CAPITAL MARKET EFFICIENCY AND TSALLIS ENTROPY

Miloslav Vošvrda

Institute of Information Theory and Automation of the AS CR
Econometrics
vosvrda@utia.cas.cz

Abstract. The concept of the capital market efficiency is a central notion in the financial markets theory. This notion is generally useful to describe a capital market in which is relevant information. If relevant information is completely processed by the capital market price mechanism then such capital market is called to be efficient. Thus the capital market efficiency accentuates the informational efficiency of capital markets. It means that in the efficient capital market investors cannot expect achieving of enormous returns by long time. In other words, the capital market is efficient if the fluctuations of returns in time are unpredictable. Thus a time series of such fluctuations can be generated by some derivations of Brownian motion. Considering long range macroeconomic forces which are non-extensive, we use Tsallis's approach for maximum entropy formulating. By this way it is possible to obtain fat tailed distributions by power laws optimizing Tsallis's entropy and to give some theoretical base to some stylized facts on capital markets. For the estimating of the capital market efficiency is used the notion Tsallis entropy (information gain).
Keywords. Capital market efficiency, expectation, Famma's approach, Tsallis entropy, macroeconomic equilibrium

JEL C01, C13, C46

1. Introduction

The capital market efficiency (CME) has roots dating back to the turn of the last century. The term CME is used to introduce a capital market in which relevant information is absorbed into its price generating system to obtain the price of capital assets. In this definition is formerly emphasized the informational CME way. If capital markets are sofisticated and competitive then economics indicators indicate that investors cannot expect achieving superior profits. If it is assumed that a capital market is in equilibrium state then it is expected that for this capital market the equilibrium price of security will be fair price. This construction is a base for trust establishing in fair functioning of the capital market mechanism. The CME is mainly constructed on the probability calculus. The price analyzing of different securities on capital markets are executed by very sofisticated methods. The base for analyzing a time series of prices is the random walk model. Thus short-term returns have fluctuations with a random nature. It means that the security prices have incorporated the market relevant information at any moment of time immediately. After information processing and risk assessing, the market mechanism generates equilibrium prices. Any deviations of these equilibrium prices should be lock-up and unpredictable. After many theoretical and empirical investigations it is sured to admit the random walk model for the time series of security prices ([3],[4],[6],[9]).
Efficient capital market is closely related to the concept of the market information. The market information is clearly essential in financial activities or trading. This one is arguably the most important determinant of success in the financial life. We shall suppose that strategic investor decisions are formulated and executed on the basis of price information in the public domain, and available to all. Let us assume that investors are considered rational. We shall further assume that information once known remains known - no forgetting - and can be accessed in real time. The ability to retain information, organize it, and access it quickly, is one of the main factors, which will discriminate between the abilities of different economic agents to react on changing market conditions. We restrict ourselves to very simple situation not differentiating between agents on the basis of their information processing abilities. Thus as time passes, new information becomes available to all agents, who continually update their information. We shall use triples $(\Omega, \Im, P)$ for expressing of a probability space and the following expression $E\left[X \middle| \Im\right]$ for the conditional expectations. The $\{\omega \in \Omega\}$ is a set of the elementary market situations. The $\Im$ is some σ-algebra of the subsets of Ω and P is a probability measure on the $\Im$. This structure gives us all the machinery for static situations involving randomness. For dynamic situations, involving randomness over time, we need a sequence of σ- algebras $\{\Im_t, t \geq 0\}$, which are increasing $\Im_t \subset \Im_{t+1}$ for all t with $\Im_t$ representing the information available to us at time t. We always suppose that all σ-algebras to be complete. Thus $\Im_0$ represents an initial information (if there is none, $\Im_0 = \{\emptyset, \Omega\}$). On the other hand, a situation that all we ever shall know is represented by the following expression

$\Im_\infty = \lim_{t\to\infty} \Im_t$. Such a family { $\Im_{t\ge 0}$ } is called a filtration; a probability space endowed with such a filtration, $(\Omega, \Im, \{\Im_t\}, P)$ is called a filtered probability space which is also called a stochastic basis. Suppose that $(\mathrm{S}, \mathcal{A})$ is a measurable space, $\{X_t(\omega)\}_{t\ge 0}$ is a sequence of independent random elements defined on a probability space $(\Omega, \Im, \{\Im_t\}, P)$ taking values in $(\mathrm{S}, \mathcal{A})$ and having the distribution $P = \boldsymbol{P}^X$. Let $\mathcal{M}$ be the set of all measurable functions from $(\mathrm{S}, \mathcal{A})$ to $(□, \Im(□))$, where $\Im(□)$ stands for the Borel σ-algebra of the subsets of □ .

2. Capital Market Efficiency and Expectations

Let $C = (\Omega, \Im, \{\Im_{t\ge 0}\}, P)$ be a capital market with distinguished flows $\{\Im_{t\ge 0}\}$ of σ-algebras filtered probability space. We also call $\{\Im_{t\ge 0}\}$ an information flow, and an expression $\{S_t\}_{t\ge 0} \in \mathcal{M}$ is a security price process.

Definition 1

A capital market $(\Omega, \Im, \{\Im_{t\ge 0}\}, P)$ is called an efficient if there exists *P* such that each security price sequence $S = \{S_t\}_{t\ge 0}$ satisfies the following condition: the sequence

$$\{S_t\}_{t\ge 0} \tag{3.1}$$

is a *P*-martingale, i.e., the variables S_t are $\Im_t$ - measurable and

$$E_P\left[|S_t|\right] < \infty,\ E_P\left[S_{t+1} | \Im_t\right] = S_t,\ t \ge 0. \tag{3.2}$$

If a sequence $\{\xi_t\}_{t\ge 1}$ is the sequence of independent random variables such that $E_P\left[|\xi_t|\right] < \infty$, $E_P\left[\xi_t\right] = 0,\ t \ge 1,\ \Im_t^\xi = \sigma(\xi_1, ..., \xi_t), \Im_0^\xi = \{\varnothing, \mathrm{S}\}$, and $\Im_t^\xi \subseteq \Im_t$ then, evidently, the security price sequence $S = \{S_t\}_{t\ge 0}$ where

$$S_t = \xi_1 + ... + \xi_t \quad \text{for} \quad t \ge 1, \text{ and } S_0 = 0,$$

is a martingale with respect to $\Im^\xi = \{\Im_t^\xi\}_{t\ge 0}$, and

$$E_P\left[S_{t+1} | \Im_t\right] = S_t + E_P\left[\xi_{t+1} | \Im_t\right]. \tag{3.3}$$

If a sequence $\{S_t\}_{t\ge 1}$ is a martingale with respect to the filtration $\{\Im_{t\ge 0}\}$ and $S_t = \xi_1 + ... + \xi_t$ with $\xi_0 = 0$ then $\{\xi_t\}_{t\ge 1}$ is a martingale difference, i.e.,

$$\xi_t \text{ is } \Im_t\text{-measurable}, E_P\left[|\xi_t|\right] < \infty, E_P\left[\xi_t | \Im_{t-1}\right] = 0. \tag{3.4}$$

We have $E[\xi_t \xi_{t+k}] = 0$ for each $t \ge 0$ and $k \ge 1$, i.e., the variables $\{\xi_t\}$ are uncorrelated, provided that $E\left[|\xi_t|^2\right] < \infty$ for $t \ge 1$. In other words, square-integrable martingale belongs to the class of random sequences with orthogonal increments:

$$E\left[\Delta S_t \cdot \Delta S_{t+k}\right] = 0, \tag{3.5}$$

where $\Delta S_t \equiv S_t - S_{t-1} = \xi_t$ and $\Delta S_{t+k} = \xi_{t+k}$. Thus, the capital market efficiency is nothing else than the martingal property of security price processes in it.

3. Capital Market Efficiency and Maximum Entropy

Consider now S_t as a security price process that is represented by the following form

$$S_t = S_0 \cdot X_t, \tag{4.1}$$

where X_t is a solution of the nonlinear Fokker-Plank equation [12]

$$\frac{\partial}{\partial t} g_t(x) = -\frac{\partial}{\partial x}\left[a(x,t)\cdot g_t(x)\right] + \frac{\partial^2}{\partial x^2}\left[b(x,t)\cdot g_t^{2-q}(x)\right] \tag{4.2}$$

with both $a(\cdot)$ and $b(\cdot)$ as a drift force and diffusion coefficient respectively and a probability density function $g_t(x)$ of X_t at time $t \geq 0$. Let us consider by Borland's modification [14] a stochastic diffusion coefficient in equation (4.2) by the following form

$$c(x,t) = \sqrt{2\cdot b(x,t)\cdot g_t(x)}. \tag{4.3}$$

This modification yields the modification of (4.2) as follows

$$\frac{\partial}{\partial t} g_t(x) = -\frac{\partial}{\partial x}\left[a(x,t)\cdot g_t(x)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{c^2(x,t)}{2}\cdot g_t^{1-q}(x)\right]. \tag{4.4}$$

A stochastic differential equation for X_t has the following form

$$dX_t = a(X_t,t)\,dt + \frac{c^2(X_t,t)}{2}\,dW_t(\omega) \tag{4.5}$$

where $W_t(\omega)$ is a standard Wiener process.

Definition 2

A generalized entropy ${}_qH_t$ (Tsallis entropy) of the capital market $\left(\Omega, \Im, \{\Im_{t\geq 0}\}, \mathrm{P}\right)$ at time *t* is introduced by the following indicator

$${}_qH_t = \frac{1 - \int\limits_{x\in X_t} g_t^q(x)\,dx}{q-1}. \tag{4.6}$$

A parameter *q* is called an entropic index. A maximization of (4.6) with the following constraints

$$\int g_t(x)\,dx = 1, \quad \frac{3-q}{5-3q}\int x_t^2 \frac{g_t^q(x)}{\int g_t^q(y)\,dy}\,dx = \sigma^2 \tag{4.7}$$

with $\sigma^2 = \int x_t^2 g_t(x)\,dx$ being the second-order moment leads to the following form of the probability density function $g_t(x)$ [6]:

$$g_t(x) = \left[1 - \frac{(1-q)\cdot x^2}{\sigma^2(5-3q)}\cdot\left\{\left[\frac{q-1}{\pi}\cdot\frac{1}{\sigma^2(5-3q)}\right]^{1/2}\cdot\frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma\left[\frac{3-q}{2q-2}\right]}\right\}^{1-q}\right]^{1/(1-q)}. \tag{4.8}$$

The expression (4.8) is denoted as a *q*-Gaussian, because for $q \to 1$ the Gaussian distribution is recovered. The parameter *q* can be used as a measure of non-Gaussianity. The probability density function $g_t(.)$ is, for *q* > 1, leptokurtic, i.e. a distribution with long tails relative to Gaussian. The 3-D versions of the *q*-Gaussian are presented in Fig.1 and Fig.2 for σ=1 and *q*=1.2 respectively.

Next, let F_t and G_t be two probability distributions on $\mathbb{R}$ with f_t and g_t as their probability density functions respectively. An information gain is a non-commutative measure of the difference between two probability distributions F_t and G_t. Generalized information divergence associated with two probability distributions F_t and G_t is defined as follows

$$I_q(F_t,G_t) = -\int\limits_{\mathbb{R}} f_t^q(x)\frac{\left[g_t(x)\right]^{1-q} - \left[f_t(x)\right]^{1-q}}{1-q}\,dx. \tag{4.11}$$

After some operations on (4.11) we get the following expression

$$I_q\left(F_t,G_t\right)=\frac{1}{1-q}\left[1-\int_{\mathbb{R}}\left(\frac{f_t(x)}{g_t(x)}\right)^q g_t(x)\,dx\right]. \tag{4.12}$$

Note: For q=1, the important quantity (4.12) is related to the Shannon entropy which is called Kullback-Leibler divergence. The Tsallis entropy is appropriate for non-equilibrium systems replacing exponential Boltzmann factors by power-law distribution[13].

Because $I_q\left(F_t,G_t\right)\neq I_q\left(G_t,F_t\right)$ unless $F_t=G_t$, the generalized information divergence $I_q\left(F_t,G_t\right)$ is thus adjusted on the following expression

$$I_q^S\left(F_t,G_t\right)=\left(I_q\left(F_t,G_t\right)+I_q\left(G_t,F_t\right)\right)/2. \tag{4.13}$$

Values of the criterion $I_q^S\left(F,G\right)$ for any F, G belong to $\left\langle 0,1\right\rangle$. If $I_q^S\left(F,G\right)$ is equal 0 then

$$F=G, \tag{4.14}$$

i.e., the stochastic processes have the same information gain.

Let F_t be an uniform probability distribution on $\mathbb{R}$. If analyzed stochastic proces $\left\{X_t,t\in T\right\}$ in period *T* has a probability distribution G_t on $\mathbb{R}$ and $I_q^S\left(F_t,G_t\right)=0$ then G_t is an information-gain-free about an interior dependence inside $\left\{X_t,t\in T\right\}$. As an illustration of the behavior of this criterion $I_q^S\left(F_t,G_t\right)$, let us put both the uniform probability distribution on $\left\langle -5,5\right\rangle$ for F_t and the G_t to be *q*-Gaussian for each *t* on $\left\langle -5,5\right\rangle$. This one is presented in the following Fig. 3 for different levels of *q*= (1.1, … ,1.6) and σ = (1, ..., 5).

Definiton 3

Let G_{dS_t} be a probability distribution of the proces $\left\{dS_t,t\in T\right\}$. Let F_t be an uniform distribution on $\mathbb{R}$. The capital market is called an efficient if for any $dS_t, t\in T$ and for *q* > 1 the following expression $I_q^S\left(F_t,G_{dS_t}\right)=0$ holds.

Theorem 1

A capital market *C* is the efficient if and only if any $dS_t\in C$ for all $t\in T$ and $G_{dS_t}=F_t$.

Proof

$(\Rightarrow)$ Let C be an efficient. Any return $dS_t\in C$, $t\in T$ is by (4.5) a standard Wiener process. Hence $I_q^S\left(G_W,G_{dS_t}\right)=0$ for $t\in T$, and therefore G_{dS_t} has not any information gain about the dependence in $\left\{X_t,t\in T\right\}$.

$(\Leftarrow)$ Draw any $dS_t\in C$, $t\in T$ and $G_{dS_t}=F_t$. Then $I_q^S\left(F_t,G_{dS_t}\right)=0$ for any $dS_t\in C$, $t\in T$. Hence C is the efficient.

4. Simulation and Empirical Analysis

Now we generate realizations of the size 500 from the following theoretical models on the one hand with and on the other without a correlation dependence inside. Considered theoretical models are

Normal Noise with μ=0, and σ = 1	ε_t
Random Walk	$S_t(\omega)=S_{t-1}(\omega)+\varepsilon_t$
Random 2-Walks	$S_t(\omega)=S_{t-1}(\omega)-S_{t-2}(\omega)+\varepsilon_t$
Moving Average (2)	$S_t(\omega)=\varepsilon_t+0.1\cdot\varepsilon_{t-1}-0.4\cdot\varepsilon_{t-2}$
Autoregressive Model (2)	$S_t(\omega)=0.7\cdot S_{t-1}(\omega)-0.3\cdot S_{t-2}(\omega)+\varepsilon_t$

Normal Noise with μ=0, and $\sigma^2 = 3^2$	$3 \cdot \varepsilon_t$
AutoRegressive Moving Average (2,2)	$S_t(\omega) - 0.2 \cdot S_{t-1}(\omega) + 0.5 \cdot S_{t-2}(\omega) = \varepsilon_t + 0.7 \cdot \varepsilon_{t-1} - 0.3 \cdot \varepsilon_{t-2}$
Deterministic Process	$S_t(\omega) = 0.9 \cdot S_{t-1}(\omega)$

Empirical data were obtained from the following daily observations of market indices PX50, DAX, BUX, SAX, WIG, S&P 500, SGX, Nikkei 225, NASDAQ, and Dow Jones Industriel Average for the period 2006.11.14 till 2008.10.23. The security price returns model $\ln\left[S_t(\omega)/S_{t-1}(\omega)\right]$ is used for a transformation of the observations. Let F_t be a uniform distribution on $\langle 0,1 \rangle$. A smooth estimation of the empirical probability distributions by Bernstein polynomials [15] is used. We estimate criterions $I_q^S(F_t, G_t)$ and $I_1^S(F_t, G_t)$ for particular models with q=1.2, and σ=1. Results are introduced in Table 1 and in Fig. 4 and Fig. 5.

Tab. 1 Results of the criterions $I_q^S(F_t, G_t)$ and $I_1^S(F_t, G_t)$ for different models.

The Criterions $I_q^S(F_t, G_t)$ and $I_1^S(F_t, G_t)$				
model	F_t	G_t	$I_q^S(F_t, G_t)$	$I_1^S(F_t, G_t)$
1	Uniform	Uniform	0.000000	0.000000
2	Uniform	Random Walk	0.260039	0.229439
3	Uniform	Random 2-Walks	0.098187	0.002911
4	Uniform	MA(2)	0.047324	0.073004
5	Uniform	AR(2)	1.000000	0.049511
6	Uniform	Normal Noise with μ=0, and σ = 3	0.061074	0.056901
7	Uniform	ARMA(2,2)	0.973694	0.011284
8	Uniform	Deterministic Process	1.000000	1.000000
9	Uniform	PX50, Prague, Czech Republic	0.541033	0.616403
10	Uniform	DAX, Frankfurt, Germany	0.542492	0.63618
11	Uniform	BUX, Budapest, Hungary	0.549074	0.577513
12	Uniform	SAX, Bratislava, Slovak	1.000000	0.596843
13	Uniform	WIG, Warsaw, Poland	0.517336	0.229463
14	Uniform	S&P 500, USA	0.575915	0.63618
15	Uniform	SGX,Singapore, Singapore	0.546653	0.520874
16	Uniform	Nikkei 225, Tokyo, Japan	0.559004	0.577505
17	Uniform	NASDAQ, USA	0.560131	0.63618
18	Uniform	Dow Jones Industriel Average, USA	0.605994	0.616401

5. Concluding Remarks

Results show that the current non-extensive approach could be more useful tool in the analysis of CME. This approach is more sensitive on detecting of any dependencies in the capital market processes. It is demonstrated that $I_q^S(F_t, G_t)$ as a measure of the divergence on capital markets pursues this fact better than $I_1^S(F_t, G_t)$. By comparison results of $I_q^S(F_t, G_t)$ from empirical and simulation data it is shown that capital markets have had a remarkable fail in an efficiency of the incorporating of the market relevant information. It is possible to obtain any farther more accurate results after estimating of the entropy index q. Consequently, the next important step is to construct an estimator $\hat{q}$ for the entropy index q. The estimator $\hat{q}$ will give a possibility to involve more pregnantly the leptokurticity of the probability distributions into the measure $I_{\hat{q}}^S(\cdot,\cdot)$.

Acknowledgments

Financial support from Czech Science Foundation 402/09/0965 and MSMT LC06075 is acknowledged.

References

[1] BORLAND, L. *Microscopic dynamics of the nonlinear Fokker-Plank equation: Phenomenological model.* Phys. Rev. E. 1998, 57,6634

[2] KRAFT, J. *Changes of macroeconomic environment at the turn of century as determinant factor for operation of state administration companies.* Acta economia No. 7, Ekonomická teória a praxa - dnes a zajtra, Bánská Bystrica, EF UMB, 2002, str. 39 – 45. ISBN 80-8055-592-3

[3] COOTNER, P.H. *The Random Character of Stock Market Price.*, MIT Press, Cambridge, MA, 1964

[4] FAMA ,E. *The Behavior of Stock Market Prices.* Journal of Business, 1965, 38, pp. 34-105.

[5] KENDALL,l M. *The Analysis of Economic Time Serie.* Journal of the Royal Statistical Society, Series A, 96, 1953, pp. 11-25.

[6] QUEIROS DUARTE, S.M. *On non-Gaussianity and dependence in financial time series: a nonextensive approach.* Quantitative Finance, Vol. 5, No. 5, October 2005, pp.475-487

[7] OSBORNE, M.F.M. *Brownian Motion in the Stock Market.* Operations Research 7, 1959, pp. 145-173

[8] PLATEN,E. *Arbitrage in continuous complete markets.* Adv. Appl. Probab.,34, 2002, pp.540-558

[9] RISKEN,H.*The Fokker-Planck Equation: Methods of Solution and Applications.* 2nd sd. 1989, Springer-Verlag, Berlin,

[10] ROBERTS, H. *Stock Market `Patterns' and Financial Analysis: Methodological Suggestion.* Journal of Finance, 1959, 44, pp. 1-10.

[11] SHIRYAEV, A.N. *Probability.* 2nd ed., Springer-Verlag, New York, 1995

[12] TSALLIS, C., Bukmann, D.J. *Anomalous diffusion in the presence of external forces* . Phys. Rev. E. 54, 1996, R2197,

[13] TSALLIS, C. *Possible Generalization of Boltzmann-Gibbs Statistics*, Journal of Statistical Physics, Vol. 52, Nos. ½, 1988,

[14] TSALLIS, C., ANTENEODO, C., BORLAND, L., OSORIO, R. *Nonextensive Statistical Mechanics and Economics,* Physica A 324, 89 -100, 2003

[15] BABU G.J.,BOYARKY A., CHAUBEY Y.P., GORA P. *A New Statistical Methods for Filtering and Entropy Estimation of a Chaotic Map from Noisy Data,* International Journal of Bifurcation and Chaos, Vol.14, No.11,3989-3994,2004

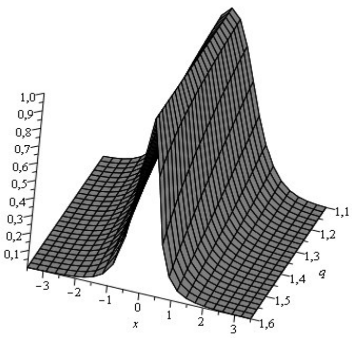

Fig. 1 q-Gaussian, σ=1

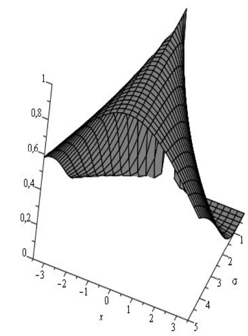

Fig.2 q-Gaussian, q=1.2

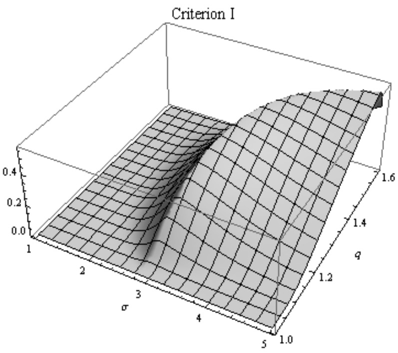

Fig.3 3D-figure of the criterion I_q

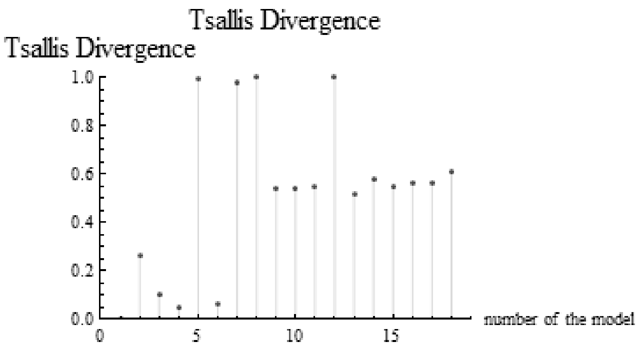

Fig. 4 Tsallis divergence among different models and uniform model

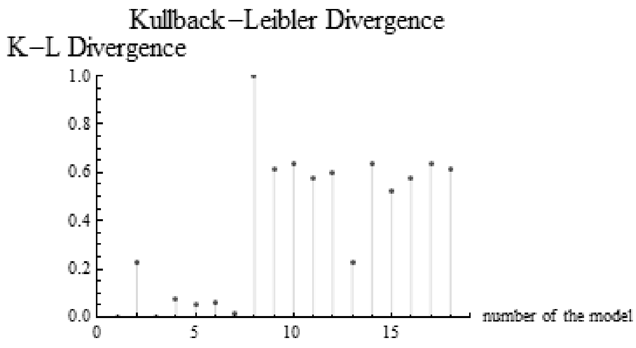

Fig.5 Kullback-Leibler divergence among different models and uniform model