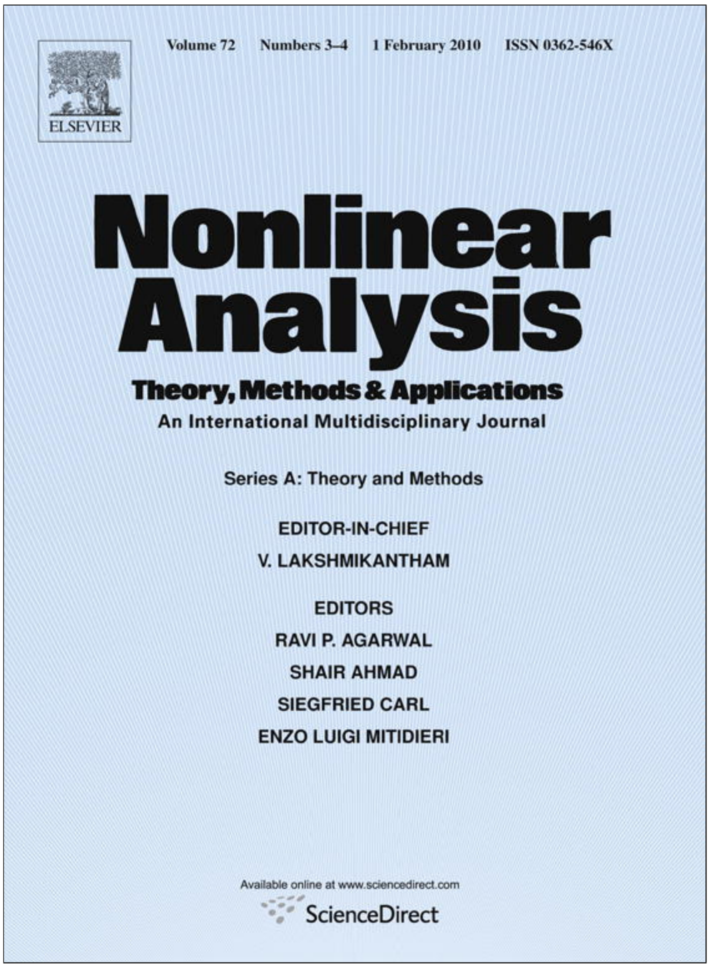
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Overlap functions

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ABSTRACT

In this paper we address a key issue in scenario classification, where classifying concepts show a natural overlapping. In fact, overlapping needs to be evaluated whenever classes are not crisp, in order to be able to check if a certain classification structure fits reality and still can be useful for our declared decision making purposes. In this paper we address an object recognition problem, where the best classification with respect to background is the one with less overlapping between the class *object* and the class *background*. In particular, in this paper we present the basic properties that must be fulfilled by overlap functions, associated to the degree of overlapping between two classes. In order to define these overlap functions we take as reference properties like migrativity, homogeneity of order 1 and homogeneity of order 2. Hence we define overlap functions, proposing a construction method and analyzing the conditions ensuring that t-norms are overlap functions. In addition, we present a characterization of migrative and strict overlap functions by means of automorphisms.

1. Introduction

In many fields, a common problem is to assign a given element to one of the several classes of objects. If the separation between the classes is not clear, the expert may not be sure about how to assign elements into a specific class. Perhaps these classes are fuzzy in nature and the expert realizes that elements are simply in between several classes (see, e.g., [1]). But perhaps it is the case that the border, being clear in reality, is simply not clear enough in our picture. In any of these situations the concept of overlap arises (see [2–7,26–28]).

In image processing, for instance, the identification of the objects in a given image is a very important problem. A widely used technique to handle this problem is thresholding. Focusing on the case of an image with a single object on a background, thresholding consists of determining an intensity threshold such that pixels whose intensity is greater than the threshold are assigned to the object (or background), and pixels whose intensity is smaller than the threshold are assigned to the background (object). Usually, these kind of problems are managed by means of functions that somehow *represent* the object and the background. Several methods can be considered to build such functions, from probabilistic techniques to fuzzy methods, including *ad hoc* constructions made by an expert.

In any case, two membership functions like the ones depicted in Fig. 1 will be always needed in this kind of problem: two functions μ_B and μ_O that, in an expert's opinion, will represent the background and the object, respectively, on the scale of L levels of gray. And there will exist two intensity values, denoted by q_i and q_j, such that for intensities lower than q_i the expert is sure that the pixels do not belong to the object, whereas for intensities greater than q_j the expert is sure that they

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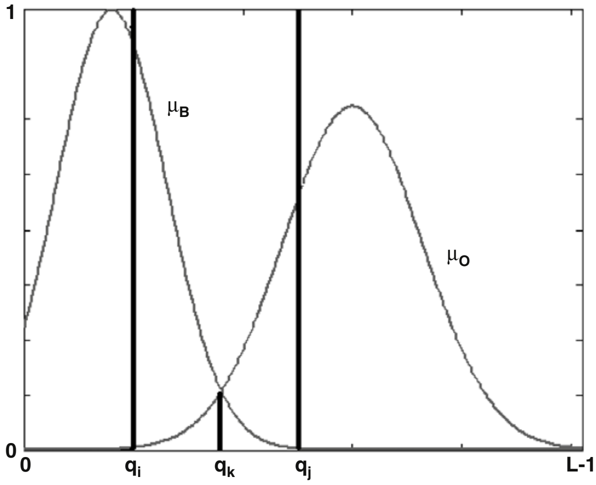

Fig. 1. Background and object representation.

do not belong to the background. However, for any intensity between q_i and q_j the expert is not positively sure of whether it belongs to one or the other. Moreover, in this case there exists an intensity q_k for which the lack of knowledge is maximum.

Overlap functions provide a mathematical model for this kind of situations.

The degree of overlap between those two functions (for object and background) can be interpreted as the representation of the lack of knowledge of the expert in determining if the pixel in question belongs to the background or to the object (see Fig. 1). Our *overlap functions* pursue to measure this overlapping degree.

Nevertheless, for a given image many different representations similar to Fig. 1 can be depicted, meaning that many pairs of mappings representing the object on one side and the background on the other side are possible. Moreover, there are different algorithms in image processing which assign a threshold to each pair of functions. Many different thresholds can be defined for a given image, and it is therefore necessary to find a method to determine which one is the best. A first approach to a possible algorithm is the following:

(1) Calculate the overlap degree for each pair of functions;
(2) Take the threshold corresponding to the pair of functions for which the overlap degree is smallest.

Hence, we can properly measure overlapping, with this kind of algorithm we can find the threshold ensuring the smallest intersection between the mappings that represent the background and the object, which can be understood as the best threshold in the sense that it is the procedure best distinguishing between the object and the background.

So the concept of overlap function should become a key mathematical tool for the modelization on overlapping indices, and a first mathematical analysis should immediately allow to:

- determine a distribution of the pixels of an image between the background and the object;
- build an algorithm to choose the best threshold for an image from a set of different possible thresholds.

From a mathematical point of view, overlap functions can be seen just as a particular instance of bivariate, continuous aggregation functions, that is, increasing functions defined over the unit square and with appropriate boundary conditions (see Definition 2 for more details). It is therefore worth to consider the possible relationships between overlap functions and other well-known examples of aggregation functions, as t-norms, copulas, semicopulas or quasi-copulas. Observe that the requirement of continuity prevents an overlap function from being a uninorm (there is no uninorm being continuous on the whole unit square). Notice also that although we will demand to our overlap functions to be defined in the unit square, this is not an essential requirement, and other domains can also be considered.

Moreover, it is clear that the properties we can require to define overlap functions will not be exhaustive at all. On the contrary, some particular assumptions may be necessary to deal with specific applications. In particular, we consider here several analytical properties (migrativity, homogeneity, Lipschitzianity) that seem to be quite natural in the image processing setting.

The structure of this paper is the following. We start by recalling some basic concepts in Section 2. In Section 3 we define overlap functions and study their first properties. In Section 4 we study under which conditions we can assure that the different aggregation functions presented in Section 2 can also be overlap functions. Section 5 is devoted to consider overlap functions which are migrative, homogeneous or Lipschitz. We end by presenting some conclusions and future lines of work.

2. Preliminaries

For the sake of completeness, in this section we recall some well-known concepts that will be useful in subsequent sections. We start with the concept of automorphism of the unit interval.

Definition 1. An automorphism of the unit interval is any continuous and strictly increasing function $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

2.1. Aggregation functions

Concerning aggregation functions, we will follow the approach and definitions given in [8–11] (but see also [12,13].)

Definition 2. An aggregation function of dimension n (n-ary aggregation function) is an increasing mapping $M : [0, 1]^n \to [0, 1]$ such that $M(0, \ldots, 0) = 0$ and $M(1, \ldots, 1) = 1$.

Definition 3. Let $M : [0, 1]^n \to [0, 1]$ be an n-ary aggregation function.

(i) M is said to have an annihilator $a \in [0, 1]$, if $M(x_1, \ldots, x_n) = a$ whenever $a \in \{x_1, \ldots, x_n\}$.
(ii) M is said to be strictly increasing if it is strictly increasing as a real function of n-variables on $[0, 1]^n$ if M has no annihilator; and on domain $([0, 1] \setminus \{a\})^n$ if M has an annihilator a.
(iii) M is said to have divisors of zero if there exists $x_1, \ldots, x_n \in]0, 1]$ such that $M(x_1, \ldots, x_n) = 0$.
(iv) M is said to be idempotent if $M(x, \ldots, x) = x$ for any $x \in [0, 1]$.

For the particular case of bivariate aggregation functions we remind the following definitions.

Definition 4. Let M be a bivariate aggregation function.

(i) M is said to be symmetric if $M(x, y) = M(y, x)$ for any $x, y \in [0, 1]$.
(ii) M is said to be associative if $M(M(x, y), z) = M(x, M(y, z))$ for any $x, y, z \in [0, 1]$.

2.2. t-norms and related concepts

We recall now the concept of t-norm, which plays a key role, for instance, to model conjunctions in fuzzy logics or intersections in fuzzy set theory.

Definition 5. A triangular norm (t-norm for short) is an associative, symmetric bivariate aggregation function $T : [0, 1]^2 \to [0, 1]$ such that $T(1, x) = x$ for all $x \in [0, 1]$. A strictly increasing continuous t-norm T is called a strict t-norm.

A particular type of continuous t-norms are Archimedean t-norms (see, e.g., [14]).

Definition 6. A continuous t-norm T is said to be Archimedean if $T(x, x) < x$ for all $x \in]0, 1[$.

It is worth to remark that this is not the usual definition of Archimedean t-norm that can be found in the literature. Nevertheless, both definitions are equivalent when dealing with continuous t-norms ([14]). Any strict t-norm (i.e., any continuous and strictly increasing t-norm) is necessarily Archimedean.

The following result shows that any strict t-norm is just the image by an automorphism of the product t-norm $T_P(x, y) = xy$ (see [15,14]).

Theorem 1. *A t-norm T is strict if and only if there exists an automorphism φ of the unit interval such that*

$$T(x, y) = \varphi^{-1}(\varphi(x)\varphi(y)), \quad x, y \in [0, 1].$$

In order to classify continuous t-norms, we need to introduce the concept of an ordinal sum as follows.

Definition 7. Suppose that $\{[a_m, b_m]\}$ is a countable family of non-overlapping, closed, non-trivial, proper subintervals of $[0, 1]$, To each $[a_m, b_m]$ in the family associate a t-norm T_m. The ordinal sum of the family $\{([a_m, b_m], T_m)\}$ is the mapping $T : [0, 1]^2 \to [0, 1]$ given by

$$T(x, y) = \begin{cases} a_m + (b_m - a_m)T_m\left(\dfrac{x - a_m}{b_m - a_m}, \dfrac{y - a_m}{b_m - a_m}\right) & \text{if } (x, y) \in [a_m, b_m]^2 \\ \min(x, y) \quad \text{otherwise} \end{cases}$$

Each T_m is called a summand.

Ordinal sums allow the following classification result [16].

Theorem 2. *Assume that T is a continuous t-norm. Then, one of the following three cases is valid for T:*

1. *$T(x, y) = \min(x, y)$;*
2. *T is Archimedean;*
3. *there exists a family $\{([a_m, b_m], T_m)\}$ such that T is the ordinal sum of this family and each T_m is a continuous Archimedean t-norm.*

For a more general introduction to t-norms and their properties we refer to [17,6,16,14,15].

2.3. Migrativity, homogeneity and Lipschitzianity

The concept of α-migrativity was introduced by Durante et al. in [18] for a class of bivariate operations having a property previously presented by Mesiar and Novak in [[19], Problem 1.8(b)], as Fodor and Rudas acknowledge in [20].

Definition 8. Let $\alpha \in [0, 1]$. A bivariate operation $G : [0, 1]^2 \to [0, 1]$ is said to be α-migrative if we have

$$G(\alpha x, y) = G(x, \alpha y) \quad \text{for all } x, y \in [0, 1].$$

Observe that any mapping G is 1-migrative, whereas 0-migrativity means that

$$G(x, 0) = G(0, y) = G(0, 0)$$

for any $x, y \in [0, 1]$. Also notice that in Definition 8 migrativity refers to a fixed, predetermined α. In [21] the concept of α-migrativity was generalized as follows.

Definition 9. A function $G : [0, 1]^2 \to [0, 1]$ is called *migrative* if and only if

$$G(\alpha x, y) = G(x, \alpha y), \quad \text{for all } x, y \in [0, 1]$$

and every $\alpha \in [0, 1]$.

In [21], an in-depth study of the migrativity property is carried out, even allowing α to take values greater than 1.

Lemma 1. *A mapping $G : [0, 1]^2 \to [0, 1]$ is migrative if and only if $G(x, y) = G(1, xy)$, for all $x, y \in [0, 1]$.*

Lemma 2. *A mapping $G : [0, 1]^2 \to [0, 1]$ is migrative if and only if there exists $g : [0, 1] \to [0, 1]$ such that $G(x, y) = g(xy)$, for all $x, y \in [0, 1]$.*

We are also going to consider two other analytical properties that can be of interest in applications of overlap functions. The first one is that of homogeneity of order $k > 0$, whereas the second is a sort of stronger continuity known as Lipschitzianity.

Definition 10. Let $G : [0, 1]^2 \to [0, 1]$ be a mapping and $k \in]0, \infty[$. G is homogeneous of order k if for any $\alpha \in [0, \infty[$ and for any $x, y \in [0, 1]$ such that $\alpha^k x, \alpha^k y \in [0, 1]$ the identity

$$G(\alpha x, \alpha y) = \alpha^k G(x, y)$$

holds.

Definition 11. Let $G : [0, 1]^2 \to [0, 1]$ be a mapping and $k \in]0, \infty[$. G is k-Lipschitz if for any $x, y, z, t \in [0, 1]$ the inequality

$$|G(x, y) - G(z, t)| \leq k(|x - z| + |y - t|)$$

holds.

2.4. Copulas, semicopulas and quasi-copulas

In Statistics, a copula is a way of formulating a multivariate distribution in such a way that various general types of dependence can be represented. By weakening the conditions required to a copula, we recover the concept of both semicopulas and quasi-copulas, that we present in the following.

Definition 12. A mapping $S : [0, 1]^2 \to [0, 1]$ is called a semicopula if it is nondecreasing in each coordinate and 1 is its neutral element, i.e., $S(x, 1) = S(1, x) = x$ for all $x \in [0, 1]$.

Definition 13. A quasi-copula is a semicopula Q which is also a 1-Lipschitz function.

Definition 14. A copula is a semicopula C which is 2-increasing, i.e.,

$$S(x, y) + S(x', y') - S(x', y) - S(x, y') \geq 0 \quad \text{for all } 0 \leq x \leq x' \leq 1, 0 \leq y \leq y' \leq 1.$$

Observe that, as stated previously, each copula is a quasi-copula. More generally, all copulas, semicopulas and quasi-copulas are aggregation functions.

3. Definition of overlap function and basic properties

We can now propose the following definition of *overlap function* as a particular type of bivariate aggregation function.

Definition 15. A mapping $G_S : [0, 1]^2 \to [0, 1]$ is an overlap function if it satisfies the following conditions:

$(G_S 1)$. G_S is symmetric.
$(G_S 2)$. $G_S(x, y) = 0$ if and only if $xy = 0$.

(G_S3). $G_S(x, y) = 1$ if and only if $xy = 1$.
(G_S4). G_S is nondecreasing.
(G_S5). G_S is continuous.

There are many possible examples of overlap functions. For instance, $G_S(x, y) = \min(x, y)$ or $G_S(x, y) = xy$. If we denote by $\mathcal{G}$ the set of all overlap functions, the following result is immediate.

Theorem 3. *$(\mathcal{G}, \leq_{\mathcal{G}})$ with the ordering $\leq_{\mathcal{G}}$ defined for $G_1, G_2 \in \mathcal{G}$ by*

$$G_1 \leq_{\mathcal{G}} G_2 \quad \text{if and only if} \quad G_1(x, y) \leq G_2(x, y)$$

for all $x, y \in [0, 1]$, is a lattice.

It is clear that the lattice $(\mathcal{G}, \leq_{\mathcal{G}})$ is not complete (no top neither bottom elements, for example). On the other hand, it is closed with respect to appropriate aggregation functions, as shown next.

Theorem 4. *Let $M : [0, 1] \times [0, 1] \to [0, 1]$ be a mapping. For $G_1, G_2 \in \mathcal{G}$, define the mapping $\mathcal{M}(G_1, G_2) : [0, 1] \times [0, 1] \to [0, 1]$ as*

$$\mathcal{M}(G_1, G_2)(x, y) = M(G_1(x, y), G_2(x, y)) \quad \text{for all } x, y \in [0, 1].$$

Then, $\mathcal{M}(G_1, G_2) \in \mathcal{G}$ for any $G_1, G_2 \in \mathcal{G}$ if and only if there is a continuous aggregation function $M^* : [0, 1] \times [0, 1] \to [0, 1]$ with no zero divisors and such that also its dual $(M^*)^d$ (that is, the mapping $(M^*)^d(x, y) = 1 - M^*(1 - x, 1 - y)$) has no zero divisors (i.e., if $M^*(x, y) = 1$ then necessarily either $x = 1$ or $y = 1$) so that $M|_E = M^*|_E$, where $E =]0, 1[^2 \cup \{(0, 0), (1, 1)\}$.

Proof. The sufficiency is obvious. To see the necessity, observe first that for any $G_1, G_2 \in \mathcal{G}$ and for any $(x, y) \in [0, 1]^2$, $G_1(x, y) = 0$ if an only if $G_2(x, y) = 0$ (if and only if $\min(x, y) = 0$), and thus necessarily $M(0, 0) = 0$. Similarly, we can show that $M(1, 1) = 1$ (observe that if $G_S(x, y) = 1$ for some $G_S \in \mathcal{G}$, then necessarily $x = y = 1$, i.e., for any $H_S \in \mathcal{G}$ we have $H(x, y) = 1$). Moreover, the values of M on $[0, 1]^2 \setminus \{]0, 1]^2 \cup (0, 0)\}$ are never applied when aggregating two overlap functions and thus they are irrelevant.

Now we show the nondecreasingness of M on $]0, 1[^2$. Suppose on the contrary that there exists $x, y, z, t \in]0, 1[$ such that $x \leq z, y \leq t$ but $M(x, y) > M(z, t)$. Then there exist $G_1, G_2 \in \mathcal{G}$, and $a, b, c, d \in]0, 1[$, $a \leq b$, $c \leq d$, so that $G_1(a, b) = x$, $G_2(a, b) = y$, $G_1(c, d) = z$ and $G_2(c, d) = t$. Then the property (G_S4) of $M(G1, G2)$ contradicts the inequality

$$\begin{aligned} M(x, y) &= M(G_1(a, b), G_2(a, b)) = \mathcal{M}(G_1, G_2)(a, b) > \mathcal{M}(G_1, G_2)(c, d) \\ &= M(G_1(c, d), G_2(c, d)) = M(z, t). \end{aligned}$$

As for any $x, y \in]0, 1[$, $M(x, y) \in]0, 1[$, nondecreasingness in E follows. Similarly, the continuity of M on E can be shown, and we can take as M^* the unique continuous aggregation function coinciding with M on E. ■

Remark. Hence, we shall assume from now on that M is continuous (remind also the arguments in [11] claiming that standard continuity may be too weak for some practical applications.) Notice also that, under this assumption,

(1) The lattice $\mathcal{G}$ is closed under the product operator.
(1) The lattice $\mathcal{G}$ is not closed under the Łukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$ For instance,

$$M(x, y) = T_L(\min(x, y), \min(x, y))$$

has zero divisors (take $x = 1/4$ and $y = 1$), thus clearly is not an overlap function.

As an important particular case we have the following result.

Corollary 1. *Let $G_1, \dots, G_m$ be overlap functions and $w_1, \dots, w_m$ be nonnegative weights with $\sum w_i = 1$. Then the convex sum $G = \sum w_i G_i$ is also an overlap function.*

In the following proposition we present some general relations between the properties required to an overlap function and some other analytical properties to be considered later.

Proposition 1. *Let $G : [0, 1]^2 \to [0, 1]$ be a bivariate mapping.*

(i) *If G satisfies the property (G_S2), then it does not satisfy the self-duality property, that is, the identity*

$$G(x, y) = 1 - G(1 - x, 1 - y)$$

does not hold.
(ii) *If G is migrative, then G satisfies (G_S1).*
(iii) *If G is migrative and $G_S(0, 0) = 0$, then G satisfies (G_S2).*
(iv) *If G is homogeneous of order $k > 0$, then $G(0, 0) = 0$.*
(v) *If G is homogeneous of order 1 and $G(1, 1) = 1$, then G is idempotent.*

(vi) *If G is migrative and idempotent, then G is also homogeneous of order* 1.
(vii) *If G is migrative and it has 1 as neutral element (i.e.,* $G(x, 1) = G(1, x) = x$ *for all* $x \in [0, 1]$*), then it is also homogeneous of order* 2.

Moreover, none of the converse implications is true.

Proof. (i) If G_S satisfies (G_S2) and the self-duality property, for all $x, y \in [0, 1]$ we have $G_S(x, y) = 1 - G_S(1 - x, 1 - y)$. In particular, if $x = 0$ and $y = 1$, then $0 = G_S(0, 1) = 1 - G_S(1, 0) = 1$ which is impossible. So G_S does not satisfy the self-duality property.

(ii) By Lemma 1 we have $G(x, y) = G(1, xy) = G(1, yx) = G(y, x)$.

(iii) (Necessity) $G(x, y) = 0 = G(0, 0) = G(0, y) = G(x, 0)$ for all $x, y \in [0, 1]$. (Sufficiency) If $xy = 0$, let's suppose that $y = 0$; then $G(x, 0) = G(x, 0 \cdot 0) = G(0, 0) = 0$

(iv) $G(0, 0) = G(\alpha \cdot 0, \alpha \cdot 0) = \alpha G(0, 0)$ for all $\alpha \in [0, 1[$. Therefore $(1 - \alpha)G(0, 0) = 0$, then $G(0, 0) = 0$.

(v) $G(x, x) = G(x \cdot 1, x \cdot 1) = xG(1, 1) = x$.

(vi) By Lemma 2 we have that there exists $g : [0, 1] \to [0, 1]$ such that $G(x, y) = g(xy)$. Besides G is idempotent, therefore $x = G(x, x) = g(x^2)$; that is, $g(x) = \sqrt{x}$. Therefore $G(\alpha x, \alpha y) = \alpha\sqrt{xy} = \alpha G(x, y)$.

(vii) By Lemma 1 we have that $G(x, y) = G(1, xy) = xy$. Therefore $G(\alpha x, \alpha y) = \alpha^2 xy = \alpha^2 G(x, y)$. ■

3.1. Characterization of overlap functions

Next result represents an alternative characterization of overlap functions.

Theorem 5. *The mapping* $G_S : [0, 1]^2 \to [0, 1]$ *is an overlap function if and only if*

$$G_S(x, y) = \frac{f(x, y)}{f(x, y) + h(x, y)}$$

for some $f, h : [0, 1]^2 \to [0, 1]$ *such that*

(1) *f and h are symmetric;*
(2) *f is nondecreasing and h is nonincreasing;*
(3) $f(x, y) = 0$ *if and only if* $xy = 0$*;*
(4) $h(x, y) = 0$ *if and only if* $xy = 1$*;*
(5) *f and h are continuous functions;*

Proof. Please note that $f(x, y) + h(x, y) \neq 0$, for all $(x, y) \in [0, 1]^2$. Then necessity is immediate just by taking $f(x, y) = G_S(x, y)$ and $h(x, y) = 1 - G_S(x, y)$.

Let us then prove sufficiency: since (G_S1), (G_S2), (G_S3) and (G_S5) are direct, let us simply check (G_S4). If $x_1 \leq x_2$ we have $f(x_1, y) \leq f(x_2, y)$ and $h(x_2, y) \leq h(x_1, y)$. Therefore,

$$f(x_1, y)h(x_2, y) \leq f(x_2, y)h(x_1, y)$$

in such a way that

$$f(x_1, y)f(x_2, y) + f(x_1, y)h(x_2, y) \leq f(x_1, y)f(x_2, y) + f(x_2, y)h(x_1, y).$$

Hence,

$$G_S(x_1, y) = \frac{f(x_1, y)}{f(x_1, y) + h(x_1, y)} \leq \frac{f(x_2, y)}{f(x_2, y) + h(x_2, y)} = G_S(x_2, y). \quad ■$$

Example 1. Take $f(x, y) = \sqrt{xy}$ and $h(x, y) = \max(1 - x, 1 - y)$, then we have that by the construction given in Theorem 5 we get an overlap function

$$G_S(x, y) = \frac{\sqrt{xy}}{\sqrt{xy} + \max(1 - x, 1 - y)}.$$

Corollary 2. *Under the conditions of Theorem 5 the following items hold:*

(i) $G_S(x, x) = x$ *for some* $x \in (0, 1)$ *if and only if*

$$f(x, x) = \frac{x}{1 - x} h(x, x).$$

(ii) G_S *is migrative if and only if* $f(\alpha x, y)h(x, \alpha y) = f(x, \alpha y)h(\alpha x, y)$ *for all* $\alpha, x, y \in [0, 1]$.
(iii) *If f and h are migrative, then* G_S *is migrative.*

(iv) *The function h cannot be homogeneous of any order.*
(v) *If f is homogeneous of order k, then G_S is homogeneous of the same order k if and only if $f + h$ is constant, i.e., $G_S(x, y) = \frac{f(x,y)}{h(0,0)}$.*

Proof. Items (i), (ii) and (iii) are direct. About (iv), if h is homogeneous of order one, for instance, then, taking, $x \neq 1$ we have $h(x, x) = xh(1, 1) = x \cdot 0 = 0$, so the condition $h(x, y) = 0$ if and only if $xy = 1$ is not fulfilled.

(v) (Necessity) By hypothesis $G_S(\alpha x, \alpha y) = \alpha^n G_S(x, y)$ and $f(\alpha x, \alpha y) = \alpha^n f(x, y)$. Therefore

$$\frac{f(\alpha x, \alpha y)}{f(\alpha x, \alpha y) + h(\alpha x, \alpha y)} = \frac{\alpha^n f(x, y)}{\alpha^n f(x, y) + h(\alpha x, \alpha y)} = \frac{\alpha^n f(x, y)}{f(x, y) + h(x, y)}.$$

If $\alpha^n f(x, y) = 0$, it is clear. In other case:

$$f(x, y) + h(x, y) = \alpha^n f(x, y) + h(\alpha x, \alpha y)$$

therefore

$$f(x, y) = \frac{h(\alpha x, \alpha y) - h(x, y)}{1 - \alpha^n}$$

for all $\alpha \in [0, 1[$. In particular $\alpha = 0$, then $f(x, y) + h(x, y) = h(0, 0)$ for all $x, y \in [0, 1]$, therefore $G_S(x, y) = \frac{f(x,y)}{h(0,0)}$.

(Sufficiency) Direct. ∎

Theorem 5 allows the definition of interesting families of overlap functions.

Corollary 3. *Let f and h be two functions in the setting of the previous theorem. Then, for $k_1, k_2 \in]0, \infty[$, the mappings*

$$G_S^{k_1,k_2}(x, y) = \frac{f^{k_1}(x, y)}{f^{k_1}(x, y) + h^{k_2}(x, y)}$$

define a parametric family of overlap functions.

Corollary 4. *In the same setting of Theorem 5, let us assume that G_S can be expressed in two different ways:*

$$G_S(x, y) = \frac{f_1(x, y)}{f_1(x, y) + h_1(x, y)} = \frac{f_2(x, y)}{f_2(x, y) + h_2(x, y)}$$

for any $x, y \in [0, 1]$ and let M be a bivariate continuous aggregation function that is homogeneous of order one. Then, if we define $f(x, y) = M(f_1(x, y), f_2(x, y))$ and $h(x, y) = M(h_1(x, y), h_2(x, y))$ it also holds that

$$G_S(x, y) = \frac{f(x, y)}{f(x, y) + h(x, y)}.$$

Proof. First observe that $f_i = h_i \frac{G_S}{1-G_S}$ for $i = 1, 2$. By the homogeneity condition on M, also $f = h\frac{G_S}{1-G_S}$ and the result follows. ∎

Example 2. If $f(x, y) = \sqrt{xy}$ and $h(x, y) = 1 - xy$, then both f and h are migrative, and thus due to Corollary 2(ii), the function G_S constructed by means of Theorem 5 and given by

$$G_S(x, y) = \frac{\sqrt{xy}}{\sqrt{xy} + 1 - xy}$$

is a migrative overlap function.

Item (v) in Corollary 2 leads us to study overlap functions G_S obtained from Theorem 5 taking f and h such that $f(x, y) + h(x, y) = 1$ for all $x, y \in [0, 1]$ (thus necessarily $f(1, 1) = 1$, and hence f is a continuous symmetric aggregation function with annihilator).

4. Specific cases

Among possible candidates for overlap functions, no uninorm can be accepted, since we already know that no uninorm is continuous in the whole unit square. Moreover, notice that the only possible neutral element for an overlap function G_S is $e = 1$; that is, if there exists $e \in [0, 1]$ such that $G_S(e, x) = G_S(x, e) = x$ for all $x \in [0, 1]$, then $e = 1$. This follows from the identity $G_S(1, e) = 1$ (which holds if e is a neutral element for G_S) and (G_S3) in the definition of overlap functions. These considerations suggest us to consider the relationship between overlap functions and specific instances of aggregation functions, such as t-norms, copulas (quasi-copulas, semicopulas), means, etc.

4.1. Specific case: t-norms

As the first result of this section, we show that any associative overlap function is indeed a t-norm.

Theorem 6. *Let G_S be and associative overlap function. Then G_S is a t-norm.*

Proof. We have to show only that 1 is the neutral element of G_S. From the continuity of G_S and the fact that $G_S(0, 1) = 0$, $G_S(1, 1) = 1$ it follows that for any $x \in]0, 1[$ there exists $y \in]0, 1[$ such that $x = G_S(y, 1)$. However, then $G(x, 1) = G_S(G_S(y, 1), 1) = G_S(y, G_S(1, 1)) = G_S(y, 1) = x$, and similarly $G_S(1, x) = x$. ∎

On the other hand, by definition t-norms fulfill (G_S1) and (G_S4). They also satisfy (G_S3), the easy proof being: $1 = T(x, y) \le \min(x, y)$, then $x = y = 1$. The reciprocal is direct, just taking into account that the neutral element of T is the one. As for the property (G_S2) we must say that when we work with t-norms, the necessary condition of double implication (of (G_S2)) coincides with the definition of *positive* t-norm, (please see page 4 in [22]). All these considerations, together with Theorem 2, allow us to get the following classification theorem.

Theorem 7. *If a t-norm T is an overlap function, then T belongs to one of the following three types:*

(1) *$T = T_M$;*
(2) *T is strict;*
(3) *T is the ordinal sum of the family $\{([a_m, b_m], T_m)\}$, with all the T_m continuous Archimedean and such that if for some m_0 $a_{m_0} = 0$, then necessarily T_{m_0} is a strict t-norm.*

Proof. We have assumed by definition that any overlap function satisfies continuity, due to condition (G_S5). By hypothesis T is a t-norm which is also an overlap function, so it is continuous. By the classification of continuous t-norms given in pg. 11 of [16] we know that for a continuous t-norm T there are three possibilities:

1. $T = T_M$;
2. T is Archimedean;
3. There exists a family $\{([a_m, b_m], T_m)\}$ such that T_m is the ordinal sum of this family in the sense of [16].

Since by hypothesis T is an overlap function, it satisfies (G_S5) and (G_S2). If T is Archimedean, we have that T is strict.
On the other hand, suppose now that T is the ordinal sum of the family $\{([a_m, b_m], T_m)\}$; that is:

$$T(x, y) = \begin{cases} a_m + (b_m - a_m)T_m\left(\dfrac{x - a_m}{b_m - a_m}, \dfrac{y - a_m}{b_m - a_m}\right) & \text{if } (x, y) \in [a_m, b_m]^2 \\ \min(x, y) & \text{otherwise}. \end{cases}$$

We know that for any t-norm the property: if $xy = 0$, then $T(x, y) = 0$ holds. Moreover, by hypothesis the previous t-norm also satisfies the reciprocal, since it is an overlap function. Then, if $T(x, y) = 0$ there are two possibilities:
(a) (x, y) does not belong to any $[a_m, b_m]^2$, so $T(x, y) = \min(x, y)$ for that (x, y).
(b) The point (x, y) belongs to $[a_m, b_m]^2$. As $T(x, y) = 0 = a_m + (b_m - a_m)T_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m})$, then $a_m = 0$ and $b_m \neq 0$, since, on the contrary, the considered interval would be $[0, 0]$ and $x = y = 0$. we know that T fulfills (G_S2) and moreover we also know that if $T(x, y) = 0$ then $T_m(\frac{x}{b_m}, \frac{y}{b_m}) = 0$, hence T_m also satisfies (G_S2). Therefore we have that the continuous, Archimedean t-norm T_m associated to the interval $[0, b_m]$ also fulfills (G_S2), so it is strict. ∎

Example 3. (1) In the construction of the following overlap function we use item (3) of Theorem 7 taking as t-norm the product, (which is strict, continuous and Archimedean), for the corresponding interval $[0, 0.5]$.

$$G_S(x, y) = \begin{cases} 2xy & \text{if } (x, y) \in [0, 0.5]^2 \\ \min(x, y) & \text{otherwise}. \end{cases}$$

(2) In the construction of the following overlap function we take the product and the Łukasiewicz t-norms (see pg 84 of [14]). Nevertheless, in this overlap function we do not consider any interval of the type $[0, b_m]$.

$$G_S(x, y) = \begin{cases} 0.1 + 2.5(x - 0.1)(y - 0.1) & \text{if } (x, y) \in [0.1, 0.5]^2 \\ 0.7 + \max(x + y - 1.6, 0) & \text{if } (x, y) \in [0.7, 0.9]^2 \\ \min(x, y) & \text{otherwise}. \end{cases}$$

(3) The following t-norm satisfies all the properties required to overlap functions, except (G_S2). This is due to the fact that in $[0, 0.25]^2$ we consider the Lukasiewicz t-norm which is continuous and Archimedean but not strict.

$$T(x, y) = \begin{cases} \max(x + y - 0.25, 0) & \text{if } (x, y) \in [0, 0.25]^2 \\ \min(x, y) & \text{otherwise}. \end{cases}$$

4.2. Specific cases: Semicopulas, quasi-copulas and copulas

As a first result we have the following.

Proposition 2. *Let S be a symmetric semicopula. Then S is an overlap function if and only if S is continuous and has not zero divisors.*

Proof. Necessity is direct, and for sufficiency it is enough to note that if $S(x, y) = 1$, monotony implies that $x = S(x, 1) = 1$ and $y = S(1, y) = 1$. ■

Corollary 5. *Let Q be a symmetric quasi-copula without zero divisors. Then Q is also an overlap function.*

Proof. Just observe that the Lipschitzianity implies the continuity of Q. ■

In general, copulas and quasi-copulas can have zero divisors. This is the case of the copula $\max(0, x+y-1)$, which is the smallest of the copulas with respect to the pointwise ordering. On the other hand, just by assuming appropriate conditions, we can recover copulas from overlap functions.

Theorem 8. *Let G_S be an overlap function being homogeneous of order $k+1$, with $k \in [0, 1]$. Suppose that there exists $e \in [0, 1]$ such that $G_S(x, e) = G_S(e, x) = x$ for all $x \in [0, 1]$, that is, that G_S has a neutral element e. Then G_S is also a copula.*

Proof. As G_S is an overlap function homogeneous of order $k + 1$, as stated at the beginning of Section 4, the neutral element e of G_S must be equal to 1. As G_S is symmetric and homogeneous, the identity

$$G_S(x, y) = G_S\left(\max(x, y), \frac{\min(x, y)\max(x, y)}{\max(x, y)}\right) = \max(x, y)^k \min(x, y) = \min(x^k y, xy^k)$$

holds. ■

Notice that the family $(\min(x^k y, xy^k))$ for $k \in [0, 1]$ is the so called Cuadras–Augé family of copulas [23].

5. Migrativity, homogeneity and k-Lipschitzianity of overlap functions

In this section we study under which conditions overlap functions are migrative, homogeneous or k-Lipschitz. Observe that these three properties can be seen as analytical properties that are stronger than continuity.

5.1. Migrative overlap functions

In this section we present a characterization theorem of migrative overlap functions. We also analyze the characterization of such migrative functions by means of automorphisms. The main consequence of this characterization is that it allows us to relate migrative overlap functions to strong negations and implication operators.

Clearly some overlap functions are not migrative (for instance, those in Example 1). There are also overlap functions which are homogeneous of order 1 or 2 but which are not migrative. In the following results we prove that there exist overlap functions which are migrative and homogeneous of order one or two.

Proposition 3. *Let $G : [0, 1]^2 \to [0, 1]$. If G is migrative, then*

$$G(x, 1) = G(\sqrt{x}, \sqrt{x}).$$

Proof. By Lemmas 1 and 2 there exists a mapping $g : [0, 1] \to [0, 1]$ such that $G(x, 1) = g(x) = g(\sqrt{x}\sqrt{x}) = G(\sqrt{x}, \sqrt{x})$. ■

Theorem 9. *A mapping $G_S : [0, 1] \to [0, 1]$ is an overlap function satisfying (G_S7) if and only if there exists a nondecreasing function $g : [0, 1] \to [0, 1]$ satisfying $g^{-1}(]0, 1[) =]0, 1[$ such that*

$$G_S(x, y) = g(xy).$$

Proof (*Necessity*). Since G_S satisfies (G_S7), then by Lemma 2 we know that there exists a function $g : [0, 1] \to [0, 1]$ such that $G_S(x, y) = g(xy)$ for all $x, y \in [0, 1]$. Besides G_S is an overlap function, hence it satisfies (G_S4) and (G_S5), therefore g is not decreasing and continuous. Besides G_S satisfies (G_S2) and (G_S3), so:

$$g(x) = g(x \cdot 1) = G_S(x, 1) = 0 \quad \text{if and only if } x = 0$$
$$g(x) = g(x \cdot 1) = G_S(x, 1) = 1 \quad \text{if and only if } x = 1.$$

(Sufficiency) By Lemma 2 we have that $G_S(x, y) = g(x, y)$ satisfies (G_S7). Taking into account Proposition 1 G_S satisfies (G_S1). On the other hand:

$$G_S(x, y) = 0 = g(xy) \quad \text{if and only if } xy = 0$$
$$G_S(x, y) = 1 = g(xy) \quad \text{if and only if } xy = 1.$$

Clearly, G_S satisfies (G_S4) and (G_S5) since g is not decreasing and continuous. ■

Example 4.

$$G_S(x,y)=\begin{cases}xy & \text{if } xy\le 0.6\\ 0.6 & \text{if } 0.6\le xy\le 0.8\\ 2xy-1 & \text{if } xy\ge 0.8.\end{cases}$$

Proof. Direct. ■

5.2. Homogeneous overlap functions

This subsection focusses on homogeneous overlap functions. First of all, from the general characterization result on homogeneous functions, we have the following result.

Theorem 10. *G_S is an homogeneous overlap function of order $k>0$ if and only if there exists a continuous mapping*

$$\psi:[0,\pi/2]\to[0,1]$$

with $\psi(0)=0$, $\psi(\frac{\pi}{4})=2^{-\frac{k}{2}}$, ψ nondecreasing in $[0,\pi/4]$ and $\psi(\theta)=\psi(\frac{\pi}{2}-\theta)$ for any $\theta\in[0,\pi/4]$ such that

$$G_S(x,y)=(x^2+y^2)^{\frac{k}{2}}\psi\left(\arctan\left(\frac{\min(x,y)}{\max(x,y)}\right)\right)$$

for any $x,y\in[0,1]$.

Proof. If G_S is homogeneous of order $k>0$, then, for any $x,y\in[0,1]$ we have

$$G_S(x,y)=(x^2+y^2)^{\frac{k}{2}}G_S\left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}},\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\right)$$

and hence, as we are reduced to consider values on the unit sphere, the result follows.

The converse is obvious. ■

Proposition 4. *Let G_1 and G_2 be overlap functions which are homogeneous of order k_1 and k_2 respectively. Then*

(i) *$G_1G_2(x,y)=G_1(x,y)G_2(x,y)$ is an overlap function homogeneous of order k_1+k_2;*
(ii) *$G_1^{\frac{1}{k_1}}(x,y)=(G_1(x,y))^{\frac{1}{k_1}}$ is an overlap function homogeneous of order 1.*

Proof. Direct. ■

Proposition 5. *Let M be a continuous aggregation function homogeneous of order $k>0$. If G_1 and G_2 are two overlap functions homogeneous of the same order l, then $\mathcal{M}(G_1,G_2)$ is also an overlap function homogeneous of order kl.*

Proof. First of all observe that, if $M(x,y)=0$, as M is increasing and homogeneous of order k

$$0=M(x,y)\ge M(\min(x,y),\min(x,y))=\min(x,y)^kM(1,1)=\min(x,y)^k,$$

so M does not have divisors of zero. In the same way, if $M(x,y)=1$, then $\max(x,y)=1$. So, by Theorem 4, $\mathcal{M}(G_1,G_2)$ is an overlap function whatever the overlap functions G_1 and G_2 are. The homogeneity of $\mathcal{M}(G_1,G_2)$ follows from an easy calculation. ■

Example 5. Let $G_S:[0,1]^2\to[0,1]$ be an overlap function which is homogeneous of order k. Then, the following items hold:

(i) $G_{S1}(x,y)=\frac{xG_S(x,y)+yG_S(x,y)}{2}$ is an overlap function homogeneous of order $k+1$ such that $G_{S1}(x,x)=x^{k+1}$ for all $x\in[0,1]$.
(ii) $G_{S2}(x,y)=\left(\frac{xG_S(x,y)+yG_S(x,y)}{2}\right)^{\frac{1}{k+1}}$ is an idempotent overlap function homogeneous of order 1.

Observe that G_{S1} and G_{S2} are not migrative. Obviously $G_{S1}(\alpha x,y)\neq G_{S1}(x,\alpha y)$ for all $\alpha\ge 0$. The same can be seen for G_{S2} in a similar way.

We can identify homogeneous migrative overlap functions as follows.

Theorem 11. *The only migrative homogeneous of order $k>0$ function $G:[0,1]^2\to[0,1]$ such that $G(1,1)=1$ is $G(x,y)=(xy)^{\frac{k}{2}}$.*

Proof. On the one hand, $G(\alpha x,\alpha y)=\alpha^kG(x,y)=\alpha^kg(xy)$. On the other hand, $G(\alpha x,\alpha y)=g(\alpha^2xy)$. Hence, $\alpha^kg(xy)=g(\alpha^2xy)$ for all $\alpha\ge 0$, $x,y\in[0,1]$, $\alpha^2xy\in[0,1]$. In particular, taking $x=y=1$ we have that $g(\alpha^2)=\alpha^k$, i.e., $g(\alpha)=\alpha^{\frac{k}{2}}$ for all $\alpha\in[0,1]$. ■

Corollary 6. *The only migrative homogeneous of order* 1 *function* $G_S : [0, 1] \to [0, 1]$ *such that* $G_S(1, 1) = 1$ *is the geometric mean.*

Corollary 7. *The only migrative homogeneous of order* 2 *function* $G : [0, 1]^2 \to [0, 1]$ *such that* $G(1, 1) = 1$ *is the product.*

Proof. On the one hand, $G(\alpha x, \alpha y) = \alpha^2 G(x, y) = \alpha^2 g(xy)$. On the other hand, $G(\alpha x, \alpha y) = g(\alpha^2 xy)$. Hence, $\alpha^2 g(xy) = g(\alpha^2 xy)$ for all $\alpha \geq 0, x, y \in [0, 1], \alpha^2 xy \in [0, 1]$. In particular, taking $x = y = 1$ we have that $g(\alpha^2) = \alpha^2$, i.e., $g(\alpha) = \alpha$ for all $\alpha \in [0, 1]$. ■

Corollary 8. *The only overlap function* $G_S : [0, 1] \to [0, 1]$ *that satisfies* $(G_S 7)$ *and* $(G_S 9)$ *is the product.*

Proof. Similar to the proof of Corollary 6. ■

5.3. *k-Lipschitz overlap functions*

First of all we write the definition of k-Lipschitzianity for overlap functions.

Definition 16. Let $k \geq 1$. An overlap function G_S is k-Lipschitz if for any $x, y, z, t \in [0, 1]$ it holds

$$|G_S(x, y) - G_S(z, t)| \leq k(|x - z| + |y - t|). \tag{1}$$

This is the usual definition of k-Lipschitz functions, and it is valid for any function, allowing any value of k greater than zero. But, in the case of overlap functions, just by taking $x = y = z = 1$ and $t = 0$ the restriction to $k \geq 1$ becomes justified.

The set of k-Lipschitz overlap functions with respect to the ordering $\leq_{\mathcal{G}}$ is bounded, as the next result shows.

Theorem 12. *Let* $k \geq 1$. *Then the supremum of the set of* k*-Lipschitz overlap functions is given by the mapping* $\min(kx, ky, 1)$, *whereas the infimum is given by* $\max(kx + ky - 2k + 1, 0)$. *That is, for any* k*-Lipschitz overlap function* G_S *the inequality*

$$\max(kx + ky - 2k + 1, 0) \leq G_S(x, y) \leq \min(kx, ky, 1)$$

holds for all $x, y \in [0, 1]$.

Proof. Suppose that $G_S(x, y) > \min(kx, ky, 1)$ for some $x, y \in [0, 1]$. Since $G_S(x, y) \leq 1$, this means that $\min(kx, ky, 1) = kx$ or $\min(kx, ky, 1) = ky$. In the first case, $y = t = 1$ and $z = 0$ in Eq. (1), we arrive at

$$kx < G_S(x, 1) \leq kx$$

which is a contradiction. The second case is analogous. On the other hand, by defining for $\epsilon > 0$ the mappings

$$\max(xy, (1 - \epsilon))(\min(kx, ky, 1))$$

we get a sequence of overlap functions which converges uniformly to $\min(kx, ky, 1)$ as $\epsilon \to 0$. The proof for the lower bound is similar.

The mapping $\max(kx + ky - 2k + 1, 0)$ is never an overlap function. On the contrary, although in general, the mapping $\min(kx, ky, 1)$ for $k > 1$ and $x, y \in [0, 1]$ such that $kx, ky \in [0, 1]$ does not define an overlap function (since by taking $x = y = \frac{1}{k}$ we see that it does not fulfill condition $(G_S 3)$), $\min(x, y)$ is an overlap function, so we have the following corollary. ■

Corollary 9. *The mapping* $\min(x, y)$ *is the strongest* 1*-Lipschitz overlap function, in the sense that for any other* 1*-Lipschitz overlap function* G_S *the inequality*

$$G_S(x, y) \leq \min(x, y)$$

holds for any $x, y \in [0, 1]$.

For associative k-Lipschitz overlap functions we have the next result which can be derived from [24,25].

Theorem 13. *If* G_S *is an associative* k*-Lipschitz overlap function, then* G_S *is a* t*-norm of the form given in Theorem 7, where each involved strict* t*-norm* T *(see item* (2) *or item* (3)*) has a* k*-convex additive generator* t*, i.e.,*

$$t(y + k\epsilon) - t(y) \leq t(x + \epsilon) - t(x)$$

for all $0 \leq y \leq x < 1$ *and* $\epsilon \in]0, \min(1 - x, (1 - y)/k)[$.

In Table 1 we present a summary of the properties fulfilled by some instances of t-norms.

6. Conclusions and future research

In this paper we have proposed the concept of overlap function, followed by a first mathematical study of the new concept. In particular, we have studied the relationship with other well-known aggregation functions, as t-norms, copulas, semicopulas and quasi-copulas. Moreover, keeping in mind potential applications, we have studied behavior under specific properties like migrativity and homogeneity, which should be of interest for image processing, for instance.

Table 1
Some functions and properties.

Expression	t-norm T_M	t-norm T_P	Geometric mean
	$\min(x, y)$	xy	$\sqrt{xy}$
G_S1	Yes	Yes	Yes
G_S2	Yes	Yes	Yes
G_S3	Yes	Yes	Yes
G_S4	Yes	Yes	Yes
G_S5	Yes	Yes	Yes
Migrativity	No	Yes	Yes
Homogeneity of order 1	Yes	No	Yes
Homogeneity of order 2	No	Yes	No
Lipschitzianity	Yes	Yes	No

Obviously, this paper can be continued in several ways, but some of them seem to us of immediate interest. On the one hand, we can search for alternative characterizations for overlap functions, specifically designed to certain applications. And on the other hand, we can explore additional properties that may hold in specific contexts. Anyway, the most relevant project should be to acknowledge some key modeling implications by extending the concept, from the present classification framework into a more general knowledge management context. In this sense, and based upon the concept of overlap function, we not only expect to develop useful tools to handle situations where the expert shows lack of information, but we expect also to revise the management of complex concepts, which are quite often learnt and their details precised by means of a sometimes long sequence of quite similar (i.e., overlapped) descriptions.

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