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Brief paper

Numerical method for the solution of the regulator equation with application to nonlinear tracking ☆

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Abstract

A numerical method to solve the so-called regulator equation is presented here. This equation consists of partial differential equations combined with algebraic ones and arises when solving the output-regulation problem. Solving the regulator equation is becoming difficult especially for the nonminimum phase systems where reducing variables against algebraic part leads to a potentially unsolvable differential part. The proposed numerical method is based on the successive approximation of the differential part of the regulator equation by the finite-element method while trying to minimize a functional expressing the error of its algebraical part. The method is analyzed to obtain theoretical estimates of its convergence and it is tested on an example of the "two-carts with an inverted pendulum" system. Simulations are included to illustrate the suggested approach.

Keywords: Nonlinear output regulation; Finite-element method; Optimization

1. Introduction

Tracking a given reference while rejecting an unknown disturbance belongs to the most prominent problems in control theory and applications. If the reference to be followed and/or disturbance to be rejected are generated by a finite-dimensional autonomous exogenous system, the corresponding framework is typically referred as the **output-regulation problem (ORP)**, first introduced in Isidori and Byrnes (1990) and Huang and Rugh (1990) and nicely summarized in the recent monograph (Huang, 2004) or the earlier tutorial paper (Byrnes & Isidori, 2000). Common feature of various modifications of ORP is that they require solving the so-called regulator equation (RE) being the set of partial differential equations combined with the algebraic ones. Geometrically, the solution of RE gives an error-zeroing manifold together with an open loop control making it invariant. Solvability of the RE was first characterized via center manifold theory (Hepburn & Wonham, 1981). Such an approach requires hyperbolic zero dynamics. The more interesting nonhyperbolic case, when RE solvability is not guaranteed, was first analyzed in Huang (1995). While all these results characterized solvability of various types of ORP in terms of RE solvability, only few of them propose methods for solution of RE as well (Čelikovský & Rehák, 2004; Huang, 2000, 2001, 2003; Rehák & Čelikovský, 2004). Solving the RE is becoming even more important for the nonminimum phase systems where its solution directly enters feedback compensator. Summarizing, the main challenge for RE solving is the **nonhyperbolic and nonminimum phase case**. The method for computing approximate solution of RE analytically via undetermined coefficients of Taylor expansion of solutions is developed in Huang (2000, 2001, 2003). Its disadvantage is given by the fact that for each plant one has to make quite nonstandard computations that are difficult to be implemented as a computer algorithm. As a certain counterpart, papers (J. Wang & J. Huang, 2001; D. Wang & J. Huang, 2001) develop numerical methods to solve the regulator equation based on its neural network approximation, including its parameters' optimization and error analysis.

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The goal of the present paper is to use more classical tools and adapt existing numerical algorithms based on finite-element method (FEM). The most PDE solvers are designed to obtain solutions of pure PDE's and cannot directly handle the algebraic part of RE. At the same time, for nonminimum phase systems it is not possible to reduce RE to pure PDE, since such an approach would lead to an unsolvable partial differential equation. The approach suggested in this paper therefore replaces the algebraic equation by a certain penalty functional which is then optimized on solutions of PDE part of RE. In such a way, solution of rather nonstandard RE is replaced by solving series of standard PDE's for which PDE solvers are designed. It will be shown that under reasonable technical assumptions the minimum of that penalty functional exists and the method converges to it. At the same time, this minimum will be shown to determine directly the ultimate bound on error of output regulation.

The paper is organized as follows. To explain the above ideas precisely, the simplest full information ORP is introduced in the next section for the single input single output system. Section 3 describes proposed algorithm in detail, including the analysis of regulation error depending on the penalty minimum, while Section 4 gives the conditions for the existence of the penalty minimum and method convergence to it. Section 5 illustrates our results on the application being the two-carts with the inverted pendulum system. Comparison between the Taylor expansion method and our new method is given in Section 6 together with further discussion of some numerical aspects. Conclusions are drawn in the final section while some necessary technical propositions and proofs are collected in the Appendix.

2. Output-regulation problem

Consider the following single input single output plant

$$\dot{x} = f(x) + g(x)u + p(x)w, \qquad y = h(x), \qquad f(0) = 0, \qquad h(0) = 0, \qquad g(0) \neq 0, \tag{1}$$

where $x \in R^n$ and $y \in R$ denote the state of the plant and its output to be controlled, respectively, while $u \in R$ stands for the controlled input and $w \in R^q$ for the disturbance input. The vector fields $f : R^n \to R^n$, $g : R^n \to R^n$, $p : R^n \to R^q$ and the function $h : R^n \to R$ are supposed to be sufficiently smooth. The standard application goal is the output y of the plant tracking asymptotically a given reference while rejecting the disturbance. Specific feature of the output-regulation problem (ORP) is that both the reference and the disturbance are generated by an autonomous system called in the sequel as the exogeneous one. More specifically, consider the following exogeneous system

$$\dot{w} = s(w), \qquad w \in R^q, \qquad s(0) = 0, \tag{2}$$

then the reference signal to be tracked is given as $\bar{q}(w)$ where $\bar{q} : R^q \to R$ is a smooth function such that $\bar{q}(0) = 0$. Without any loss of generality, the state w of the exosystem is assumed to generate the disturbance on the right-hand side of the plant (1) as well. Moreover, as in Huang (2004), the exosystem (2) is assumed to be neutrally stable.

To give the formal definition of the ORP define the tracking error as

$$e(t) = y(t) - \bar{q}(w(t)). \tag{3}$$

Definition 1. The State Feedback-Regulation Problem for (1) and (2) is said to be **locally solvable** if there exists a function $\alpha(x, w) \in C^k(R^n \times R^q, R)$, $k \geq 2$, satisfying the following conditions:

(1) $\alpha(0, 0) = 0$,
(2) the system (usually called as the "disconnected system") $\dot{x} = f(x) + g(x)\alpha(x, 0)$ is asymptotically stable in the first approximation,
(3) there exists a neighborhood $U \subset R^n \times R^q$ of the origin such that for every $(x(0), w(0)) \in U$ the solution of the Eqs. (1) and (2) with $u(t) = \alpha(x(t), w(t))$ satisfies $\lim_{t\to+\infty} e(t) = 0$, where $e(t)$ is given by (3).

Theorem 2. *Suppose (1) with $w \equiv 0$ has asymptotically stabilizable linear approximation. Then the state feedback ORP for (1) and (2) is locally solvable if and only if there exist a neighborhood W of the origin in R^q and a pair of smooth functions, denoted by $\mathbf{x}, c$ ($\mathbf{x} : W \to R^n$, $c : W \to R$) such that $\mathbf{x}(0) = 0$, $c(0) = 0$,*

$$\begin{aligned} &h(\mathbf{x}(w)) - \bar{q}(w) = 0 \\ &\frac{\partial \mathbf{x}(w)}{\partial w} s(w) = f(\mathbf{x}(w)) + g(\mathbf{x}(w))c(w) + p(\mathbf{x}(w))w \end{aligned} \tag{4}$$

are satisfied. Moreover, if the solution of (4) exists, the corresponding state feedback compensator can be taken as $u = \alpha(x, w) = c(w) + K(x - \mathbf{x}(w))$ where $u = Kx$ stabilizes the linear approximation of (1).

Proof of Theorem 2 and further facts on ORP may be found in Isidori and Byrnes (1990), Byrnes and Isidori (2000) and Huang (2004). In particular, the Eq. (4) are usually called as the **regulator equation (RE)**. As a matter of fact, the solution of the RE may be interpreted as follows. Putting in (1) $u = c(w)$ and choosing for (1) and (2) any initial conditions $x(0) = x_0$, $w(0) = w_0$ with $x_0 = \mathbf{x}(w_0)$ guarantees that $x(t) = \mathbf{x}(w(t))\ \forall t \geq 0$, and consequently $h(x(t)) - \bar{q}(w(t)) = 0$, $\forall t \geq 0$. Using the geometric terminology, the submanifold of R^{n+q} given by $x = \mathbf{x}(w)$ is the so-called error-zeroing manifold which is forward invariant when $u = c(w)$. To study the RE, the following terminology will be useful in the sequel.

Definition 3. Relative degree of the ORP is the usual relative degree (Isidori, 1995) of the so-called extended system (1) and (2) having the state $\left[x^\top, w^\top\right]^\top \in R^{n+q}$, the input u and the output $h(x) - \bar{q}(w)$. Analogously, zero dynamics of the ORP, the minimum, resp. the nonminimum phase and the hyperbolic ORP are also defined via the extended system (1) and (2).

3. Numerical solution of the regulator equation

The main contribution of this paper is to develop a numerical method for solving the RE and analyze its convergence and applicability for output-regulation design. From the PDE theory point of view, the regulator Eq. (4) appears to be rather nonstandard due to several reasons:

(1) It contains a first-order PDE. These are much less investigated than the second-order PDE's.
(2) Auxiliary boundary conditions have to be chosen as one cannot solve this PDE on the whole space R^q. Rather it must be solved on a fixed bounded domain $\Omega_b \subset R^q$, $0 \in \Omega_b$, $w(0) \in \Omega_b$.
(3) Most of existing software (like FEMLAB) cannot treat algebraic part, which must be therefore handled in a special way.

When the ORP for a SISO plant has a well-defined relative degree r, it is possible to reduce the RE (being a PDE plus one algebraic equation for $n+1$ variables) into $n-r$ pure PDE's for $n-r$ variables. Nevertheless, such a reduced PDE is guaranteed to be solvable only if the zero dynamics of the ORP is hyperbolic, (Huang, 2004).

Example 1. To illustrate the solvability issue for nonhyperbolic nonminimum phase plant, consider the plant and the exosystem ($a \in R$ is a parameter discussed later on): $\dot{x}_1 = x_2 - u$, $\dot{x}_2 = aw_1^2 - u + x_2$, $\dot{w}_1 = w_2$, $\dot{w}_2 = -w_1$ with the error $e = x_1 - w_1$. The corresponding ORP has nonhyperbolic zero dynamics $\dot{x}_2 = aw_1^2$. Its regulator equation has the form

$$\mathbf{x}_1(w_1) - w_1 = 0, \qquad \frac{\partial \mathbf{x}_1}{\partial w_1} w_2 - \frac{\partial \mathbf{x}_1}{\partial w_2} w_1 = \mathbf{x}_2 - c(w),$$

$$\frac{\partial \mathbf{x}_2}{\partial w_1} w_2 - \frac{\partial \mathbf{x}_2}{\partial w_2} w_1 = aw_1^2 - c(w) + \mathbf{x}_2(w), w = (w_1, w_2)^\top$$

and therefore $\mathbf{x}_1(w) = w_1$, $\mathbf{x}_2(w) = c(w) + w_2$, while

$$\frac{\partial \mathbf{x}_2}{\partial w_1} w_2 - \frac{\partial \mathbf{x}_2}{\partial w_2} w_1 = w_2 + aw_1^2.$$

Notice, that if $a = 0$ the latter equation has a solution $\mathbf{x}_2(w) = -w_1$, if $a \neq 0$ no solution exists.

Theorem 4. *Suppose in (4) that f has Hurwitz Jacobian at 0. Then PDE part of (4) has local solution $\mathbf{x}(w)$ for every given continuous function $c(w)$, $c : R^q \to R$, such that $c(0) = 0$. Moreover, $\mathbf{x}(0) = 0$.*

Proof. Follows by a standard application of center manifold theorem, cf. e.g. Carr (1981), Huang (2004) and Isidori (1995). Actually, for any given $c(w)$ PDE part of (4) defines center manifold for the extended system (1) and (2) with $u \equiv c(w)$ which locally exists as the Jacobian of f is Hurwitz. Moreover, first equation of (4) turns into $f(\mathbf{x}(0)) = 0$ for $w = 0$. The Jacobian of f is Hurwitz, thus it is regular. Hence one obtains $\mathbf{x}(0) = 0$, consequently $h(\mathbf{x}(0)) - \bar{q}(0) = 0$. □

Theorem 5. *Suppose there exists a bounded region $\Omega_b \ni 0$ and real numbers $\varepsilon_1 > \varepsilon_0 \geq 0$ such that for every $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ there exist sufficiently smooth mappings $c_\varepsilon(w)$, $\mathbf{x}_\varepsilon(w)$, $w \in \Omega_b$ such that*

$$J(c_\varepsilon(w)) = \int_{\Omega_b} (h(\mathbf{x}_\varepsilon(w)) - \bar{q}(w))^2 \mathrm{d}w_1 \ldots \mathrm{d}w_\mu = \varepsilon, \tag{5}$$

where $\mathbf{x}_\varepsilon(w)$ is the corresponding solution of PDE part of (4) with $c_\varepsilon(w)$, $c_\varepsilon(0) = 0$. Then there exist positive constants C, β, R such that for all $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ $|e(t)| \leq C \exp(-\beta t) + R\varepsilon$ where the error $e(t)$ is given by (1)–(3) with sufficiently small initial conditions and $u = \alpha(x, w) = c_\varepsilon(w) + K(x - \mathbf{x}_\varepsilon(w))$.

Proof. Denote by $\mathbf{x}(w)$, $c(w)$ the exact solution of the Eq. (4). The true state of the controlled system is denoted by $x(t)$. Finally, $y(t) = h(x(t))$, $y_\varepsilon(t) = h(\mathbf{x}_\varepsilon(w(t)))$ and $y_p(t) = h(\mathbf{x}(w(t)))$. (Due to the last relation in (4) the equality $y_p(t) = \bar{q}(w(t))$ holds.) We assume that the initial conditions of the system were set as $x(0) = x_0$ which may not lie on the zero-output manifold. To estimate the error $e(t) = y(t) - y_p(t)$ triangle inequality yields $|y(t) - y_p(t)| \leq |y(t) - y_\varepsilon(t)| + |y_\varepsilon(t) - y_p(t)|$. Since $|y_\varepsilon(t) - y_p(t)| = |y_\varepsilon(t) - \bar{q}(w(t))|$, Proposition 12 (see Appendix) guarantees that (5) implies existence of a constant $R > 0$ such that $|h(\mathbf{x}_\varepsilon(w(t))) - \bar{q}(w(t))| = |y_\varepsilon(t) - y_p(t)| < R\varepsilon$ for each $t > 0$. This together with Proposition 11 (see Appendix) completes the proof. □

Remark 6. Obviously, Theorem 5 gives for $\varepsilon = 0$ the well-known Theorem 2 as its particular case. Notice, that the origin of the "error" $\varepsilon > 0$ is indifferent: it can be both a consequence of inevitable numerical inaccuracy of computational algorithm for otherwise well solvable RE and the best available inaccuracy of the algebraic part of some unsolvable RE. It can be even combination of both mentioned inaccuracy aspects.

Algorithm 1. Based on the previous theorems, the following algorithm for the output regulation was designed. First, define $\hat{f}$ by $\hat{f}(x) = f(x) + g(x)Kx$, where static state feedback $u = Kx$ stabilizes the linear approximation of (1) with $w = 0$. Then, consider the equation

$$\frac{\partial \mathbf{x}(w)}{\partial w} s(w) = \hat{f}(\mathbf{x}(w)) + g(\mathbf{x}(w))\hat{c}(w) + p(\mathbf{x}(w))w, \tag{6}$$

$$0 = h(\mathbf{x}(w)) - \bar{q}(w). \tag{7}$$

Obviously, $(\mathbf{x}(w), c(w))$ is the solution of (4) if and only if $(\mathbf{x}(w), \hat{c}(w))$, $\hat{c}(w) = c(w) - K\mathbf{x}(w)$, is the solution of (6) and (7). To solve (6) and (7), the following procedure is applied:

(1) Choose the first iteration of the function $\tilde{c}_0(w)$.
(2) Suppose the ith iteration is known, say $\tilde{c}_i(w)$. Using a numerical PDE solver, solve the Eq. (6) with $\tilde{c}(w) = \tilde{c}_i(w)$ to obtain $\mathbf{x}_i(w)$. This is always possible by Theorem 4. Compute penalty (5) for $\mathbf{x}_i(w)$.
(3) If the penalty value is not satisfactory, compute the gradient of the functional (5) and based on it the next iteration $\tilde{c}_{i+1}(w)$, then go to step 2.
(4) If the penalty value is satisfactory and equal to some $\varepsilon > 0$, stop the procedure to obtain $\mathbf{x}_{\text{final}}, \tilde{c}_{\text{final}}$ solving (6) and generating penalty (5) equal to ε.

By Theorem 5, upon terminating of the above algorithm, the following inequality holds:

$$|e(t)| \leq C \exp(-\beta t) + R\varepsilon.$$

Here, $e(t)$ is given by (2) and (3) and (1) with $u = Kx + \tilde{c}_{\text{final}}(w)$.

Remark 7. As a matter of fact, other approaches to design output-regulation feedback do basically the following: (1) assume RE is somehow (**but how**?!) solved; (2) then, using gains K stabilize the corresponding error-zeroing manifold. On the contrary, our algorithm **at first** asymptotically stabilizes the approximate linearization and only afterwards tries to solve RE for a much more convenient system. Due to the well recognized peculiarity of solving RE, we find our approach more reasonable. Notice that the solution to original nonstabilized RE (4) is $\mathbf{x}(w)$, $c(w) = K\mathbf{x}(w) + \tilde{c}(w)$.

The data produced by the corresponding numerical method finding an approximation of the RE is in a numerical form. Thus some postprocessing of these data is necessary. The software package FEMLAB (used to evaluate the solution of RE in the example presented in Section 5) offers for this purpose a possibility of application of some built-in procedures. The results can be also converted into a look-up table suitable for interpolation. This allows to avoid necessity of using these predefined functions. Notice also that at each minimization step, a certain PDE is solved approximately via the finite-element method which is the source of further minor small numerical errors. Nevertheless, the crucial regulation error is imposed by the penalty minimum which is investigated in the next section.

4. Convergence to a penalty minimum

The minimum of the functional J is sought on a finite-dimensional space in practice. This is due to the fact that the function $\mathrm{x}(w)$ is computed numerically using FEM. (For details on implementation of the FEM see e.g. Ciarlet (1978).) It is applied as follows: first, the domain Ω_b is expressed as a union of triangles such that intersection of two distinct triangles is either empty, a common vertex or a common edge. This set of triangles is called the mesh, the vertices (denoted by $w_k,\ k = 1,\ldots,N$) are called the nodes.

The functions $\varphi_{i,j}, i = 1,\ldots N, j = 1,\ldots,n$ can be defined by $\varphi_{i,j}(w_k) = (0,\ldots,0,\delta(i,k),0,\ldots,0)^{\mathrm{T}}$, $\delta(i,k)$ being on the $iN+j$th position for all nodes of the mesh $w_k, k = 1,\ldots,N$. Then, the evaluated function can be expressed as $\mathbf{x}(w) = \sum_{i=1}^{N}\sum_{j=1}^{n}\tilde{x}_{i,j}\varphi_{i,j}(w)$. The unknown parameters are the values $\tilde{x}_{i,j}$ for $i = 1,\ldots,N,\ j = 1,\ldots,n$. Define $\tilde{x} = \begin{pmatrix}\tilde{x}_{1,1} & \ldots & \tilde{x}_{1,n} & \ldots & \tilde{x}_{N,1} & \ldots & \tilde{x}_{N,n}\end{pmatrix}^{\mathrm{T}}$. The control u is also replaced by a vector $\tilde{u} \in R^M$ for a $M > 0$. The Eq. (4) is evaluated in the nodes of the mesh and the derivatives of the function $\mathbf{x}(w)$ are replaced by a linear term. Hence one obtains a set of nN algebraic equations (the unknown variable being $\tilde{x}$). For further purpose its right-hand side is split into the linear terms (denoted by $\tilde{A}\tilde{x}$, $\tilde{B}\tilde{u}$) and the higher-order terms denoted by $\tilde{f}$, $\tilde{g}$. The discretized equation then reads

$$\tilde{M}(s(w))\tilde{x} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} + \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u}. \tag{8}$$

Here one can notice that $\tilde{A} = \mathrm{diag}(A,\ldots,A)$ (N times). The term $\tilde{M}(s(w))\tilde{x}$ is the numerical approximation of the derivative $\frac{\partial \mathrm{x}}{\partial w}s(w)$. An estimate for the left-hand side term is: $\|\tilde{M}(s(w))\tilde{x}\| \le \varrho\ \mathrm{diam}\Omega\|\tilde{x}\|$. As $f(0) = 0,\ g(0) \ne 0$ the same holds for the functions $\tilde{f}$, $\tilde{g}$. Moreover the Jacobi matrix of the function f evaluated at 0 is regular, this holds also for the matrix $\tilde{A}$ and due to the previous estimate it holds for the term $\tilde{A} - \tilde{M}(s(w))$ as well. As $M < Nn$ the set $\{\xi \in R^{nN}, \xi$ is the discrete solution of RE with right-hand side $\tilde{u}$, $\tilde{u} \in R^M\}$ is a smooth M-dimensional manifold in R^{Nn}. This will be denoted by Λ. For the sake of simplicity we assume in the following text that the function h is linear. This is not restrictive as, upon the condition that $\mathrm{d}h(0) \ne 0$, the system can be transformed so that the output mapping is linear. In this case the Eq. (8) is written for the transformed system. Then the vector $\tilde{h}$ can be defined as $\tilde{h} = (h(\varphi_{1,1}),\ldots,h(\varphi_{n,N}))$.

Assumption A. The manifold Λ does not contain the cone $\{x \in R^{nN}, \langle x, y\rangle < \alpha\|x\|\|y\|, y \in \mathrm{Ker}\, h\}^{\perp}$ for a fixed $\alpha > 0$.

Assumption R. The matrix $\tilde{A} - \tilde{M}(s(w))\tilde{x}$ is regular. The cost functional has to be discretized as well. The discretization of the mapping $\mathrm{x}(w) \mapsto \int_{\Omega}(h(\mathrm{x}(w)) - \bar{q}(w))^2\mathrm{d}w$ will be denoted by $\tilde{I}$ while the symbol $\tilde{J}$ stands for the discretization of the mapping $\tilde{u} \mapsto \tilde{J}(\tilde{x})$ where $\tilde{x}, \tilde{u}$ satisfy (8).

Lemma 8. *Let the Assumptions A and R hold. Moreover, let* $\sup_{\tilde{x}\in R^n}\frac{\|\tilde{f}(\tilde{x})\|}{\|\tilde{x}\|} < +\infty$ *and let there exists a constant* $k > 0$ *such that* $\sup_{\tilde{x}\in R^n}\|\tilde{B} - \tilde{g}(\tilde{x})\| \ge k$. *Then* $\tilde{J}(u) \to +\infty$ *as* $\|u\| \to +\infty$.

Corollary 9. *Under Assumptions A and R there exists a minimum of the functional* $\tilde{I}$.

The corollary follows from the nonnegativity of the functional $\tilde{I}$ and the previous lemma. The proof of the lemma is given in the Appendix.

To prove convergence to the above existing minimum, certain convexity properties should be guaranteed. In the case of linearity of the function h the mapping $\mathbf{x} \mapsto \int_{\Omega_b}(h(\mathbf{x}(w)) - \bar{q}(w))^2\mathrm{d}w$ is strictly convex. It is assumed that strict convexity is preserved even after the discretization of this functional. Then there a nonnegative function $\gamma : R^{nN} \times R^{nN} \to R$ so that for every $\lambda \in (0,1)$ holds $\lambda\tilde{I}(\tilde{x}_1) + (1-\lambda)\tilde{I}(\tilde{x}_2) \ge \tilde{I}(\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2) + \lambda(1-\lambda) + \gamma(\tilde{x}_1,\tilde{x}_2)$. The function γ is called the **convexity modulus** of the function $\tilde{I}$. Further, let us investigate the convexity of the penalty optimization problem. Define $\tilde{e}_\lambda$ as $\tilde{e}_\lambda = \lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2 - \tilde{x}_\lambda$. The function $\tilde{e}_\lambda$ satisfies the equation

$$\begin{aligned}\tilde{M}(s(w))\tilde{e}_\lambda &= \tilde{A}\tilde{e}_\lambda + \lambda\tilde{f}(\tilde{x}_1) + (1-\lambda)\tilde{f}(\tilde{x}_2) - \tilde{f}(\tilde{x}_\lambda) + \\ &\quad \lambda\tilde{g}(\tilde{x}_1)\tilde{u}_1 + (1-\lambda)\tilde{g}(\tilde{x}_2)\tilde{u}_2 - \tilde{g}(\tilde{x}_\lambda)\tilde{u}_\lambda.\end{aligned}$$

One has $\tilde{I}(\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2) = \tilde{I}(\tilde{x}_\lambda) + \nabla\tilde{I}(\hat{x})\tilde{e}_\lambda$ with a point $\hat{x}$ lying in the segment $S := [\lambda\tilde{x}_1 + (1-\lambda)\tilde{x}_2, \tilde{x}_\lambda]$. On the other hand the definition of the functional $\tilde{I}$ implies that its gradient is equal to $2(\int_{\Omega_b}(\tilde{h}(\hat{x}) - \bar{q}(w))\varphi_{1,1}(w)\mathrm{d}w\ h_{1,1}\ldots, 2\int_{\Omega_b}(\tilde{h}(\hat{x}) - \bar{q}(w))\ \varphi_{n,N}(w)\mathrm{d}w\ h_{n,N})$. Hence, using the Holder inequality twice, one gets

$$|\nabla\tilde{I}(\hat{x})\tilde{e}_\lambda| \le 2|\tilde{I}(\hat{x})|\,|\tilde{h}(\tilde{e}_\lambda)|\sum_{n,N}\|\varphi_{i,j}\| =: D.$$

Moreover, $|\tilde{I}(\hat{x})| \leq \sup_{\hat{x}\in S} =: \hat{C}$ as the segment S is a compact set. Now, it is sufficient to find $\tilde{u}_1, \tilde{u}_2$ so that $D \leq \lambda(1-\lambda)\gamma(\tilde{x}_1, \tilde{x}_2)$. The matrix $\tilde{A} - \tilde{M}(s(w))$ is regular thanks to the Assumption R. Hence

$$\tilde{h}(\tilde{e}_\lambda) = \tilde{h}\Big((\tilde{A} - \tilde{M}Sw)^{-1}(\lambda\tilde{f}(\tilde{x}_1) + (1-\lambda)\tilde{f}(\tilde{x}_2) - \tilde{f}(\tilde{x}_\lambda) + \lambda\tilde{g}(\tilde{x}_1)\tilde{u}_1 + (1-\lambda)\tilde{g}(\tilde{x}_2)\tilde{u}_2 - \tilde{g}(\tilde{x}_\lambda)\tilde{u}_\lambda)\Big).$$

As $\tilde{x}$ is a function of $\tilde{u}$ one can define $G(\tilde{u}) = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{u}$ where $\tilde{x}$ solves (8) with right-hand side $\tilde{u}$. According to the assumption, the mapping $\tilde{u} \mapsto \tilde{x}$ is a C^2-mapping. Hence, using differentiability of the function G one gets $\tilde{h}(\tilde{e}_\lambda) = \lambda(1-\lambda)\tilde{h}\left((\tilde{A} - \tilde{M}Sw)^{-1}\frac{1}{2}D^2G(\hat{u})(\tilde{u}_1 - \tilde{u}_2, \tilde{u}_1 - \tilde{u}_2)\right)$ where $\hat{u}$ lies in the segment $[\tilde{u}_1, \tilde{u}_2]$. This implies existence of a matrix P such that $\tilde{h}(\tilde{e}_\lambda) \leq \lambda(1-\lambda)|\tilde{h}(P(\tilde{u}_1 - \tilde{u}_2))|$. On the other hand, using the definition of the functional $\tilde{I}$, one can easily see that $\gamma(\tilde{x}_1, \tilde{x}_2) = \|\tilde{h}(\tilde{x}_1) - \tilde{h}(\tilde{x}_2)\|^2$. This can be summarized using the following lemma.

Lemma 10. *Let $2\hat{C}\sum_{n,N}\|\varphi_{i,j}\|\,|\tilde{h}(P(\tilde{u}_1 - \tilde{u}_2))| \leq \|\tilde{h}(\tilde{x}_1) - \tilde{h}(\tilde{x}_2)\|^2$. Then the functional $\tilde{J}$ is convex in the neighborhood of the optimum.*

The lemmas presented above make up a basis for the successful application of a gradient-based method for the minimization of the cost functional. Convexity is often a sufficient condition for convergence of such methods. Hence, the convexity region around the minimum is actually a set of initial guesses for the control $\tilde{u}$ that give rise to a convergent iterative minimization process.

5. Applications

We illustrate our approach on a two-cart system with an inverted pendulum, considered first by Devasia (1996) and later used by Huang (2003). The system consists of two elastically connected carts. An inverted pendulum is placed on the first cart while the input is the force F acting on this cart. This system has six states $x = (x_1, x_2, x_3, x_4, x_5, x_6)^{\mathrm{T}}$, correspondingly: position and velocity of the first cart, position and angular velocity of the pendulum, position and velocity of the second, passive cart. The output of the system is $y = x_1$, a single input is denoted as u, thereby giving the standard nonlinear plant of the form (1) with

$$f = \begin{pmatrix} x_2 \\ \dfrac{mlx_4^2\sin x_3 - bx_2 - mg\cos x_3\sin x_3 + K(x_5 - x_1)}{M + m(\sin x_3)^2} \\ x_4 \\ \dfrac{(M+m)g\sin x_3 + \cos x_3[bx_2 - mlx_4^2\sin x_3] - K(x_5 - x_1)}{l(M + m(\sin x_3)^2)} \\ x_6 \\ \dfrac{K}{M}(x_1 - x_5) \end{pmatrix}$$

$$g = \left(0, \frac{1}{M + m(\sin x_3)^2}, 0, \frac{-\cos x_3}{l(M + m(\sin x_3)^2)}, 0, 0\right)^{\mathrm{T}}$$

$$h = x_1, \quad p(x) \equiv 0.$$

We adopted the same physical constants from Huang (2003): the mass of the cart $M = 1.378$, the coefficient of friction $b = 12.98$, the length of the pendulum $l = 0.352$, mass of the pendulum $m = 0.051$, spring constant $K = 10$ and the gravitational constant $g = 9.81$. For the same reason, we also aim to track the reference $w(t)$ generated by the following exosystem $\dot{w}_1 = w_2$, $\dot{w}_2 = -w_1$, *reference* $= w_1$. To follow our Algorithm 1, the linear approximation of the system in question around the equilibrium point $(0, 0, 0, 0, 0, 0)^\top$ was computed and asymptotically stabilized by feedback gains $K = (-97.2, -72.0065, -172.027, -31.252, 13.0584, -50.58)$, thereby obtaining the following asymptotically stable in linear approximation system $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = f(x(t)) - g(x(t))Kx(t) + g(x(t))\tilde{u}$. Here, $\tilde{u} = u - Kx$ stands for the new control input. It remains to determine its feedforward part to ensures the tracking the desired reference. To do so, let us obtain the RE for the last system, i.e. $\forall\,(w_1, w_2) \in R^2$

$$\left(w_2\frac{\partial x_1(w)}{\partial w_1} - w_1\frac{\partial \mathbf{x}_1(w)}{\partial w_2}, \ldots, w_2\frac{\partial \mathbf{x}_6(w)}{\partial w_1} - w_1\frac{\partial \mathbf{x}_6(w)}{\partial w_2}\right)^{\mathrm{T}} = f(\mathbf{x}(w)) - g(\mathbf{x}(w))(K\mathbf{x}(w) - \tilde{u}), \quad w_1 = \mathbf{x}_1(w), \tag{9}$$

where $\mathbf{x}(w) = (\mathbf{x}_1(w), \ldots, \mathbf{x}_6(w))^{\mathrm{T}}, \mathbf{x}_1(0) = 0, \ldots, \mathbf{x}_6(0) = 0$. To solve this RE, we used the fact that the signal to be tracked, being sine, has absolute value less or equal to one and therefore it was sufficient to find $x(w)$ for $\|w\| \leq 1$, only. Therefore the domain $\Omega_b = \{w \in R^2|\ \|w\| < 2\}$ was chosen and the algebraic condition assumed to hold on $\partial\Omega_b$ giving $\mathbf{x}_1(w) = w_1$ for all $w \in \partial\Omega_b$. The boundary conditions for the other components of the solution are chosen as $\mathbf{x}_i(w) = 0$ on $\partial\Omega_b$ and Algorithm 1 applied to RE (9). The corresponding penalty functional was chosen as

$$J = \int_{\{w\in R^2;\ w_1^2 + w_2^2 \leq 1\}} (w_1 - \mathbf{x}_1(w))^2 \mathrm{d}w_1\mathrm{d}w_2. \tag{10}$$

When evaluating all its steps, the values of the feedforward function iterations $c_i(w)$ were set at points of a rectangular grid with edge 0.5. Its values at other points are calculated via first-order interpolation. Thus there is a finite amount of design parameters only. This allows to use a modified direct search method (see Bertsekas (1995)) to adjust the values of the feedforward iteration $c_i(w)$ at the points of the grid. The values of the functional (10) depending on iteration number are shown in Fig. 1. The output $\mathbf{x}_1$ of the compensated plant and the desired reference are shown at Fig. 2. The solid line represents the plant output while the dashed one the reference. The optimization of the penalty functional is a rather complex though straightforward technical issue and its detailed description is therefore omitted here.

6. Comparative study and further discussion on numerical aspects of the proposed method

The first purpose of this section is to provide a comparative study of the new method proposed by this paper and the well-known method based on computation of undetermined coefficients of the Taylor expansion. To do so, recall the basics

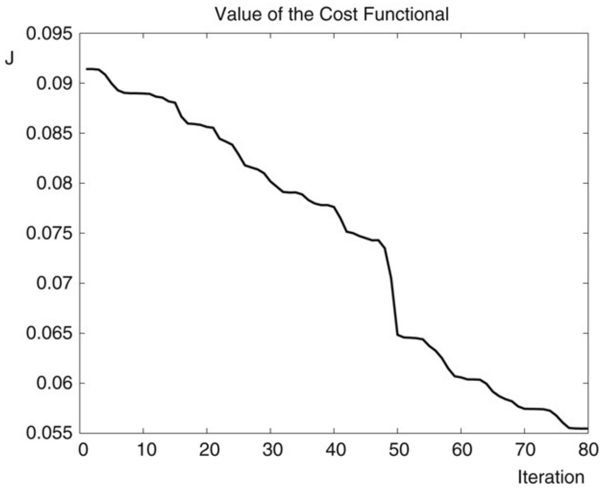

Fig. 1. The value of the error functional.

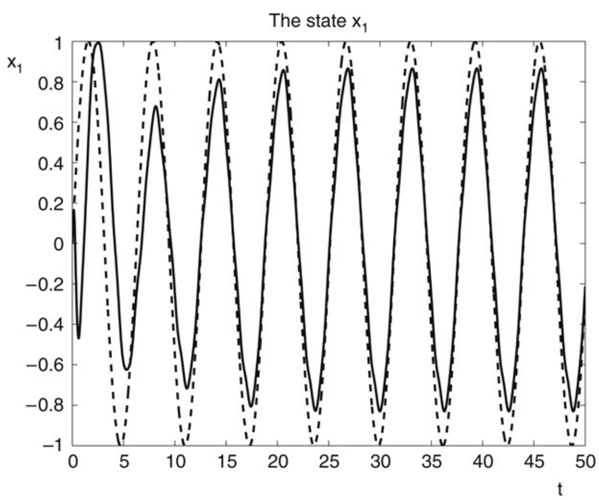

Fig. 2. Regulation results.

of that method, further referred as the "classical" one. For its full description, see Huang (2000). According to this classical method the approximation $\mathbf{z}(w)$ and $\bar{c}(w)$ of the RE solution are to be determined using Taylor series of Kth order:

$$\mathbf{z}(w) = \sum_{k=1}^{K} Z_k w^{[k]}, \qquad \bar{c}(w) = \sum_{k=1}^{K} C_k w^{[k]},$$

$$w^{[1]} = [w_1, \ldots, w_q],$$

$$w^{[2]} = [w_1^2, w_1 w_2, \ldots, w_1 w_q, w_2^2, \ldots, w_2 w_q, \ldots, w_q^2],$$

etc. Substituting these expansions into RE and identifying coefficients of $w^{[k]}$ yields the corresponding approximation of the center manifold.

As far as the FEM-based method presented here is concerned, the crucial idea is to replace algebraic part by a penalty to solve each time less peculiar PDE then in the case of classical method, cf. Remark 7. The common feature of both the classical method and the method developed by the present paper is the need for lengthy calculations off-line. Nevertheless, while the FEM-based method uses computer numerical algorithm easily applicable by any nonspecialized user, the classical method requires sophisticated manual and laborious computations by a skilled mathematician. Up to our best knowledge, no symbolic code for these computations has been developed yet. Moreover, while numerical computations are addressed via standard convergence and numerical analysis, those manual or symbolic computations usually suffer in that respect serious drawbacks. In particular, our FEM-based algorithm provides center manifold data on possible large domain, while the classical method only on a potentially very small neighborhood with no guarantee of its size. Notice also that while the classical method is just a comparison of coefficients in the Taylor series at the origin providing no estimate of the error caused by this approximation, the error estimate of the FEM-based method valid on the whole domain Ω_b is given by the Theorem 5. Last, but not least, the classical method is applicable only if all involved functions can be expanded into the Taylor series. This assumption is not required by our method. For instance, some functions can be defined as interpolations of values given by a table. The implementation of the control law (which is based on the computed center manifold) is fairly straightforward in both cases. To demonstrate the practical viability of the FEM-based approach, remind its real-time implementation to a laboratory gyroscopical platform, (Rehák, Čelikovský, Orozco-Mores, & Ruiz-León, 2007).

The second purpose of this section is to further discuss practical computer and numerical aspects of our algorithm. The extensive list of literature concerning problems arising from numerical solution of the partial differential equations, might be represented e.g. by Ciarlet (1978) and references within there. At the very end, the finite-element method converts the problem of numerical solution of partial differential equations into the problem of finding a solution of a large set of linear algebraic equations. These systems are usually badly conditioned. A thorough analysis of such errors which can be found in Higham (1996). During extensive numerical testing of our method, these software tools appeared as the reliable ones.

The detailed sensitivity analysis of the numerical optimization of the penalty induced by the algebraic part of RE is out of scope of the present paper. Nevertheless, one can proceed as follows. First, under the assumptions of the Lemma 10 and Corollary 9 the optimization problem, being the least square problem, is convex and existence of its solution is guaranteed. Secondly, as the system and the exosystem are described by smooth functions, differentiating the RE with respect to a parameter one can obtain an equation whose solution is the derivative of the function $\mathbf{x}$ with respect to this parameter. Now, to conclude the sensitivity analysis, one can derive the expression for the derivative of the functional (5). Moreover, if this parameter is the design variable defining the feedforward c, one can use smoothness of this problem to employ gradient-based optimization methods.

7. Conclusions

A new approach to the solution of the nonlinear output-regulation problem based on the solution of the regulator equation using the finite-element method was presented. Use of this method is enabled by two key steps proposed in this paper: pre-stabilization of the plant and consequently replacing the algebraic part of RE by a penalty functional. A convergence analysis was carried out for the proposed method which was also compared with other existing approaches in detail and demonstrated by the applied case study.

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Appendix

Proposition 11. *There exist positive constants C, β so that (notation of Theorem 5 and its proof is being used)*

$$\|y(t) - y_\varepsilon(t)\| \le C\mathrm{e}^{-\beta t}\|y(0) - y_\varepsilon(0)\|. \tag{11}$$

Proof. Denote by $x(t)$ the true state of the controlled system and by $\mathbf{x}_\varepsilon(w(t))$ the value of the solution of the regulator Eq. (4) at the point $w(t)$. Then according to the Theorem 2.28 in Huang (2004) the relation $\|x(t) - \mathbf{x}_\varepsilon(w(t))\| \le \bar{C}\mathrm{e}^{-\beta t}\|x(0) - \mathbf{x}(w(0))\|$ holds for some positive constants $\bar{C}, \beta$. Then (11) follows from the smoothness of the function h. □

Proposition 12. *Let us use the notation of Theorem 5 and its proof. Then, (5) implies existence of a constant $R > 0$ such that $\max_{w\in\Omega_b} |h(\mathbf{x}_\varepsilon(w)) - \bar{q}(w)| < R\varepsilon$.*

Proof. First, note that the error at the origin equals zero according to Theorem 4.

It is assumed that the solution of the regulator equations is a C^1 function on the closure of the set Ω_b. Thus the absolute value of the derivatives $\|\frac{\partial \mathbf{x}_\varepsilon}{\partial w}(w)\|$ is bounded by a constant k uniformly for all w. Let $\bar{w} \in \Omega_b$ be arbitrary fixed. Denote $M = (h(\mathbf{x}_\varepsilon(\bar{w})) - \bar{q}(\bar{w}))^2$. Let the set $U_{\bar{w}}$ be defined as $U_{\bar{w}} = \{w \in \Omega_b | \|w - \bar{w}\| < \frac{M}{2k}\}$. Due to the boundedness of the derivatives of the function x one gets: $w \in U_{\bar{w}}$ implies $(h(\mathbf{x}_\varepsilon(\bar{w})) - \bar{q}(\bar{w}))^2 > \frac{M}{2}$. On the other hand

$$\varepsilon > \int_{\Omega_b} (h(\mathbf{x}_\varepsilon(\bar{w})) - \bar{q}(\bar{w}))^2 \mathrm{d}w_1 \ldots \mathrm{d}w_\mu \ge \frac{M}{2}\kappa\left(\frac{M}{2k}\right), \tag{12}$$

where the expression $\kappa(r)$ represents the volume of a μ-dimensional ball of radius r. The function $\xi(M) = \frac{M}{2}\kappa(\frac{M}{2k})$ is continuous and increasing with M. Thus there exists an inverse function $\omega = \xi^{-1}$. Obviously, (12) implies $M < \omega(\varepsilon)$. Moreover, $\xi = O(M^{\mu+1})$ and $M^{\mu+1} = O(\xi)$ close to 0. Hence $\omega = O(M^{\frac{1}{\mu+1}})$. Since $\mu+1 > 1$ there exists $R > 0$ and $0 < \varepsilon_R$ such that $\omega(\varepsilon) < R\varepsilon$ for all $\varepsilon < \varepsilon_R$. □

Proof of Lemma 8. According to the Assumption **R**, the Eq. (8) has a solution $\tilde{x}$ for each $\tilde{u}$. Hence $\|\tilde{A} - MSw\|\|\tilde{x}\| + \|\tilde{f}(\tilde{x})\| \ge \|(\tilde{B} - \tilde{g}(\tilde{x}))\tilde{u}\| \ge k\|\tilde{u}\|$. Thus if $\|\tilde{u}\| \to +\infty$ then also $\|\tilde{x}\| \to +\infty$.

It is to prove that $\|\tilde{x}_k\| \to +\infty$ implies $|h(\tilde{x}_k)| \to +\infty$. The vector $\tilde{x}_k$ can be decomposed so that $\tilde{x}_k = \tilde{x}_k^1 + \tilde{x}_k^2$ where $\tilde{x}_k^1 \in \operatorname{Ker} h$, $\tilde{x}_k^2 \in (\operatorname{Ker} h)^\perp$. As $\|\tilde{x}_k\| \to +\infty$ then either $\|\tilde{x}_k^1\| \to +\infty$ or $\|\tilde{x}_k^2\| \to +\infty$. In the latter case one can easily see that $\tilde{J}(\tilde{u}_k) \to +\infty$. The Assumption **A** however implies that, in the first case, $\|\tilde{x}_k^2\| > \alpha\|\tilde{x}_k^1\|$ which implies that $\|\tilde{x}_k^2\| \to +\infty$ also. As the sets $L_c = \{\tilde{x} \in (\operatorname{Ker} h)^\perp | h(\tilde{x}) \le c\}$ are bounded for every $c > 0$ one gets $h(\tilde{x})_k \to +\infty$ and thus also $J(\tilde{u}_k) \to +\infty$.

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