Correspondence

Affine Invariants of Convex Polygons

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Abstract—**In this correspondence, we prove that the affine invariants proposed recently by Yang and Cohen [1] are algebraically dependent. We show how to select an independent and complete set of the invariants. The use of this new set leads to a significant reduction of the computing complexity without decreasing the discrimination power.**

Index Terms—**Affine invariants, completeness, convex polygons, independence.**

I. Introduction

Numerous invariants have been proposed to describe the shape of an object undergoing a general affine transform. The following classes of invariants have been studied in the literature: affine area moment invariants [2]–[4], affine curve moment invariants [5], cross-weighted moments [6], and Fourier descriptors [7], among others. Most of the aforementioned invariants are global, i.e., the whole object must be visible and accessible to calculate them. However, this implies significant limitations when recognizing occluded or locally distorted objects. To overcome this, Yang and Cohen [1] proposed local affine invariants of convex polygons. Although they proved the invariance of their descriptors, they paid only little attention to the *independence* and *completeness*. Both these properties are very important issues that influence significantly the discriminative power and computing complexity of the invariants.

In this correspondence, we revise the theory of Yang and Cohen from this viewpoint. We show that most invariants published in [1] are algebraically dependent. Moreover, we show that for a convex quadruplet the complete and independent set contains only two invariants. Since the dependent invariants are useless for object recognition, our results lead to significant reduction of the computing complexity.

II. Invariants by Yang and Cohen

In this section, we briefly recall the theory presented in [1].

A two-dimensional (2-D) affine transformation is a mapping from coordinate system $r = (x, y)^T$ to coordinate system $r' = (x', y')^T$ given by

$$r' = \mathbf{A} \cdot r + b \tag{1}$$

where $\mathbf{A}$ is a nonsingular matrix and b is a translation vector.

Two important observations mentioned in [1] are

- affine transform does not change the convexity, i.e., the convex hull of the transformed point set equals the transformed convex hull of the original set;
- the (oriented) area S of any object is a relative affine invariant, i.e.,

$$S' = \det(\mathbf{A}) \cdot S.$$

Let us have a convex polygon C of n ordered vertices $(r_1, \ldots, r_n)$ where $r_i = (x_i, y_i)^T$. Now, n convex quadruplets $(r_i, r_{i+1}, r_{i+2}, r_{i+3})$, $i = 1, \ldots, n$, are formed along the boundary of C. The invariants of all quadruplets are grouped together to form the invariant vector of C.

The invariants of a single quadruplet $(r_i, r_{i+1}, r_{i+2}, r_{i+3})$ whose area is denoted as $S(i)$ are defined in [1] as follows. Let us divide the quadruplet into four triangles: (r_i, r_{i+1}, r_{i+3}), (r_i, r_{i+2}, r_{i+3}), (r_i, r_{i+1}, r_{i+2}), and $(r_{i+1}, r_{i+2}, r_{i+3})$ the areas of which are $S_1(i)$, $S_2(i)$, $S_3(i)$ and $S_4(i)$, respectively. Then two sets of invariants of the quadruplet $(r_i, r_{i+1}, r_{i+2}, r_{i+3})$ are defined

$$\mathbf{I}_1 = \left\{ \frac{S_1}{S_2}, \frac{S_1}{S_3}, \frac{S_1}{S_4}, \frac{S_2}{S_3}, \frac{S_2}{S_4}, \frac{S_3}{S_4} \right\} \tag{2}$$

$$\mathbf{I}_2 = \left\{ \frac{S_1}{S}, \frac{S_2}{S}, \frac{S_3}{S}, \frac{S_4}{S} \right\}. \tag{3}$$

(We omit here the index i for simplicity.)

III. Independence and Completeness

In [1], only little attention was paid to studying the dependence/independence of the sets $\mathbf{I}_1$ and $\mathbf{I}_2$. The authors only observed that "$\mathbf{I}_2$ is not totally independent of $\mathbf{I}_1$" and recommended to use $\mathbf{I}_1$ for object recognition purposes. Actually, $\mathbf{I}_2$ is *totally dependent* on $\mathbf{I}_1$ because

$$\frac{S_1}{S} = \frac{S_1}{S_1 + S_4} = \frac{1}{1 + \left(\frac{S_1}{S_4}\right)^{-1}}$$

and similarly for all other elements of $\mathbf{I}_2$.

However, much more important (although not mentioned in [1] at all) is intrinsic dependence *inside* $\mathbf{I}_1$. If we choose two arbitrary elements of $\mathbf{I}_1$, the other four elements can be expressed as their functions. To illustrate this assertion, let us choose S_1/S_2 and S_1/S_3 as a basis. To prove the dependence of S_1/S_4, we recall that, for a convex quadruplet, $S_1 + S_4 = S_2 + S_3$. Thus,

$$\frac{S_1}{S_4} = \left(\left(\frac{S_1}{S_2}\right)^{-1} + \left(\frac{S_1}{S_3}\right)^{-1} - 1 \right)^{-1}.$$

The dependence of the other three invariants is clear

$$\frac{S_3}{S_4} = \frac{S_1}{S_4} \cdot \left(\frac{S_1}{S_3}\right)^{-1}$$

$$\frac{S_2}{S_4} = \frac{S_1}{S_4} \cdot \left(\frac{S_1}{S_2}\right)^{-1}$$

$$\frac{S_2}{S_3} = \frac{S_1}{S_3} \cdot \left(\frac{S_1}{S_2}\right)^{-1}.$$

Let us note that this is in accordance with our intuitive expectation. Since four points $r_i, r_{i+1}, r_{i+2}, r_{i+3}$ have eight degrees of freedom

Manuscript received April 3, 2000; revised December 14, 2001. This work was supported by the Grant Agency of the Czech Republic under Grant 102/00/1711. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Josiane B. Zerubia.

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Publisher Item Identifier 10.1109/TIP.2002.802524.

and affine transform has six parameters, we can expect $8 - 6 = 2$ independent invariants.

Moreover, any two elements of $\mathbf{I}_1$ form a *complete* set of invariants. Knowing them, we can recover not only the other invariants but also the shape of the original quadruplet. More precisely, we can recover a class of affine-equivalent convex quadruplets.

The completeness can be proven as follows. Let us assume two different convex quadruplets (r_0, r_1, r_2, r_3) and (r'_0, r'_1, r'_2, r'_3) are divided into triangles with the areas S_1, S_2, S_3, S_4 and S'_1, S'_2, S'_3, S'_4, respectively. Let $S_1/S_2 = S'_1/S'_2$ and $S_1/S_3 = S'_1/S'_3$. Without loss of generality, we consider the quadruplets in standard positions, i.e., $r_0 = r'_0 = (0, 0)$, $r_1 = r'_1 = (1, 0)$ and $r_3 = r'_3 = (0, 1)$. Every quadruplet can be brought into its standard position by an affine transform. Thanks to the standard positions it holds $S_1 = S'_1$ and, consequently, $S_2 = S'_2$, $S_3 = S'_3$. Using the formula for calculation of triangle areas we get

$$\begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x'_2 & y'_2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

and

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ x'_2 & y'_2 & 1 \end{vmatrix}.$$

Evaluating the determinants we obtain the constraints

$$y_2 = y'_2$$
$$x_2 = x'_2.$$

Thus, $r_2 = r'_2$ and the quadruplets in their standard positions are identical. In other words, if any two convex quadruplets have identical invariant feature vectors then there exist an affine transform which maps one quadruplet on the other.

IV. Conclusion

In this correspondence, we have shown that the affine invariants of convex polygons presented earlier in [1] are mutually dependent. We have proven there are only two independent invariants of a convex quadruplet. These two invariants, when evaluating them on each boundary quadruplet, generate the feature vector of the original polygon. Using only two invariants instead of six proposed in [1] we get exactly the same discrimination power with no redundancy and with reduced computing complexity.

In a broader sense, this illustrates the necessity of checking carefully not only the invariance but also mutual dependence/independence and completeness of the features when constructing a feature vector.

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