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STOCHASTIC CATASTROPHE THEORY

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Stochastic Catastrophe Theory: Transition Density*

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Abstract

The so called Cusp deterministic catastrophe model extends the classical linear regression adding nonlinearity into a model. A property of a stochastic catastrophe model connected with stochastic differential equation could be described by density, which is known in closed-form only in stationary case. The approximation of the transition density is done here by finite difference method.

JEL: C01,C53

Keywords: stochastic catastrophe theory, cusp, transition density, finite difference.

1 Catastrophe Theory

1.1 Introduction

Catastrophe theory is a special case of singularity theory part of the study of *nonlinear dynamical systems*. *Bifurcation theory* is considered to have been discovered by Henri Poincaré, as part of his qualitative analysis of systems of nonlinear differential equations (1880-1890). After onward studying of differential equations René Thom (1972) described the seven elementary catastrophes going up through six dimensions in control and state variables. This became standard catastrophe theory.

In general catastrophe theory enables to estimate such a model, where the small continuous change in exogenous variables could cause the discontinuous

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change of endogenous variable. This theory could be used for modelling of the sudden collapse of a bridge under gradually increasing pressure or freezing of water in decreasing temperatures. The earliest economic application was published by Zeeman (1974), who tried to model bubbles and crashes in stock markets. He used the so called CUSP model with two types of investors but only for qualitative analysis. This was one reason why the paper was very criticized especially by Zahler and Sussman. Despite the criticism the catastrophe theory has still being developed. Stochastic catastrophe theory was introduced in late seventieth by Loren Cobb [3]. He also developed maximum likelihood estimation of catastrophe models. There are two other methods based on least-squares and regression technique, but Cobb's method is the most cited and further used and developed. Recently Rosser [10] summed up the discussions about catastrophe theory and concluded that the catastrophe theory definitely is worth further attention while investigating nonlinear systems.

1.2 Cusp

Commonly used way how to introduce Cusp Surface Analysis is to compare it to multiple linear regression. As in regression, there is one dependent variable ($\mathbf{Y}$) and a state variables ($\mathbf{Z} = (\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$):

$$\mathbf{Y} = b_0 + b_1\mathbf{Z}_1 + \ldots + b_n\mathbf{Z}_n + \mathbf{U}, \tag{1}$$

where the random variable $\mathbf{U}$ is assumed to be normally distributed with zero mean and constant variability. This regression model has n+2 degrees of freedom.

To obtain greater than linear flexibility it is necessary to add $2n + 2$ degrees of freedom into linear regression model to define three control factors:

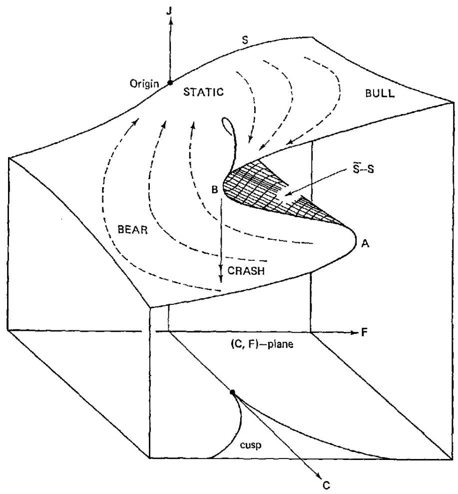
$$\begin{aligned} A(\mathbf{Z}) &= A_0 + A_1\mathbf{Z}_1 + \ldots + A_n\mathbf{Z}_n \\ B(\mathbf{Z}) &= B_0 + B_1\mathbf{Z}_1 + \ldots + B_n\mathbf{Z}_n \\ L(\mathbf{Z}) &= L_0 + L_1\mathbf{Z}_1 + \ldots + L_n\mathbf{Z}_n \end{aligned} \tag{2}$$

These factors plus coefficient C determine the predicted values of $\mathbf{Y}$ given $\mathbf{Z}$. The predicted values of $\mathbf{Y}$ are values $\mathbf{Y}$ for which holds this equation:

$$0 = A(\mathbf{Z}) + B(\mathbf{Z})\ [\mathbf{Y} - L(\mathbf{Z})] - C\ [\mathbf{Y} - L(\mathbf{Z})]^3. \tag{3}$$

Factors A, B, and L are used to be called the asymmetry, bifurcation, and linear factor, respectively. The catastrophe model defined by (3) can be seen as a generalization of the regression model (1): the two models coincide if (i) $A = 0$, (ii) $B = 1/\mathrm{Var}[\mathbf{U}]$, and (iii) $C = 0$. When these conditions are satisfied the coefficients L_i of L are the same as the coefficients b_i of (1).

The model based on (3) is, like the regression model described by (1), a static random model. The static catastrophe model is related to a dynamic model. The (deterministic) dynamic cusp catastrophe model is described by a differential equation:



Obrázek 1: Zeemans original cusp surface

$$dy(t)/dt = a(z) + b(z)\ [y(t) - l(z)] - c\ [y(t) - l(z)]^3, \tag{4}$$

where the term on the right hand side could be expressed like

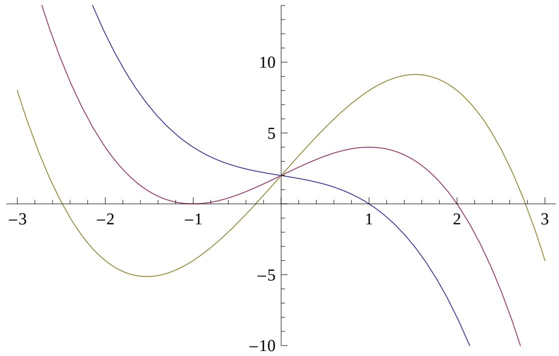
$$-\frac{dV(y)}{dy} = a(z) + b(z)\ [y(t) - l(z)] - c\ [y(t) - l(z)]^3. \tag{5}$$

The function $V(y)$ is called potential function and enable us to study the behavior of catastrophe models. When the left hand side of the equation (4) equals 0, the system is in equilibrium. We distinguish between stable and unstable equilibrium states, which correspond to the lokal minima and lokal maxima of potential function. We can observe one stable or two stable and one unstable equilibria of cusp model.

The cusp catastrophe model in canonical form

$$0 = \alpha + \beta x - x^3 \tag{6}$$

shapes the canonical cusp surface (figure 1). The equation does not descibe the relation between the original control variables and the original behavioral variable. The variables α, β and x are derived from the original one by diffeomorphic transfornations. This transformations adjust the coordinate system so that the shape of original response surface matches to the canonical cusp surface near the cusp catastrophe point ($\alpha = \beta = x = 0$). The variable x is a function of the original behavioral variable and the original control variables, while the variables α, β are each functions of all of the original control variables.



Obrázek 2: Drift: $a = 2$, $c = 1$, $b = \{-1 \text{ (blue)}, 3 \text{ (red)}, 7 \text{ (yellow)}\}$

2 Stochastic catastrophe theory

The basic assumption of catastrophe modelling is the existence of a mean equilibrium state between the variable of interest y (e.g., market index, long-term interest rate) and market foundamentals z (e.g., dividends, trading volume, Put/Call ratio, short-term interest rate) which drive the parameters of diffusion proces ($z(t)$ is a vector of market foundamentals strictly exogenous with respect to explained variable $y(t)$).

The CUSP stochastic model originally advanced by Creedy et al. [6] arises from a general diffusion proces for the variable of interest with a cubic drift and an arbitrary volatility functions. When choosing the drift function one has to consider the strength of mean reversion. Aït-Sahalia [1] has shown that a stronger mean-reverting drift is more likely to pull the process back towards its mean even in high volatile dynamics. The cubic drift can take the shape illustrated in figure (2).

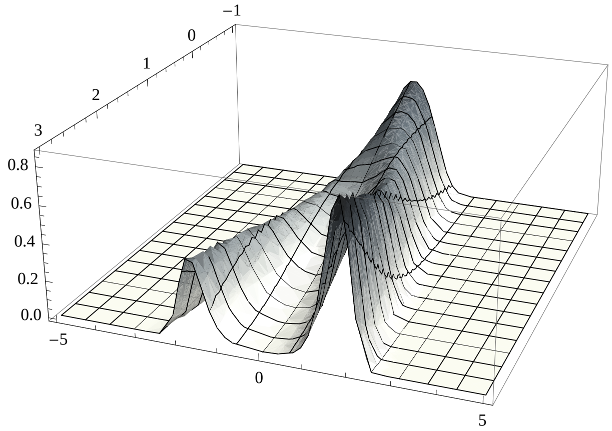
In the basic Cusp stochastic model the volatility function $\sigma(y,t)$ is assumed to be a constant σ^2. However this is quite limiting this assumption does not restrict the crucial features of underlying model. There are several other commonly used diffusion specification:

$$\begin{aligned}\sigma(y,t) &= \sigma\sqrt{x}\\ \sigma(y,t) &= \sigma\ x\\ \sigma(y,t) &= \sigma\sqrt{x\ (1-x)}\\ \sigma(y,t) &= \sigma_0 + \sigma_1\ x.\end{aligned} \tag{7}$$

In our configuration the stochastic differential equation is following:

$$dy(t) = \big(a(z,t) + b(z,t)\ [y(t)-\lambda] - c\ [y(t)-\lambda]^3\big)\, dt + \sigma\ dW(t). \tag{8}$$

where $W(t)$ denotes a standard Brownian motion.



Obrázek 3: Stationary density: $a = 0.1$, $b \in [-1, 3]$, $c = 1$, $y \in [-5, 5]$

2.1 Density

The use of Itô stochastic differential equation (8) allowed Cobb to relate the potential function of a deterministic catastrophe system with stationary probability density functin of the stochastic process. By solving the corresponding Fokker–Planck equation

$$\frac{\partial}{\partial t} f(y,t) = -\frac{\partial}{\partial y} [\mu(y,t) f(y,t)] + \frac{1}{2} \frac{\partial^2}{\partial y^2} [\sigma(y,t)^2 f(y,t)] \tag{9}$$

one could obtain the transition density $f(y)$, but in most cases the-closed form solution does not exist. Therefore the research has been done using the stationary density

$$f_I(y) = N_s \exp\left[2 \int_s^y \frac{\{\mu(x) - (1/2)[\sigma^2(x)]'\}}{[\sigma^2(x)]} dx\right], \tag{10}$$

where N_s is normalizing constant, s is an arbitrary interior point of the state space and the prime denotes differentiation with respect to x. Maximimum likelihood estimation using stationary density is appropriate only under random sampling, which is not the case of financial time series. Another approach is to minimize the distance between the nonparametric kernel density estimate and the parametric stationary density.

The problem of Cobb's interpretation in Itô sense inhere in fact, that stationary density is not invariant under diffeomorphic transformations, which is essential in deterministic catastrophe theory. Wagenmakers et al. [15] came with the solution of this problem. They prooved that the use of Stratonovich interpretation of stochastic differential equation is invariant under diffeomorphic transformations. The invariant density could be derived from the stationary density from an Itô SDE by multiplying it by diffusion function $\sigma(y)$:

$$f_S(y) = f_I(y)\sigma(y) = N_i \ \exp\left[2\int_i^y \frac{\{\mu(x) - (1/4)[\sigma^2(x)]'\}}{[\sigma^2(x)]}dx\right] \tag{11}$$

When the diffusion function is constant, $\sigma(y) = \sigma$, we can derive the following form of stationary pdf

$$f_\infty(y|z;\theta) = N \ \exp\left[2 \ a(z) \ \frac{(y-\lambda)}{\sigma^2} + b(z) \ \frac{(y-\lambda)^2}{\sigma^2} - \frac{c}{2} \ \frac{(y-\lambda)^4}{\sigma^2}\right], \tag{12}$$

where θ are parameters of $a(z)$ and $b(z)$, which are linear functions of z, c is parameter of mean-reversion strenght and λ position parameter. This stationary density corresponds to the stochastic differential equation (8) irrespective of its Itô or Stratonovich interpretation. The parameters can be estimated using maximum likelihood procedures for example by the invariant one developed by Hartelman [7].

This stationary density (12) belongs to the class of generalized normal distributions, which is highly flexible with respect to skewness and kurtosis, and exibits, at most, two modes (Cobb [4]).

2.2 Cardan's discriminant

A procedure to identify bimodality of the stationary densiy consist in assessing the sign of the Cardan's discriminant:

$$\delta_C(z;a,b) \equiv \frac{a(z)^2}{4} - \frac{b(z)^3}{27 \ c}. \tag{13}$$

A necessary and sufficient condition for unimodality is $\delta_C(z;a,b) \geq 0$. In this case $a(z)$ and $b(z)$ measure skewness and kurtosis of the distribution. In bimodality case $\delta_C(z;a,b) < 0$ $a(z)$ and $b(z)$ determine the relative heights and separateness of the two modes. In case of transition density Cardan's discriminant is saying to which steady state the system is in the moment driven. The similarity between the stationary density and the transition density is particularly driven by the length of the time step.

3 Approximation of transition density

3.1 Finite Differences

The Fokker-Planck equation usually does not have closed-form solutin and our case of cusp stochastic model is not an exception. Finite difference method is one of the possible numerial estimation. We are not limited by irregular shape of the boundary and there is no flux across the boundary as well. In the light of investigating cusp model we can set the boundary conditions for the density during the time to zero in sufficiently distance from zero on both sides. Starting density in the zero time does not play importat role and in practical applications is omitted because of its negligible influence when maximizing the likelihood. We need to discretized the Fokker-Planck equatin

$$\frac{\partial}{\partial t}\ [f(y,t)] = -\frac{\partial}{\partial y}\ [\mu(y,z,t)f(y,t)] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\ [f(y,t)] \tag{14}$$

in the nodes of rectangular mesh with time step Δt going from 0 to the number of observation T and value step Δy splitting the values $y^{min} = y^0, y^1, \ldots, y^N = y^{max}$, where we consider the drift

$$\mu(y,z,t) = a(z,t) + b(z,t)\ y(t) - c\ y(t)^3. \tag{15}$$

For approximating derivatives of equation (14) the finite difference method has many possibilities which are of different degrees of accuracy and with specific stability conditions. This two characteristic would be subordinated to other two desired properties of approximation. First is suppressing of the undesirable oscillation of the solution. This requirement shuts out the central differences for the first derivation with respect to y (Strikwerda [12]). Secondly, there is natural requirement to keep the solution above zero which rules out the compounded approximations for time derivation. Bearing all requirements in mind we finally arrive to pertinent dicretization

$$\begin{aligned}\frac{f(y,t)-f(y,t-\Delta t)}{\Delta t} = \quad & -[\mu(y,z,t)]'\,f(y,t) - \frac{\mu(y,z,t)}{2\ \Delta y}\ I_p \\ & \cdot(3f(y,t) - 4f(y - I_p\ \Delta y, t) + f(y - 2I_p\ \Delta y, t)) \\ & + \frac{\sigma^2}{2\ \Delta y^2}(f(y-\Delta y,t) - 2f(y,t) + f(y+\Delta y,t)),\end{aligned} \tag{16}$$

where the $I_p = 1$ if $\mu(y,z,t) > 0$ and $I_p = -1$ otherwise and the prime denotes the derivation with respect to y. This method is suggested by Shapira [11] here is just adapted for second order accuracy and specific situation. For the time derivation the backward difference is used because of its unconditional stability which means that we have first order accuracy in time discretization.

Then for every time layer we have to solve the set of equation generally written down as:

$$\begin{aligned}f(y,t-\Delta t) = \quad & f(y,t)\left(1 + \Delta t\ [\mu(y,z,t)]' + \frac{3\ \Delta t}{2\ \Delta y}\ |\mu(y,z,t)| + \frac{\sigma^2\ \Delta t}{\Delta y^2}\right) \\ & + f(y-\Delta y, y)\left(-\frac{4\ \Delta t}{2\ \Delta y}\ \mu P(y,z,t) - \frac{\sigma^2\ \Delta t}{2\ \Delta y^2}\right) \\ & + f(y+\Delta y, y)\left(-\frac{4\ \Delta t}{2\ \Delta y}\ \mu N(y,z,t) - \frac{\sigma^2\ \Delta t}{2\ \Delta y^2}\right) \\ & + f(y-2\Delta y, t)\ \frac{\Delta t}{2\ \Delta y}\ \mu P(y,z,t) \\ & + f(y+2\Delta y, t)\ \frac{\Delta t}{2\ \Delta y}\ \mu N(y,z,t),\end{aligned} \tag{17}$$

where

$$\mu P(y,z,t) := (|\mu(y,z,t)| + \mu(y,z,t))/2$$
$$\mu N(y,z,t) := (|\mu(y,z,t)| - \mu(y,z,t))/2.$$

This general equations could be rewritten into compact matrix notation:

$$f^{(i-1)} = \mathbf{A} f^{(i)},$$

where $f^{(i)} = \{f(y^0, i\ \Delta t), f(y^1, i\ \Delta t), \dots, f(y^N, i\ \Delta t)\}$ is a vector of density approximation values in i-th time layer and $\mathbf{A}$ is compounded from five vectors (one diagonal and two ofdiagonals on both sides at diagonal) such that the product correspond to the equation (17).

After all its possible to estimate the parameters of a diffusion proces with discretely observed process values $y_0, y_1, \dots, y_T$ and values of market foundamentals denoted z_t using appoximate maximum likelihood:

$$\sum_{s=1}^{T} \log f(y_s|z_s, y_{s-1}; \theta), \tag{18}$$

where θ is a vector of parameters a_i, b_i, c, σ.

3.2 Other possible approaches

More precise and flexible but still quite similar method to finite differences is finite element method. The main advantage is, that it aproximates the result not the solved equation. Other advantages are possibility to refine the mesh during the calculation to achieve higher accuracy. This method is prefered by Hurn *et al.* [8].

Aït-Sahalia [2] has proposed a method based on Hermite polynomials to derive explicit sequence of closed-form expansion for the transition density of diffusion processes. Still, it has to be adapted for diffusion processes with exogenous variables. Thanks to closed-form the maximum likelihood estimation of the parameters is considerably less computationally demanding and achieving at least comparable accuracy with other approximation methods ([9]).

Reference

[1] Aït-Sahalia, Y. (1996) *Testing continuous-time models of the spot interest rate.* Review of Financial Studies 9, pp. 385-426.

[2] Aït-Sahalia, Y. (2002) *Maximum likelihood estimation of discretely sampled diffusion: A closed-form approximation approach.* Econometrica, Vol. 70, No. 1, pp. 223-262.

[3] Cobb, L. (1978) *Stochastic catastrophe models and multimodal distributions.* Behavioral Science 23, pp. 360-374

[4] Cobb, L. (1983) *Estimation and Moment Recursion Relations for Multimodal Distributions of the Exponential Family.* Journal if American Statistical Association 78, pp. 124-130.

[5] Cobb, L. (1985) *An Introduction to Cusp Surface Analysis.*

[6] Creedy, J., Lye, J.N., Martin, V.L. (1996) *A non-linear model of the real US/UK exchange rate.* Journal of Applied Econometrics 11, pp. 669-686.

[7] Hartelman, P. (1997) *Stochastic catastrophe theory.* Dissertatie reeks, Faculteit Psychologie, Universiteit van Amsterdam.

[8] Hurn, A., Jeisman, J., Lindsay, K. (2007) *Teaching an Old Dog New Tricks Im- proved Estimation of the Parameters of Stochastic Differential Equations by Numerical Solution of the Fokker-Planck Equation.* Working Paper #9, National Centre for Econometric Research, Queensland University of Technology, Brisbane.

[9] Jensen, B., Poulsen, R. (2002) *Transition Densities of Diffusion Processes: Nu- merical Comparison of Approximation Techniques.*, Journal of Derivatives, pp. 18-34.

[10] Rosser Jr., J.B. (2007) *The rise and fall of vatastrophe theory applications in economics: Was the baby thrown out with the bathwater?* Journal of Economic Dynamics and Control, Elsevier, vol. 31, No. 10, pp. 3255-3280.

[11] Shapira, Y. (2006) *Solving PDEs in C++* Siam, Philadelphia.

[12] Strikwerda, J.C. (2004) *Finite Difference Schemes and Partial Differential Equations, Second Edition* Siam, Philadelphia.

[13] Varian, H.R. (1979) *Catastrophe theory and the Business Cycle.* Economic Inquiry 17:1, pp. 14-28.

[14] Vosvrda, M., Barunik, J. (2009) *Can a stochastic cusp catastrophe model explain stock market crashes?* Preprint submitted to Elsevier Science

[15] Wagenmakers, E.J., Molenaar, P.C.M., Grasman, R.P.P.P., Hartelman, P.A.I., van der Maas, H.L.J. (2005) *Transformation invariant stochastic catastrophe theory.* Physica D 211, pp. 263-276.

[16] Zeeman, E.C. (1974) *On the unstable behavior of stock exchanges.* Journal of Mathematical Economics, Elsevier, vol. 1, pp. 39-49.