Rotation Moment Invariants for Recognition of Symmetric Objects

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***Abstract*—In this paper, a new set of moment invariants with respect to rotation, translation, and scaling suitable for recognition of objects having N-fold rotation symmetry are presented. Moment invariants described earlier cannot be used for this purpose because most moments of symmetric objects vanish. The invariants proposed here are based on complex moments. Their independence and completeness are proven theoretically and their performance is demonstrated by experiments.**

***Index Terms*—Complex moments, moment invariants, N-fold rotation symmetry, symmetric objects.**

I. Introduction

MOMENT invariants have become a classical tool for object recognition during last 40 years. No doubt they are one of the most important and most frequently used shape descriptors. Even if they suffer from some intrinsic limitations (the most important of which is their globality, which prevents them from being used for recognition of occluded objects), they frequently serve as a reference method for evaluation of the performance of other shape descriptors. Despite of large amount of effort and huge number of published papers, there are still open problems to be resolved.

A. State-of-The-Art in Brief

The history of moment invariants begun in the 19th century, many years before the appearance of the first computers, under the framework of the theory of algebraic invariants. The theory of algebraic invariants probably originate from the famous German mathematician David Hilbert [1] and was thoroughly studied also in [2], [3].

Moment invariants were firstly introduced to the pattern recognition community in 1962 by Hu [4], who employed the results of the theory of algebraic invariants and derived his seven famous invariants to in-plane rotation of 2-D objects and further studied in classical papers [6], [5], [7]. Since that time, numerous works have been devoted to various improvements and generalizations of Hu's invariants and also to its use in many application areas.

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Dudani [8] and Belkasim [9] described their application to aircraft silhouette recognition, Wong and Hall [10], Goshtasby [11], and Flusser and Suk [12] employed moment invariants in template matching and registration of satellite images, Mukundan [13] applied them to estimate the position and the attitude of the object in 3-D space, Sluzek [14] proposed to use local moment invariants in industrial quality inspection and many other authors used moment invariants for character recognition [9], [15]–[18]. Maitra [19] and Hupkens [20] made them invariant also to contrast changes, Wang [21] proposed illumination invariants particularly suitable for texture classification. Li [22] and Wong [23] presented the systems of invariants up to the orders nine and five, respectively. Lin and Tianxu [24] published another technique how to derive higher-order moment invariants. Unfortunately, the invariant sets presented in their papers are algebraically dependent. Flusser [25], [26] proposed a unified method how to derive independent sets of rotation invariants of any orders and proved that the original Hu's invariants are dependent and incomplete.

The aforementioned papers dealt with invariants to translation, rotation, and scaling (TRS). In addition to them, there have been numerous papers on moment invariants to affine and projective transforms, to photometric (color) changes and to linear filtering of an image. Other groups of papers have dealt with reconstruction power of moments, with their numerical properties, and also with moments with respect to special orthogonal polynomials. However, these aspects of moment invariants are not the topics of this paper, so we do not cite particular references.

B. Topic of This Paper

As many authors have pointed out, objects having certain degree of symmetry may cause problems in moment-based recognition systems. The reason is that many moments and, consequently, many moment invariants vanish for symmetric objects. For example, all odd-order moments of a centrosymmetric object equal identically zero.

The goal of this paper is to develop TRS moment invariants which are particularly suitable for objects having N-fold rotation symmetry. This is very important when recognizing man-made objects and natural shapes. To achieve this, we substantially generalize our recent theory published in [25] and [26], taking into account N-fold symmetric objects for arbitrary N including infinity.

The rest of the paper is organized as follows. In Section II, we briefly recall the derivation of TRS invariants from complex moments, introduced in [25] and [26]. Section III performs the core of the paper. We generalize the construction of the invariant basis to make it suitable for recognition of symmetric objects. Numerical experiments on both artificial as well as real data are presented in Section IV.

II. Deriving Rotation Invariants by Means of Complex Moments

There are various approaches to theoretical derivation of moment-based rotation invariants. One can employ the theory of algebraic invariants, Lie groups, tensor calculus, Fourier–Mellin transform, Zernike orthogonal polynomials, or properties of trigonometric functions. In this paper, we use a scheme which is based on *complex moments*. The idea of using complex moments for deriving invariants was firstly proposed by Mostafa and Psaltis [27] and finalized by Flusser [25]. In comparison with other approaches, this one is more transparent and allows to study mutual dependence/independence of the invariants in a readable way. However, it should be noted that all the above approaches differ from each other formally by mathematical tools and notation used but the general idea behind them is common and the results are similar or even equivalent.

Complex moment c_{pq} of order $(p+q)$ of an integrable image function $f(x,y)$ is defined as

$$c_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x,y)\,dx\,dy \quad (1)$$

where i denotes imaginary unit. Each complex moment can be expressed in terms of geometric moments m_{pq} as

$$c_{pq} = \sum_{k=0}^{p}\sum_{j=0}^{q} \binom{p}{k}\binom{q}{j} (-1)^{q-j} \cdot i^{p+q-k-j} \cdot m_{k+j,p+q-k-j} \quad (2)$$

where geometric moments are defined as

$$m_{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\,dx\,dy \quad (3)$$

and vice versa

$$m_{pq} = \frac{1}{2^{p+q} i^q} \sum_{k=0}^{p}\sum_{j=0}^{q} \binom{p}{k}\binom{q}{j} (-1)^{q-j} \cdot c_{k+j,p+q-k-j}. \quad (4)$$

It follows from the definition that only the indexes $p \geq q$ are meaningful when dealing with complex moments because $c_{pq} = c_{qp}^*$ (the asterisk denotes complex conjugate).

In polar coordinates, (1) becomes the form

$$c_{pq} = \int_{0}^{\infty}\int_{0}^{2\pi} r^{p+q+1} e^{i(p-q)\theta} f(r,\theta)\,dr\,d\theta. \quad (5)$$

Equation (5) implies rotation invariance of the moment magnitude $|c_{pq}|$ while the phase is shifted by $(p-q)\alpha$, where α is the angle of rotation. More precisely, it holds for the moment of the rotated image

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}. \quad (6)$$

Any approach to the construction of rotation invariants must be based on a proper kind of phase cancellation. The simplest method proposed by many authors (see [28] for instance) is to use the moment magnitudes themselves as the invariants. However, they do not generate a complete set of invariants. Flusser [25] proposed to achieve phase cancellation by multiplication of appropriate moment powers, as is explained in the following Theorem.

Theorem 1: Let $n \geq 1$ and let k_i, p_i, and $q_i (i = 1, \ldots, n)$ be nonnegative integers such that

$$\sum_{i=1}^{n} k_i(p_i - q_i) = 0.$$

Then

$$I = \prod_{i=1}^{n} c_{p_i q_i}^{k_i} \quad (7)$$

is invariant to rotation.

To achieve also translation and scaling invariance, we use central coordinates in the definition of the complex moments (1) and standard normalization by a proper power of c_{00}, respectively.

Theorem 1 allows us to construct an infinite number of the invariants for any order of moments, but only few of them are mutually independent. It was shown in [25] that there exist relatively small complete and independent subset called *basis*, by means of which all other rotation invariants can be expressed. Such a basis $\mathcal{B}$ is defined in Theorem 2.

Theorem 2: : Let us consider complex moments up to the order $r \geq 2$. Let a set of rotation invariants $\mathcal{B}$ be constructed as follows:

$$\mathcal{B} = \left\{ \Phi(p,q) \equiv c_{pq} c_{q_0 p_0}^{p-q} \,|\, p \geq q \bigwedge p+q \leq r \right\}$$

where p_0 and q_0 are arbitrary indexes such that $p_0 + q_0 \leq r, p_0 - q_0 = 1$ and $c_{p_0 q_0} \neq 0$ for all images involved. Then, $\mathcal{B}$ is a basis of a set of all rotation invariants created from the moments up to the order r.

The knowledge of the basis is a crucial point in all pattern recognition tasks because the basis provides the same discriminative power as the set of all invariants and minimizes the computational cost.

III. Recognition of Symmetric Objects

If we want to recognize symmetric objects, we either cannot apply Theorem 2 at all or many rotation invariants might be identically zero. Let us imagine an illustrative example. We want to recognize three shapes—square, cross, and circle—independently of their orientation. Because of symmetry, all complex moments of the second and third orders except c_{11} are zero. If the shapes are appropriately scaled, c_{11} can be the same for all of them. Consequently, neither the Hu's invariants nor the invariants constructed by means of Theorem 2 provide any discrimination power, even if the shapes are easy to recognize visually. Appropriate invariants in this case would be $c_{22}, c_{40}c_{04}, c_{51}c_{04}, c_{33}, c_{80}c_{04}^2, c_{62}c_{04}, c_{44}$, etc.

The above simple example shows the necessity of having different systems of invariants for objects with different types of symmetry. In this Section we consider so called N-fold rotation symmetry (N-FRS). An object is said to have N-FRS ($N \geq 1$) if it repeats itself when it rotates around its centroid by $2\pi j/N$ for all $j = 1, \cdots, N$. We will use this definition not only for finite N but also for $N = \infty$. Thus, in our terminology, objects having circular symmetry $f(x,y) = f(\sqrt{x^2+y^2})$ are said to have ∞-FRS.

Rotation symmetry of the object determines the vanishing moments.

Lemma 1: If object $f(x,y)$ has N-fold rotation symmetry (N finite), then all its complex moments with noninteger $(p-q)/N$ equal zero.

Proof: Let us rotate the object around its origin by $2\pi/N$. Due to its symmetry, the rotated object must be the same as the original. In particular, it must hold $c'_{pq} = c_{pq}$ for any p and q. On the other hand, it follows from (6) that

$$c'_{pq} = e^{-2\pi i(p-q)/N} \cdot c_{pq}.$$

Since $(p-q)/N$ is assumed not to be an integer, this equation can be fulfilled only if $c_{pq} = 0$. □

Lemma 1a: If object $f(x,y)$ has ∞-fold rotation symmetry, then all its complex moments with $p \neq q$ equal zero.

Proof: Let us rotate the object around its origin by arbitrary angle α. The rotated object must be the same as the original for any α and, consequently, its moments cannot change under rotation. Equation (6) implies

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}.$$

Since $p \neq q$ and α may be arbitrary, this equation can be fulfilled only if $c_{pq} = 0$. □

In order to derive invariants for recognition of objects with N-fold rotation symmetry, we propose the following generalization of Theorem 2.

Theorem 3: Let us consider a set of objects having N-FRS, N finite, and their complex moments up to the order $r \geq N$. Let a set of rotation invariants $\mathcal{B}_N$ be constructed as follows:

$$\mathcal{B}_N = \Big\{ \Phi(p,q) \equiv c_{pq} c^k_{q_0 p_0} \mid p \geq q \bigwedge p+q \leq r \bigwedge k \equiv (p-q)/N \text{ is integer} \Big\}$$

where p_0 and q_0 are arbitrary indexes such that $p_0 + q_0 \leq r$, $p_0 - q_0 = N$, and $c_{p_0 q_0} \neq 0$ for all objects involved. Then, $\mathcal{B}_N$ is a basis of a set of all nontrivial invariants for objects with N-FRS, created from the moments up to the order r.

Proof: Rotation invariance of all elements of $\mathcal{B}_N$ follows immediately from Theorem 1. The independence of $\mathcal{B}_N$ follows from the mutual independence of the complex moments themselves. To prove its completeness, it is sufficient to resolve so-called *inverse problem*, which means recovering all complex moments (and, consequently, all geometric moments) up to the order r when knowing the elements of $\mathcal{B}_N$.

Since $\mathcal{B}_N$ is a set of rotation invariants, it does not reflect the orientation of the object. Thus, there is one degree of freedom when recovering the object moments which corresponds to the choice of the object orientation. Without loss of generality, we can choose such orientation in which $c_{p_0 q_0}$ is real and positive. As can be seen from (6), if $c_{p_0 q_0}$ is nonzero then such orientation always exists. Thus, the equation

$$\Phi(p_0, q_0) = c_{p_0 q_0} c_{q_0 p_0}$$

can be immediately resolved for $c_{p_0 q_0}$

$$c_{p_0 q_0} = \sqrt{\Phi(p_0, q_0)}.$$

Consequently, using the relationship $c_{q_0 p_0} = c_{p_0 q_0}$, we get the solutions

$$c_{pq} = \frac{\Phi(p,q)}{c^k_{q_0 p_0}}$$

for any p and q such that $(p-q)/N$ is integer and

$$c_{pq} = 0$$

for the other indexes. Recovering the geometric moments is then straightforward from (4). □

Note that most invariants $\mathcal{B}_N$ are complex. If we want to have real-valued features, we only take real and imaginary parts of each of them.

One can see that the basis defined in Theorem 3 is generally not unique. It depends on the particular choice of p_0 and q_0. How shall we, in practical applications, select these indexes? On one hand, we want to keep p_0 and q_0 as small as possible because lower-order moments are less sensitive to noise than the higher order ones. On the other hand, close-to-zero value of $c_{p_0 q_0}$ may cause numerical instability of the invariants. Thus, we propose the following algorithm. We start with $p_0 = N$ and $q_0 = 0$ and check if $|c_{p_0 q_0}|$ exceeds a predefined threshold for all objects (in practice this means for all given training samples). If yes, we accept this choice, otherwise we increase both p_0 and q_0 by one and repeat the above procedure.

It may happen that, for the given set of objects and for the given moment order r, we do not find such $c_{p_0 q_0}$. This may indicate that the objects actually have higher number of folds than N.

For $N = 1$, which means no rotation symmetry, Theorem 3 is reduced exactly to Theorem 2. The following modification of Theorem 3 deals with the case $N = \infty$.

Theorem 3a: Let us consider a set of objects having ∞-FRS and their complex moments up to the order $r \geq 2$. Then the basis of all nontrivial rotation invariants $\mathcal{B}_\infty$ is

$$\mathcal{B}_\infty = \{ c_{pp} \mid p \leq r/2 \}.$$

The Proof of Theorem 3a follows immediately from Theorem 1 and Lemma 1a.

Theorems 3 and 3a have several interesting consequences. Some of them are summarized in the following Lemma.

Lemma 2: Let us denote all rotation invariants which can be expressed by means of elements of basis $\mathcal{B}$ as $\langle \mathcal{B} \rangle$. Then, the following holds for any order r.

1) If M and N are finite and L is their least common multiple, then

$$\langle \mathcal{B}_M \rangle \cap \langle \mathcal{B}_N \rangle = \langle \mathcal{B}_L \rangle.$$

In particular, if M/N is integer then $\langle \mathcal{B}_M \rangle \subset \langle \mathcal{B}_N \rangle$.

2)
$$\bigcap_{N=1}^{\infty} \langle \mathcal{B}_N \rangle = \langle \mathcal{B}_\infty \rangle.$$

3) If N is finite, the number of elements of $\mathcal{B}_N$ is
$$|\mathcal{B}_N| = \sum_{j=0}^{n} \left[\frac{r - jN + 2}{2} \right]$$
where $n = [r/N]$ and symbol $[a]$ means integer part of a. For $N = \infty$ it holds
$$|\mathcal{B}_\infty| = \left[\frac{r+2}{2} \right].$$

In practical pattern recognition experiments, the number of folds N may not be known *a priori*. In that case, we can apply a fold detector (see [29], [30], and [31] for algorithms detecting the number of folds) to all elements of the training set before we choose an appropriate system of moment invariants. In case of equal fold numbers of all classes, proper invariants can be chosen directly according to Theorem 3 or 3a. However, it is not realistic to meet such a simple situation in practice. Different shape classes use to have different numbers of folds. The previous theory does not provide a solution to this problem.

As can be seen from Lemma 2, we cannot simply choose one of the numbers of folds detected as the appropriate N for constructing invariant basis according to Theorem 3 (although one could intuitively expect the highest number of folds to be a good choice, it is not that case). More sophisticated choice is to take the least common multiple of all finite fold numbers and then to apply Theorem 3. Unfortunately, taking the least common multiple often leads to high-order instable invariants. This is why, in practice, one may prefer a decomposition of the problem into two steps—first, preclassification into "groups of classes" according to the number of folds is performed and then final classification is done by means of moment invariants, which are defined separately in each group. This decomposition can be performed explicitly in a separate preclassification stage or implicitly during the classification. The word "implicitly" here means that the number of folds of an unknown object is not explicitly tested, however, at the beginning we must test the numbers of folds in the training set. Let us explain the latter version.

Let us have C classes altogether such that C_k classes have N_k folds of symmetry; $k = 1, \ldots, K$; $\quad N_1 > N_2 > \cdots N_K$. The set $\mathcal{M}$ of proper invariants can be chosen as follows.

Algorithm Select Inv

Set $\mathcal{M} = \emptyset$.
for $F = 1 : K$

1) Compute the discriminability D among all classes with fold numbers $N_1, \ldots, N_F$ by means of $\mathcal{M}$. Discriminability D can by defined in terms of Euclidean, Mahalanobis, or another metric.

2) **If** $\{D < \text{threshold}\}$ **then** $\{\mathcal{M} = \mathcal{M} \cup \{\phi\}$, where $\phi \in \mathcal{B}_{N_F}$, **goto** 1$\}$

endfor

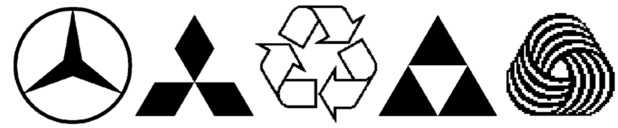

Fig. 1. Test trademarks (from left to right): Mercedes-Benz, Mitsubishi, Recycling, Fischer, and Woolen Stuff.

Starting from the highest symmetry, this algorithm selects in each loop those invariants which are able to distinguish among objects with the fold numbers N_F and higher, but which may equal zero for some (or all) other objects. Note that for some F the algorithm need not to select any invariant because the discriminability can be assured by the invariants selected before or because $C_F = 1$.

In addition to rotation symmetry, axial symmetry appears often in practical experiments and it also contributes to the vanishing of some moments. There is a close connection between axial and rotation symmetry—if an object has K axes of symmetry $(K > 0)$ then it is also rotationally symmetric and K is exactly its number of folds [32]. Thus, we will not discussed the choice of invariants for axially symmetric objects separately.

IV. Numerical Experiments

In order to illustrate how important is a careful choice of the invariants, in particular, pattern recognition tasks, we carried out the following experimental study.

A. Trademark Recognition

In the first experiment, we tested the capability of recognizing objects having the same number of folds, particularly $N = 3$. As a test set we used three trademarks of major companies (Mercedes-Benz, Mitsubishi, and Fischer) and two commonly used symbols ("recycling" and "woolen stuff"). All trademarks were downloaded from the respective websites, resampled to 128×128 pixels and binarized. We decided to use trademarks as the test objects because most trademarks have certain degree of symmetry and all commercial trademark recognition systems face the problem of symmetry. A comprehensive case study on trademark recognition and retrieval [33] used the Hu's moment invariants as a preselector; here we show that Theorem 3 yields more discriminative features.

As can be seen in Fig. 1, all our test marks have threefold rotation symmetry. Each mark was rotated ten times by randomly generated angles. Since the spatial resolution of the images was relatively high, the discretization effect was negligible. Moment invariants from Theorem 3 ($N = 3, p_0 = 3$, and $q_0 = 0$) provide an excellent discrimination power even if we take only two simplest of them (see Fig. 2), while the invariants from Theorem 2 are not able to distinguish the marks at all (see Fig. 3).

B. Recognition of Simple Shapes

In the second experiment, we used nine simple binary patterns with various numbers of folds: capitals F and L $(N = 1)$, rectangle and diamond $(N = 2)$, equilateral triangle and tripod $(N = 3)$, cross $(N = 4)$, and circle and ring $(N = \infty)$ (see Fig. 4). As in the previous case, each pattern was ten times rotated by ten random angles.

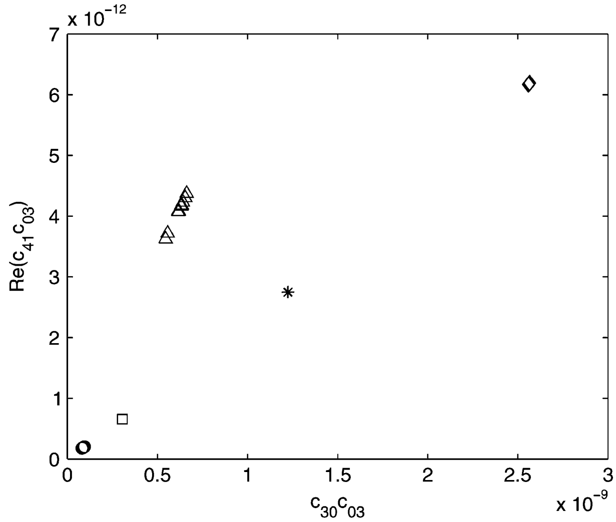

Fig. 2. Trademark positions in the space of two invariants $c_{30}c_{03}$ and $\mathrm{Re}(c_{41}c_{03})$ showing good discrimination power. □: Mercedes-Benz; ◇: Mitsubishi; △: Recycling; $*$: Fischer; and ∘: Woolen Stuff.

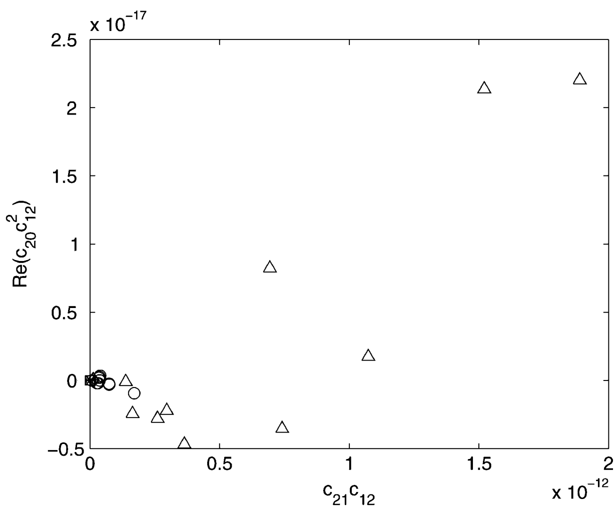

Fig. 3. Trademark positions in the space of two invariants $c_{21}c_{12}$ and $\mathrm{Re}(c_{20}c_{12}^2)$ introduced in Theorem 2. These invariants have no discrimination power with respect to this trademark set. □: Mercedes-Benz; ◇: Mitsubishi; △: Recycling; $*$: Fischer; and ∘: Woolen Stuff.

First, we applied rotation invariants according to Theorem 2 choosing $p_0 = 2$ and $q_0 = 1$. The positions of our test patterns in the feature space are plotted in Fig. 5. Although only a 2-D subspace showing the invariants $c_{21}c_{12}$ and $\mathrm{Re}(c_{20}c_{12}^2)$ is visualized here, we can easily observe that the patterns form one dense cluster around the origin (the only exception is the tripod, which is slightly biased because of its nonsymmetry caused by quantization effect). Two nonsymmetric objects—the letters F and L—are far from the origin, out of the displayed area. The only source of nonzero variance of the cluster are spatial quantization errors. All other invariants of the form $c_{pq}c_{12}^{p-q}$ behave in the same way. Thus, according to our theoretical expectation, we cannot discriminate among symmetric objects (even if they are very different) by means of invariants defined in Theorem 2.

Second, we employed the invariants introduced in Theorem 3 choosing $N = 4$ (the highest finite number of folds among the test objects), $p_0 = 4$, and $q_0 = 0$ to resolve the above

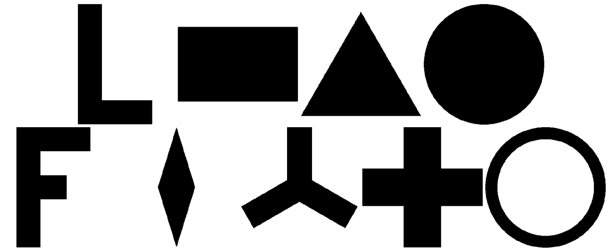

Fig. 4. Test patterns: capital L, rectangle, equilateral triangle, circle, capital F, diamond, tripod, cross, and ring.

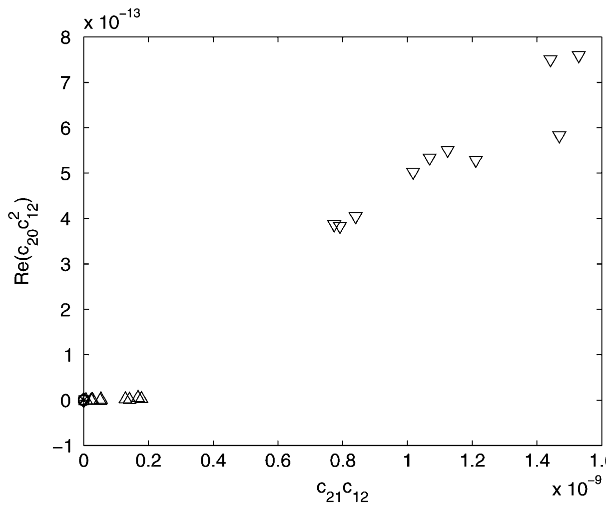

Fig. 5. Space of two invariants $c_{21}c_{12}$ and $\mathrm{Re}(c_{20}c_{12}^2)$ introduced in Theorem 2. ×: rectangle; ◇: diamond; △: equilateral triangle; ▽: tripod; $+$: cross; •: circle; and ∘: ring.

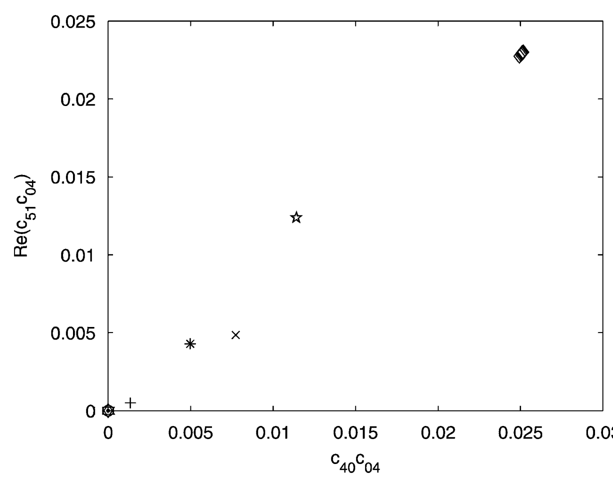

Fig. 6. Space of two invariants $c_{40}c_{04}$ and $\mathrm{Re}(c_{51}c_{04})$ introduced in Theorem 3, $N = 4$. ×: rectangle; ◇: diamond; △: equilateral triangle; ▽: tripod; $+$: cross; •: circle; ∘: ring; $*$: capital F; and ⋆: capital L.

recognition experiment. The situation in the feature space looks different from the previous case (see the plot of two simplest invariants $c_{40}c_{04}$ and $\mathrm{Re}(c_{51}c_{04})$ in Fig. 6). Five test patterns formed their own very compact clusters which are well separated from each other. However, the patterns circle, ring, triangle, and tripod still made a mixed cluster around the origin and remained nonseparable. This is also fully in accordance with the theory, because the number of folds used here is not optimal for our test set.

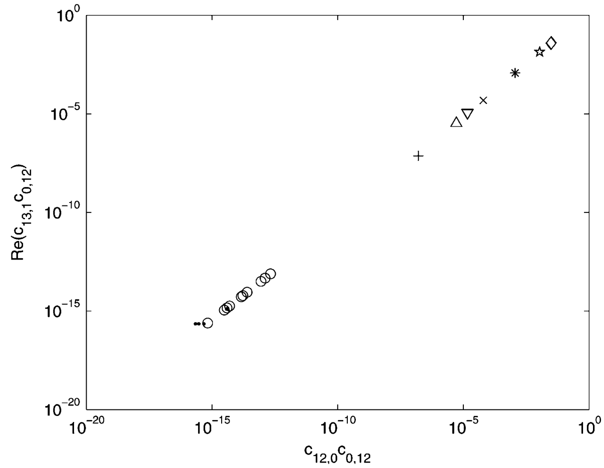

Fig. 7. Space of two invariants $c_{12,0}c_{0,12}$ and $\mathrm{Re}(c_{13,1}c_{0,12})$ introduced in Theorem 3, $N = 12$ (logarithmic scale). $\times$: rectangle; $\diamondsuit$: diamond; $\triangle$: equilateral triangle; ∇: tripod; $+$: cross; $\bullet$: circle; $\circ$: ring; $*$: capital F; and $\star$: capital L.

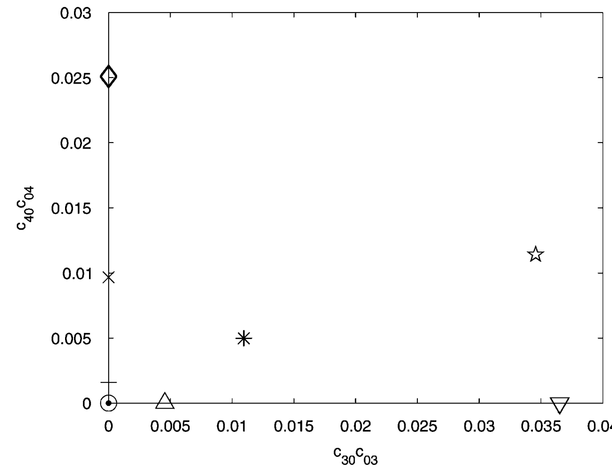

Fig. 8. Space of two invariants $c_{3,0}c_{0,3}$ and $c_{40}c_{04}$. $\times$: rectangle; $\diamondsuit$: diamond; $\triangle$: equilateral triangle; ∇: tripod; $+$: cross (it is on vertical axis close to zero); $\bullet$: circle; $\circ$: ring; $*$: capital F; and $\star$: capital L. Comparing to Fig. 7, note less correlation of the invariants and lower dynamic range.

Third, we repeated this experiment again with invariants according to Theorem 3 but selecting N as the least common multiple of all finite fold numbers involved, i.e., $N = 12$. One can learn from Fig. 7 that now all clusters are well separated (because of high dynamic range, logarithmic scale was used for visualization purposes). The only exception are two patterns having circular symmetry—the circle and the ring—that still made a mixed cluster. If we wanted to also separate these two patterns one from the other, we could use the invariants c_{pp}. On the other hand, using *only* these invariants for the whole experiment is not a good choice from the practical point of view—since there is only one such invariant for each order, we would be pushed into using high-order noise-sensitive moments.

Finally, we used the algorithm described at the end of Section 3. In this case, two invariants $c_{30}c_{03}$ and $c_{40}c_{04}$ are sufficient to separate all classes (of course, with an exception of the circle and the ring), see Fig. 8. Comparing to the previous case, note less correlation of the invariants, their higher robustness, and lower dynamic range. On the other hand, neither $c_{30}c_{03}$ nor $c_{40}c_{04}$ provide enough discrimination power when used individually while the twelfth-order invariants are able to distinguish all classes.

C. Real Data Experiment

We demonstrate the performance of the invariants in object matching task. We used a popular baby toy (see Fig. 9) which is also commonly used in testing computer vision algorithms and robotic systems. The toy consists of a hollow sphere having twelve holes and of twelve objects of various shapes. Each object matches with one particular hole. The baby (or the algorithm) should assign the objects to the corresponding holes and insert them into the sphere. The baby can employ both the color and shape information; however, in our experiment, we disregarded the colors at all to make the task more difficult.

First, we took the pictures of the holes (one picture per each hole) and we binarized them by simple thresholding. Binarization was the only preprocessing, we did not make any sphere-to-plane corrections.

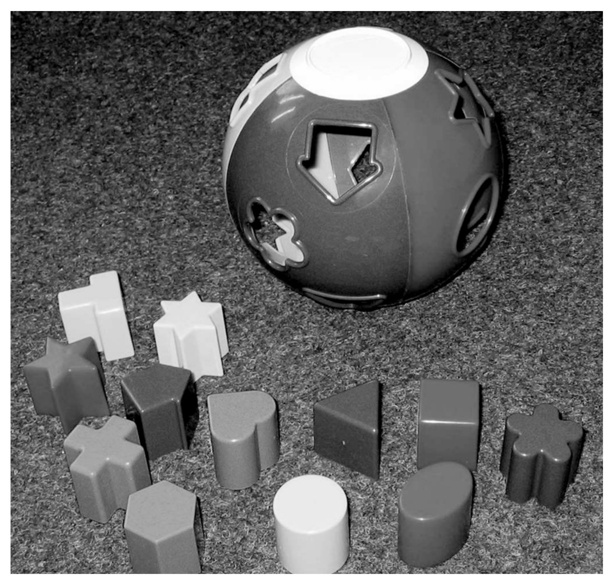

Fig. 9. Toy set used in the experiment.

To select proper invariants, we applied the algorithm from Section 3 on the images of the holes. As a discriminability measure, we took weighted Euclidean distance, where the weights were set up to normalize the dynamic range of the invariants. As one can observe, the highest finite number of folds is 6. The algorithm terminated after passing three loops and selected the three following invariants: $c_{60}c_{06}$, $c_{50}c_{05}$, and $c_{40}c_{04}$.

Then, we took ten pictures of each object with random rotations, binarized them, and run the classification. This task is not so easy as it might appear because the holes are a bit larger than the objects but this relation is rather morphological than linear and does not preserve the shapes exactly. Fortunately, all 120 unknown objects were recognized correctly and assigned to proper holes. It should be emphasized that only three invariants without any other information yielded 100% recognition rate for 12 classes, which is a very good result even though the shapes are relatively simple.

We repeated the classification once again with invariants constructed according to Theorem 2 setting $p_0 = 2, q_0 = 1$. Only three objects having one-fold symmetry were recognized correctly, the others were classified randomly.

V. Conclusion

In this paper, we proposed a new set of rotation moment invariants designed for description and recognition of objects having certain degree of rotation symmetry. Moment invariants described earlier cannot be used for this purpose because most moments of symmetric objects vanish. The solution to this problem is given in Theorem 3, which is in our opinion the first definition of rotation moment invariants for symmetric objects. Moreover, Theorem 3 proves the independence and completeness of the proposed invariants.

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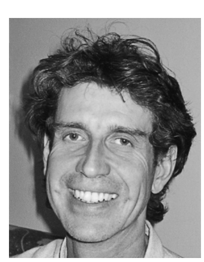

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