results may be used for further development of the theory of games with fuzzy coalitions characterized by fuzzy sets of crisp coalitions. They show that the procedure developed in [11] appears to be the most adequate.

1 Introduction

In this paper, we deal with fuzzification of coalitional games with transferable utility, briefly TU-games. Generally, TU-games can be fuzzified in several ways depending upon which of the data specifying the game are uncertain. Here we are concerned with modeling the situations in which coalitions may be fuzzy whereas the total payoff to each coalition remains to be known precisely. Such models, which allow players to participate in several coalitions have been studied since the seventies of the last century, see [1, 2] and [6]. For more recent studies, see, for example, [3, 4, 8, 9].

The following sections are motivated by some specific uncertainties connected with the interpretation of that model and with its "translation" into the reality of cooperative behavior. Namely, if each (fuzzy) coalition is considered to be a fuzzy subset of the set of

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all players, with membership function defined for all players (even if sometimes vanishing) then the real structure of cooperation in coalitions including the partition of the set of players into some "groups dealing the same interests" can become rather hidden and the conflict of motivations can be unclear. Some very brief comments on this were mentioned in [8] and partly in [9], too, and in [11] one of its possible versions is suggested, as well. From the formal point of view, there may appear some doubts about the sense of the concept of disjointness of coalitions and, consequently, the concepts of superadditivity, additivity, coalitional structure, and related notions.

An attempt to handle these difficulties can be based on a modification of the formalism describing the concept of fuzzy coalition. The "traditional" fuzzy coalitions defined as fuzzy sets of players can easily be transformed into fuzzy classes of crisp coalitions. This transformation preserves the main advantages of the fuzzy sets of players and, moreover, it offers even more refined diversification of the cooperative bounds objectively existing in the game. Moreover, it offers a possibility to use the well-known properties of crisp coalitions (and deterministic TU-games) even for the applications in which the structure of cooperation is rather vague. In the main parts of the paper, we study three potentially possible versions of such model. More exactly, the eventual models of the considered game may be treated as certain mixtures of the two models of fuzzy coalitions – their representations by fuzzy sets of players or by fuzzy sets of crisp sub-coalitions. Their analysis is focused on the mutual compatibility of these two approaches and on their proportions in one eventual model. The main results regard very elementary (and, consequently, fundamental) concepts of the cooperation theory, namely the disjointness of coalitions and the superadditivity. The following text represents a rather discussion paper contributing to the methodological analysis of the adequacy of particular approaches to the reality of cooperative behaviour as well as to the formal operability of the model.

2 Coalitional Games with Transferable Utility

By a coalitional game with transferable utility we mean an ordered pair $G = (I, v)$ where I is a nonempty finite set and v is a real valued function defined on the set of all subsets of I such that, for the empty set $\emptyset$, $v(\emptyset) = 0$. The members of I are called players, the subsets of I are called coalitions, and the function v is called the characteristic function of G. The value $v(K)$ of the characteristic function at a coalition K is interpreted as the total payoff that is available to coalition K for division among the members of K.

For every coalition K, each vector $\boldsymbol{x}^K = (x_i)_{i \in K} \in R^K$ such that

$$\sum_{i \in K} x_i \le v(K) \tag{1}$$

represents an *achievable* distribution of the total payoff of coalition K among the members of K.

The game is called *superadditive* iff, for each pair of disjoint coalitions K and K',

$$v(K \cup K') \ge v(K) + v(K'). \tag{2}$$

For a detailed treatment, see, for example, [5, 13, 14].

3 Fuzzy Coalitions – Classical Model

As already explained in the Introduction, in many cases, it is not realistic to assume that each player participates in exactly one coalition which consumes all his potential "power". In fact, we often part our endeavor into cooperative activities within the frame of several groups sharing common interest in a social or economic process. One of these "groups" may be even the one-player coalition. This distribution of player's endeavor is modelled by the tools of fuzzy set theory – each coalition is considered to be a fuzzy subset of the set I.

In the following sections, for every set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X and by $\mathcal{F}(X)$ the class of all fuzzy subsets of X. To differentiate between subsets of X and fuzzy subsets of X, we sometimes say "crisp subsets" of X instead of "subsets" of X. Thus crisp coalitions are elements of $\mathcal{P}(I)$.

Every *fuzzy coalition* $L \in \mathcal{F}(I)$ is characterized by its membership function $\tau_L : I \to [0, 1]$ with the usual interpretation (cf. [1, 2, 3, 10, 9]). Since the participation level $\tau_L(i)$ is a number from the unit interval $[0, 1]$, we can identify $\mathcal{F}(I)$ with the unit hypercube $H^n = [0, 1]^n$ where n denotes the number of players in the game. Consequently, the vertices of H^n represent crisp coalitions. In other words, a crisp coalition K can be identified with a fuzzy coalition whose membership function has value 1 for $i \in K$ and 0 otherwise. To simplify the orientation in the following text, we denote the crisp coalitions by K (possibly with indices), and the fuzzy coalitions by letters L, J, M (possibly with indices). For the crisp coalition of all players, we also use the letter I.

Without loss of generality, we suppose that $I = \{1, 2, \ldots, n\}$, where n denotes the number of players in the game. In addition we set $N = 2^n - 1$ and label the crisp coalitions as follows: $K_0, K_1, K_2, \ldots, K_N$ where $K_0 = \emptyset$. From many of possible labellings, we fix one of them throughout the paper.

The characteristic function of a TU-game with fuzzy coalitions is defined (see [1]) as a function $v : \mathcal{F}(I) \to R$ such that $v(K_0) = 0$. Its properties are investigated in numerous works; see, for example, [1, 2, 3, 4].

The extension of some basic concepts of the deterministic TU-games model to TU-games with fuzzy coalitions is quite inspirative. Nevertheless, there are some open topics which deserve attention. The roots of some of them can be found in a concealed but natural intuitive expectation that any TU-game with fuzzy coalitions could be interpreted as an extension of some crisp coalitional TU-game. This approach was used in [10] and [9]. In the same spirit, in the following sections, we discuss one of the basic concepts; namely, the disjointness of fuzzy coalitions and its immediate consequences. On this basic level, some potentially admissible approaches to the fuzzy coalitions are illustrated.

4 Fuzzy Coalitions – Extension of Crisp Cooperation

Let $b = (b_0, b_1, \ldots, b_N)$ be an ordered $(N + 1)$-tuple of nonnegative numbers such that $b_0 + b_1 + \cdots + b_N = 1$. The set of all such vectors is an N-dimensional simplex in $(N + 1)$-dimensional space, and it is denoted by Δ^N. Note that Δ^N is a proper subset of the hypercube H^{N+1}.

Using the notation introduced in the previous section and representing the crisp coalitions $K_0, K_1, \dots, K_N$ by the $\{0,1\}$-membership functions $\tau_0, \tau_1, \dots, \tau_N$, respectively, we associate with every $b \in \Delta^N$ the fuzzy subset $L(b)$ of I whose membership function $\tau_{L(b)}$ is defined by

$$\tau_{L(b)}(i) = b_0\tau_0(i) + b_1\tau_1(i) \cdots + b_N\tau_N(i). \tag{3}$$

It has been shown in [8] and [9] that, for every fuzzy coalition $L \in \mathcal{F}(I)$, there is at least one $b \in \Delta^N$ with the property $L = L(b)$, and that, for a given fixed L, there usually exist many b's for which $L(b)$ is equal to L.

This fact offers an interesting interpretation. Namely, for each fuzzy coalition L, its membership function τ_L does not contain the complete information about the structure of cooperation of a player i with other members of L. It specifies only, that player i "invests" the part $\tau_L(i)$ of his total endeavor in the interests of coalition L. The possibility of representing L by $L(b)$ where b has several positive components shows that L itself is a structure of several cooperating crisp groups of players where, usually, each cooperating group participates in the common goals of L with only some part of its "power". The fact that there may exist many b's for which $L(b)$ is equal to the same L indicates that any resulting choice of b conveys a piece of new information about the existing structure of relations inside L. It appears to be useful to represent the fuzzy cooperation in TU-games not only by fuzzy coalitions but also by involving this additional information.

Before doing so, we describe some relations between the values of characteristic functions for crisp and fuzzy coalitions. Since the equality $L = L(b)$ can be valid for several different elements b from Δ^N, it is natural to extend the characteristic function v of a coalitional game from crisp to fuzzy coalitions by defining the value $\bar{v}(L)$ for each fuzzy coalition L as follows: For each $b = (b_0, b_1, \dots, b_N)$ from Δ^N, let v_b denote the value of the sum

$$b_0 v(K_0) + b_1 v(K_1) + \cdots + b_N v(K_N).$$

Then we define $\bar{v}(L)$ by

$$\bar{v}(L) = max(v_b : b \in \Delta^N, L(b) = L). \tag{4}$$

It is easy to verify that $\bar{v}$ is an extension of v, that is, $\bar{v}(L) = v(L)$ for each crisp coalition L.

Since the components of each b from Δ^N are numbers from the unit interval $[0,1]$, they can be interpreted as values of membership functions. In other words, each b from Δ^N determines uniquely not only a fuzzy subset $L(b)$ of I but also a fuzzy subset $\mathcal{L}(b)$ of $\mathcal{P}(I)$. The former is defined by (3), the latter by

$$\beta_{\mathcal{L}(b)}(K_j) = b_j \text{ for } j = 0, 1, \dots, N. \tag{5}$$

We say that a fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$ *reflects cooperation* in L from $\mathcal{F}(I)$ if there exists $b \in \Delta^N$ such that both $L = L(b)$ and $\mathcal{L} = \mathcal{L}(b)$.

Lemma 1. For every fuzzy coalition L from $\mathcal{F}(I)$, there exists at least one $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ reflecting cooperation in L.

Lemma 2. A fuzzy subset $\mathcal{L}$ of $\mathcal{P}(I)$ reflects cooperation in some fuzzy coalition from $\mathcal{F}(I)$ iff

$$\sum_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}}(K) = 1.$$

The following example illustrates the already mentioned fact that there may exist many fuzzy subsets of $\mathcal{P}(I)$ reflecting cooperation in a fuzzy coalition from $\mathcal{F}(I)$.

Example 1. Let $I = \{1,2,3,4\}$ and let L be the fuzzy coalition such that $\tau_L(i) = 1/2$ for all $i \in I$. Then the following subsets (and many others) of $P(I)$ reflect cooperation in L.

$$\begin{array}{llll}
\beta_{\mathcal{L}}: & \beta_{\mathcal{L}}(\{1,2\}) = 1/2, & \beta_{\mathcal{L}}(\{3,4\}) = 1/2, & \beta_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
\beta'_{\mathcal{L}}: & \beta'_{\mathcal{L}}(\{1,3\}) = 1/2, & \beta'_{\mathcal{L}}(\{2,4\}) = 1/2, & \beta'_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
\beta''_{\mathcal{L}}: & \beta''_{\mathcal{L}}(I) = 1/2, & \beta''_{\mathcal{L}}(\{K_0\}) = 1/2, & \beta''_{\mathcal{L}}(K) = 0 \text{ otherwise},\\
\beta^*_{\mathcal{L}}: & \beta^*_{L}(\{1,2,3\}) = 1/2, & \beta^*_{\mathcal{L}}(\{4\}) = 1/2, & \beta^*_{\mathcal{L}}(K) = 0 \text{ otherwise}.
\end{array}$$

As already mentioned, the classical concept of a fuzzy coalition is based only on information about the level of participation of each player. On the other hand, the paradigm that the fuzziness of a coalition means that it itself is a combination of homogeneous groups more or less contributing to L opens the possibility to analyze the relations of players to other partners in the coalition in a more sophisticated way.

For every player $i \in I$ and every fuzzy coalition $L \in \mathcal{F}(I)$ for which a fuzzy set $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ reflects cooperation in L, we define the mapping ${}^{(i)}\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0,1]$, which we call a *structure of contacts* of i in characterization $\beta_{\mathcal{L}}$, as follows:

$$\begin{aligned}
{}^{(i)}\beta_{\mathcal{L}}(K) &= \beta_{\mathcal{L}}(K) \quad \text{if } i \in K,\\
&= 0 \qquad\quad\; \text{if } i \notin K.
\end{aligned} \tag{6}$$

Lemma 3. For each $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ that reflects cooperation in a coalition $L \in \mathcal{F}(I)$, and each player $i \in I$,

$$\sum_{K\in\mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K) = \tau_L(i).$$

P r o o f. The equality follows directly from (3), (5) and (6). □

This Lemma illustrates the fact that the conceptions of a fuzzy coalition as a fuzzy subset of I and as a fuzzy subset of $\mathcal{P}(I)$ are somewhat related. However, this relation is not very tight – there is no one-to-one correspondence between both types of fuzzy coalitions. Among other consequences, it means that the method of extension of the characteristic function v used in Section 3 cannot be simply transmitted to $\mathcal{F}(\mathcal{P}(I))$. Nevertheless, it is possible to define other extensions (one of them is suggested in [11]) and the assumption that function v oculd be extended from $\mathcal{P}(I)$ to $\mathcal{F}(\mathcal{P}(I))$ is fully justified.

5 Fuzzy Coalitions – Disjointness

In this section we use the concepts of disjointness to illustrate differences between the approaches to the fuzziness of coalitions discussed in the previous sections and their consequences for the concept of superadditivity.

5.1 Fuzzy subsets of I

First let us recall that the disjointness of crisp coalitions K and K' can be expressed in terms of membership functions by the requirement

$$min(\tau_K(i), \tau_{K'}(i)) = 0 \text{ for } i = 1, 2, \ldots, n.$$

Then it seems natural to say that the fuzzy coalitions L and M from $\mathcal{F}(I)$ are disjoint iff

$$min(\tau_L(i), \tau_M(i)) = 0 \text{ for } i = 1, 2, \ldots, n.$$

In this strict view, the disjointness of fuzzy coalitions is a binary relation on $\mathcal{F}(I)$, that is, a crisp subset of the Cartesian product $\mathcal{F}(I) \times \mathcal{F}(I)$. We will call it "crisp disjointness" of fuzzy coalitions.

This view appears to be too rigid when the level of participation of players is negligibly low. To relax this rigidity, we consider the concept of disjointness of fuzzy coalitions as a fuzzy subset of $\mathcal{F}(I) \times \mathcal{F}(I)$. We denote its membership function by $\overline{\delta}$ and define it by

$$\overline{\delta}(L, M) = 1 - \max_{i \in I} \left(\min(\tau_L(i), \tau_M(i))\right). \tag{7}$$

Remark 3. In view of Lemma 3, it can easily be seen that

$$\overline{\delta}(L, M) = 1 - \max_{i \in I} \left(\min\left(\textstyle\sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K), \sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{M}}(K)\right)\right),$$

where $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ are any fuzzy sets from $\mathcal{F}(\mathcal{P}(I))$ that reflect the cooperation in L and M, respectively. Lemma 3 also implies that this equality is independent of the choice of $\mathcal{L}$ and $\mathcal{M}$ among those which reflect cooperation in L and M.

Lemma 4. If $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ reflect the cooperation in $L, M \in \mathcal{F}(I)$, respectively, and if for some $K \in \mathcal{P}(I)$, $\beta_{\mathcal{L}}(K) > 0$ and $\beta_{\mathcal{M}}(K) > 0$ then, evidently, $\overline{\delta}(L, M) < 1$.

Proof. The statement follows directly from Remark 3. □

Theorem 1. Fuzzy coalitions L and M from $\mathcal{F}(I)$ are crisply disjoint if and only if $\overline{\delta}(L, M) = 1$.

Proof. If L and M from $\mathcal{F}(I)$ are crisply disjoint, then $min(\tau_L(i), \tau_M(i)) = 0$ for each $i \in I$. Therefore the maximum of these minima is also equal to zero, which implies $\overline{\delta}(L, M) = 1$. On the oter hand, if L and M from $\mathcal{F}(I)$ are not crisply disjoint, then necessarily $min(\tau_L(i), \tau_M(i)) > 0$ for some $i \in I$. Consequently, $\max_{i\in I}(\min(\tau_L(i), \tau_M(i)))$ is positive, and therefore $\overline{\delta}(L, M) < 1$. $\square$

As a direct corollary of this theorem, we obtain that if the coalitions L, M are crisp, then $\overline{\delta}(L, M) = 1$ if $L \cap M = \emptyset$, and $\overline{\delta}(L, M) = 0$ if $L \cap M \neq \emptyset$.

Now, it is natural to define the superadditivity as a fuzzy property defined on the class of all TU-games (I, v) over the set of players I. We define its membership function (denoted by $\overline{\sigma}_I$) as follows:

$$\overline{\sigma}_I(v) = 1 - \max\left(\overline{\delta}(L, M) : L,\ M \in \mathcal{F}(I),\ \bar{v}(L \cup M) < \bar{v}(L) + \bar{v}(M)\right), \tag{8}$$

where $\bar{v}$ is the characteristic function defined by (4) and where the union $L \cup M$ is the fuzzy coalition whose membership function is

$$\tau_{L\cup M}(i) = \max\left(\tau_L(i), \tau_M(i)\right). \tag{9}$$

It can easily be verified that the definition of $\overline{\sigma}_I$ together with Lemma 4 implies the validity of the following lemma.

Lemma 5. For the games in which only crisp coalitions are admissible, the superadditivity introduced above reduces to the classical superadditivity specified in Section 2, formula (2).

5.2 Fuzzy subsets of $\mathcal{P}(I)$ – respecting also I

The above approach to the superadditivity respects the classical model of fuzzy coalitions as fuzzy subsets of I. If we wish to amend the weakness of links between the crisp and fuzzy coalitions connected with this model, we have to consider the paradigm in which fuzzy coalitions are, rather than some independent objects, extensions of the crisp coalitions, and to involve a more complex structure of cooperative relations. We have already done something similar in the previous section by introducing the concepts of reflection of cooperation, structure of contacts, and, especially, in formula (4) where a close relation between crisp and fuzzy coalitions is manifested.

If we accept the principle that a fuzzy coalition is not to be described as a fuzzy subset of I but as a fuzzy class of crisp coalitions, then its impact on the concept of disjointness (and other concepts involving it) is quite significant.

If the cooperation in a fuzzy coalition L is identified with those $\mathcal{L}$ that reflect cooperation in L, then also the disjointness may be understood as a relation between the crisp coalitions with positive values of the membership function $\beta_{\mathcal{L}}$. The disjointness remains to be a fuzzy relation between fuzzy coalitions from $\mathcal{F}(I)$. We denote its membership function by $\delta : \mathcal{F}(I) \times \mathcal{F}(I) \to [0, 1]$ but now it is defined by

$$\delta(L, M) = 1 - \max_{i\in I}\left(\max_{K,K'\in\mathcal{P}(I)}\left(\min\left({}^{(i)}\beta_{\mathcal{L}}(K), {}^{(i)}\beta_{\mathcal{M}}(K')\right)\right)\right), \quad L,\ M \in \mathcal{F}(I), \tag{10}$$

where $\mathcal{L}$ with $\beta_{\mathcal{L}}$ and $\mathcal{M}$ with $\beta_{\mathcal{M}}$ reflect the cooperation in L and M, respectively, and ${}^{(i)}\beta_{\mathcal{L}}$, ${}^{(i)}\beta_{\mathcal{M}}$ are defined by (6).

Let us note that this formulation represents a hybrid approach to the phenomenon of disjointness in the sense that it is formally based on the fuzzy subsets of $\mathcal{P}(I)$ but, as a consequence of the application of the structures of contacts ${}^{(i)}\beta_{\mathcal{L}}$, it does not contradict the traditional interpretation of coalitions as fuzzy or crisp subsets of I.

Theorem 2. If L, M are crisp then $\delta(L, M) = 1$ iff $L \cap M = \emptyset$ and $\delta(L, M) = 0$ iff $L \cap M \neq \emptyset$.

Proof. The statement follows from (10). If L and M are crisp, that is, L, $M \in \mathcal{P}(I)$, then $\beta_{\mathcal{L}}(L) = 1$, $\beta_{\mathcal{L}}(K) = 0$ for $K \neq L$, and $\beta_{\mathcal{M}}(M) = 1$, $\beta_{\mathcal{M}}(K) = 0$ for $K \neq M$. Moreover, for all $i \in L$, ${}^{(i)}\beta_{\mathcal{L}}(L) = 1$ and for all $i \in M$, ${}^{(i)}\beta_{\mathcal{M}}(M) = 1$, and the values of ${}^{(i)}\beta_{\mathcal{L}}(\cdot)$ and ${}^{(i)}\beta_{\mathcal{M}}(\cdot)$ vanish in other cases. It follows that, for disjoint L, M, always at least one of the values ${}^{(i)}\beta_{\mathcal{L}}(K)$, ${}^{(i)}\beta_{\mathcal{M}}(K)$ for any $K \in \mathcal{P}(I)$ and any $i \in I$ is equal to 0 and, consequently, $\delta(L, M) = 1$. On the other hand, if there exists $i \in L \cap M$ then ${}^{(i)}\beta_{\mathcal{L}}(K) = {}^{(i)}\beta_{\mathcal{M}}(K) = 1$ for $K = L$ and $K' = M$ and, consequently, $\delta(L, M) = 0$. □

Now it is easy to modify formula (8) by means of modifying the condition of disjointness and to define the *fuzzy superadditivity* as a fuzzy property of the TU-games over the set of players I. We denote its membership function σ_I, and define it for a game (I, v) by

$$(11) \qquad \sigma_I(v) = 1 - \max\left(\delta(L, M) : L,\, M \in \mathcal{F}(I),\, \bar{v}(L \cup M) < \bar{v}(L) + \bar{v}(M)\right).$$

where again $\bar{v}$ is the characteristic function defined by (4).

Remark 4. If L, $M \in \mathcal{F}(I)$ in (11) and $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ reflecting their cooperation are such that $\min(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)) = 0$ for all $K \in \mathcal{P}(I)$ then it is possible (and, perhaps, natural) to use special expression

$$\beta_{\mathcal{L}\cup\mathcal{M}}(K) = \max\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right)/\,2, \quad K \in \mathcal{P}(I).$$

In accordance with (4), then

$$\bar{v}(L \cup M) \geq \sum\nolimits_{K \in \mathcal{P}(I)} \beta_{\mathcal{L}\cup\mathcal{M}}(K) \cdot v(K).$$

Remark 5. Analogously to the previous case, it is easy to verify that for a TU-game with only crisp coalitions the previous definition of fuzzy superadditivity corresponds with the classical deterministic one (cf. Lemma 5 and (2)).

5.3 Exclusively fuzzy subsets of $\mathcal{P}(I)$

The last approach to the disjointness (and, consequently, superadditivity) of fuzzy coalitions follows consequently from their representation by fuzzy subsets of the set $\mathcal{P}(I)$. It means that their *fuzzy disjointness* keeps being a fuzzy relation, i. e., fuzzy subset of $\mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I))$, with membership function $\delta^* : \mathcal{F}(\mathcal{P}(I)) \times \mathcal{F}(\mathcal{P}(I)) \to [0, 1]$, defined by

$$(12) \qquad \delta^*(\mathcal{L}, \mathcal{M}) = 1 - \max_{K \in \mathcal{P}(I)} \left[\min\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right)\right]$$

for $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$.

This definition of fuzzy disjointness essentially differs from the classical and intuitively accepted one. Let us note, for illustration, that fuzzy coalition of 4 players described in Example 1 as fuzzy subset of I, has several different representations by fuzzy subsets of $\mathcal{P}(I)$. Many of them are completely disjoint in the sense of (12) i. e. $\delta^*(\cdot,\cdot) = 1$, even if they represent the same fuzzy coalition in the sense of Section 3, i. e., the values of $\overline{\delta}(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ are equal to 0.

For this consequent acceptation of fuzzy coalitions from $\mathcal{F}(\mathcal{P}(I))$, also their union and intersection gains completely different sense. Namely, for $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$, $\mathcal{L} \cup \mathcal{M}$ and $\mathcal{L} \cap \mathcal{M}$ are from $\mathcal{F}(\mathcal{P}(I))$, too, and for any $K \in \mathcal{P}(I)$,

$$\beta_{\mathcal{L}\cup\mathcal{M}}(K) = \max\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right), \quad \beta_{\mathcal{L}\cap\mathcal{M}}(K) = \min\left(\beta_{\mathcal{L}}(K), \beta_{\mathcal{M}}(K)\right). \tag{13}$$

Except very special degenerated cases, the sums

$$\sum\nolimits_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}\cup\mathcal{M}}(K) \quad \text{and} \quad \sum\nolimits_{K\in\mathcal{P}(I)} \beta_{\mathcal{L}\cap\mathcal{M}}(K) \tag{14}$$

are not equal to 1. With respect to Lemma 2 it means that they have no counterparts in the class $\mathcal{F}(I)$. Consequently, in this model, we have definitely left the environment of fuzzy coalitions extending the class of subsets of I by its fuzzy subsets.

It is worth mentioning that if K, K' are different crisp coalitions, then the corresponding fuzzy coalitions $\mathcal{L}_K$ and $\mathcal{L}_{K'}$ defined by

$$\beta_{\mathcal{L}_K}(K) = 1, \quad \beta_{\mathcal{L}_K}(\bar{K}) = 0 \text{ for } \bar{K} \neq K$$

$$\beta_{\mathcal{L}_{K'}}(K') = 1, \quad \beta_{\mathcal{L}_{K'}}(\bar{K}) = 0 \text{ for } \bar{K} \neq K'$$

are completely disjoint in the sense that $\delta^*(\mathcal{L}_K, \mathcal{L}_{K'}) = 1$.

While the disjointness is a property of the inter-coalitional relation, the superadditivity is to respect also specific properties of the characteristic function v. Till now, we have considered v as a mapping $v : \mathcal{P}(I) \to R$ extended to $v : \mathcal{F}(I) \to R$ by means of (4). In this subsection, where we consider fuzzy subsets of $\mathcal{P}(I)$, i. e., fuzzy sets from $\mathcal{F}(\mathcal{P}(I))$ for the main representation of coalitional cooperation, it is desirable to extend v also on the mapping $v : \mathcal{F}(\mathcal{P}(I)) \to R$.

Let us stress the fact that the consistency of this extension with the original characteristic function $v : \mathcal{P}(I) \to R$ is desirable.

Let us consider a fuzzy set of crisp coalitions $\mathcal{L} \in \mathcal{F}(\mathcal{P}(I))$ with membership function $\beta_{\mathcal{L}} : \mathcal{P}(I) \to [0, 1]$. Then we define the value

$$v(\mathcal{L}) = \max\left\{v(K) \cdot \beta_{\mathcal{L}}(K) : K \in \mathcal{P}(I)\right\}. \tag{15}$$

Remark 6. It is easy to see that if the fuzzy set $\mathcal{L}$ is formed by a single possible crisp coalition $K \in \mathcal{P}(I)$, where $\beta_{\mathcal{L}}(K) = 1$, $\beta_{\mathcal{L}}(K') = 0$ for $K' \neq K$, $K' \in \mathcal{P}(I)$, then evidently $v(\mathcal{L}) = v(K)$.

Let us note that (15) is not the single possibility of extension of v on the set $\mathcal{F}(\mathcal{P}(I))$. The alternative approach, extending v into a fuzzy function, is considered in [12].

Even in this model the definition of *fuzzy superadditivity* preserves the classical pattern, and it is defined as a fuzzy property of TU-games with membership function σ^*_I such that for any (I, v) the value $\sigma^*_I(v)$ denoting the possibility that (I, v) is superadditive is defined by

$$(16) \quad \sigma^*_I(v) = 1 - \max\left(\delta^*(\mathcal{L}, \mathcal{M}) : \mathcal{L},\, \mathcal{M} \in \mathcal{F}(\mathcal{P}(I)),\, v(\mathcal{L} \cup \mathcal{M}) < v(\mathcal{L}) + v(\mathcal{M})\right).$$

In this formula, the consequent and formally pure definition of union by (13) enables to define the value of $v(\mathcal{L} \cup \mathcal{M})$ (rather similarly to (4)) by the formula used in Remark 4.

The procedure described in this subsection has some evident discrepancies. Their roots consist in the fact that this model of fuzzy coalition as a fuzzy subset of $\mathcal{P}(I)$ separates that notion from the natural demand due to which even the fuzzy coalition is to be a set of players, in some sense extending the crisp coalition model. It means that it is to characterize the distribution of each player's endeavor among the coalitions in which he participates. This demand is not respected. For example, even if $\mathcal{L}$, $\mathcal{M} \in \mathcal{F}(\mathcal{P}(I))$ fulfil the statement of Lemma 3 and for some $i \in I$

$$\textstyle\sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L}}(K) \le 1, \quad \sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{M}}(K) \le 1$$

the union of both fuzzy coalitions need not respect that limitation, and then

$$\textstyle\sum_{K \in \mathcal{P}(I)} {}^{(i)}\beta_{\mathcal{L} \cup \mathcal{M}}(K) > 1,$$

in such case, player $i \in I$ distributes more of his “energy” than he disposes with.

Nevertheless, the approach used in [11] follows from 5.3 with some modifications reflecting the individual motivation of particular players, and the monotonicity of the pay-off function for the fuzzy coalitions.

6 Conclusion

The definition of fuzzy coalition as a fuzzy subset of the class of all crisp coalitions is, itself, formally acceptable, and it can be closely connected with the fuzzy coalitions defined as fuzzy sets of players. However, their further processing closely analogous to the processing of the fuzzy coalitions from $\mathcal{F}(I)$ leads to some paradoxes, mostly following from the attempts to manage the concepts of union and intersection of such fuzzy coalitions.

The acceptance of the alternative model of fuzzy coalition given here and in [11] does not mean that its further development can follow without alternatives. We have tested three of them on the very basic concept of superadditivity. It is obvious that all of them are in certain limits possible but each of them is connected with formal problems demanding other and more essential interventions in the model. The methodological principles presented here in Subsection 5.3 were further developed in [11] and the results are quite optimistic. They appear to be an adequate reflection of the realistic cooperation with vague participation in coalitions.

Anyhow, the definition of the fuzzy coalitions as fuzzy subsets of the class $\mathcal{P}(I)$ appears inspirative and perspective. It effectively extends the existing model and brings its new interpretations, and it also offers a qualitatively new view at the structure of fuzziness in cooperative behaviour. Hence, it appears to be an interesting topic of the further development of the theory of TU-games with fuzzy coalitions.

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