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J. J. Loiseau [a]; P. Zagalak [b]
[a] Institut de Recherche en Communications et en Cybernétique de Nantes, UMR 6597 CNRS, Ecole Centrale de Nantes, F-44 321, Nantes, Cedex 03, France [b] Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, PO Box 18, 182 08 Praha, Czech Republic

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On pole structure assignment in linear systems

J.J. Loiseau[a] and P. Zagalak[b*]

[a]*Institut de Recherche en Communications et en Cybernétique de Nantes, UMR 6597 CNRS, Ecole Centrale de Nantes, Ecole des Mines de Nantes, Université de Nantes, BP 92 101, F–44 321, Nantes, Cedex 03, France;* [b]*Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, PO Box 18, 182 08 Praha, Czech Republic*

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The problem of pole structure assignment (PSA) by state feedback in implicit, linear and uncontrollable systems is discussed in the article. It is shown that the problem is solvable if the system is regularisable. Then necessary and sufficient conditions for characteristic polynomial assignment are established. In the case of PSA (invariant polynomials assignment) just necessary conditions have been obtained. But it turns out that these conditions are also sufficient in some special cases. This happens, for example, when the system does not possess any non-proper controllability indexes. A possible application of the achieved results to modelling a constrained movement of a robot arm is outlined, too.

Keywords: linear systems; linear state feedback; pole structure assignment

1. Introduction

Let a linear system governed by the equation

$$E\dot{x} = Ax + Bu, \tag{1}$$

where E and A are $n \times n$ matrices and B is an $n \times m$ matrix over $\mathbb{R}$, the field of real numbers, be given. The matrix B is, without any loss of generality, supposed to be of rank m. Frequently the system (1) will also be referred to as the triple (E, A, B). Next let

$$u = Fx + v, \tag{2}$$

where F is an $m \times n$ matrix over $\mathbb{R}$ and v denotes a new external input, be a state feedback around the system (1), which yields the closed-loop system

$$E\dot{x} = (A + BF)x + Bv. \tag{3}$$

In linear control it is of fundamental importance to characterise all possible pole structures of the system (3) generated by changing the state feedback gain F in (2) since the pole structure of (3) determines its dynamical behaviour, the thing we frequently want to modify. The pole structure of (3) is a complex concept that is defined as the zero structure of the pencil $sE - A - BF$ – see the definitions below.

This problem, hereafter called the problem of PSA by state feedback, has been intensively studied for more than two decades. The seminal work of Rosenbrock (1970) should be recalled first. In that work necessary and sufficient conditions for the existence of a state feedback (2) such that the system (1) with $E = I_n$, and $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] = n$ has its (finite) pole structure given by monic polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_n(s)$ (here $\psi_i(s) \rhd \psi_{i+1}(s)$ means that $\psi_{i+1}(s)$ divides $\psi_i(s)$), were formulated. The result is often referred to as the fundamental theorem of state feedback for explicit (or state-space, proper) controllable systems.

Rosenbrock's result has been widely commented in the control literature. Alternative proofs have been proposed by Dickinson (1974) who used a state-space approach, Kučera (1981) applied the theory of polynomial equations, and Flamm (1980) and Özcaldiran (1990) studied the problem in a geometric framework. Many authors have also tried to generalise this result and one can find that these generalisations go in two lines.

First, Zaballa (1987) established a result concerning the PSA in explicit and uncontrollable systems, i.e. the systems with $E = I_n$, $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] < n$. Based on the so-called interlacing inequalities, which tie together the invariant polynomials characterising the uncontrollable subspace of (1) with those describing the overall state space of (1), necessary and sufficient conditions for the existence of state feedback (2) assigning a prescribed set of monic polynomials as the invariant polynomials of $sI_n - A - BF$ have been given.

Second, Kučera and Zagalak (1988) and Zagalak and Loiseau (1992) generalised Rosenbrock's theorem

*Corresponding author. Email: zagalak@utia.cas.cz

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to the implicit (E is singular), square, and controllable systems (1). They have established necessary and sufficient conditions under which there exists a state feedback (2) such that the pencil $sE - A - BF$ has prescribed structures of its finite and infinite zeros. Moreover, these conditions also describe all the possible ranks of the pencil $sE - A - BF$.

Not surprisingly, it is of great interest and theoretical importance to establish a similar result for implicit systems that are not completely controllable. Such a result would mainly be important from theoretical point of view, but – as Example 2 shows – there are also some practical problems where similar questions arise. Thus, the article is devoted to the problem of PSA by state feedback in a very general case of linear systems, the case of implicit and uncontrollable systems (1).

The article is organised as follows. Section 2 is devoted to defining some basic concepts. Especially the feedback canonical form and normal external description (NED) of (1) are detailed. The problem formulation is introduced in § 3. The discussion starts in § 4 with the problem of characteristic polynomial assignment, which can be viewed as a simpler version of the invariant polynomials assignment problem. Under the assumption of regularisability, necessary and sufficient conditions of solvability for this problem are established. The core of the article lies in § 5 where the problem of PSA in regularisable systems is treated. First, the already solved special cases of that problem are recalled (the case of implicit and controllable systems and the case of explicit and uncontrollable systems), which enables us to introduce the mathematical tools needed for approaching the main problem. This is done in two steps. First, the problem of finite PSA is considered and then the assignment of both finite and infinite pole structures is studied. As a result, necessary conditions of solvability are established (Theorem 5). These conditions become also sufficient if the system does not have non-proper controllability indices (Corollary 1). The last section is devoted to possible applications of the obtained results; linear equations of a constrained movement of a robot arm are studied therein.

As far as notation is concerned, standard symbols and concepts of linear control theory, see Kailath (1980) for instance, are used throughout the article. For the reader's convenience some of them are now introduced.

The pole structure (finite or infinite) of the system (1), the main concept of the article, is a synonym for the zero structure (finite or infinite) of the pencil $sE - A$ – see the definition below. The degree of a polynomial vector $x(s) \in \mathbb{R}^k[s]$, $\deg x(s)$ stands for the greatest degree of all its entries $x_i(s)$. Accordingly, the degree of the i-th column of a polynomial matrix $M(s) \in \mathbb{R}^{p\times m}[s]$ is denoted by $\deg_{ci} M(s)$. Such a matrix is called *column reduced* if it can be written in the form $M(s) = M_{lc}\mathrm{diag}\{s^{c_i}\}_{i=1}^m + \bar{M}(s)$, where $M_{lc} \in \mathbb{R}^{p\times m}$ is of full column rank and $\bar{M}(s) \in \mathbb{R}^{p\times m}[s]$ is such that $\deg_{ci}\bar{M}(s) < c_i := \deg_{ci} M(s)$.

2. Background

The main concept that plays a key role when shifting the poles of the system (1) is the concept of controllability. We use this concept in the sense of Verghese, Lévy and Kailath (1981) and Cobb (1984), where it is called strong or impulse controllability. There are many definitions of controllability in the literature; see Özcaldiran and Lewis (1990) and the discussion therein. Here, as suggested in the later reference, controllability means reachability of the origin. Some other basic definitions needed in the sequel are recalled, too. For a more detailed treatment of these and other basic concepts of linear implicit (singular) systems, the reader is referred to Dai (1989), Lewis (1992) and the references therein.

Given a pencil $sH - Q$, $H, Q \in \mathbb{R}^{p\times q}$, *the finite zero structure* of $sH - Q$ is given by the invariant polynomials of $sH - Q$ while *the infinite zero structure* is defined by the negative powers of s occurring in the Smith–McMillan form at infinity of $sH - Q$; see Vardulakis, Limebeer and Karcanias (1982).

The system (1) is *controllable* if $\mathrm{rank}\,[sE - AB] = n$ for all complex s (finite and infinite), i.e. the pencil has neither finite nor infinite input decoupling zeros. The absence of finite and infinite input decoupling zeros means that both exponential and impulsive modes of (1) can be excited by non-impulsive inputs. A useful consequence of this property is, for example, the existence of a state feedback (2) eliminating impulses in (3) (Cobb 1984); see also Theorem 2 for a more complete answer.

The pencil $sE - A$ (and analogously the system (1)) is called *regular* if $\det(sE - A)$ is not identically equal to zero.

The pencil $sE - A$ (or the system (1)) is called *regularisable* if there exists a state feedback (2) such that the pencil $sE - A - BF$ is regular; see Özcaldiran and Lewis (1990).

2.1 *Feedback canonical form*

The system (1) can be transformed to another system by many types of transformations, among which the transformations involving the state feedback (2) are of special importance. Let P, Q and G be $n\times n$, $n\times n$ and $m\times m$ invertible matrices over $\mathbb{R}$ and let further F be an $m \times n$ matrix. Then the action of the (proportional) feedback group upon the system (1) is defined, see Loiseau, Özcaldiran, Malabre and Karcanias (1991), by

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P(A + BF)Q, PBG).$$

Under this action, the system (1) can be brought into *the feedback canonical form* (E_C, A_C, B_C) described below.

$$sE_C - A_C := \text{block diag}\,\{sE_{ci} - A_{ci}\}_{i=1}^{6}, \tag{4}$$

where

$$sE_{C1} - A_{C1} := \text{block diag}\left\{ \left. \underbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}}_{\epsilon_i + 1} \right\} \epsilon_i \right\}_{i=1}^{k_\epsilon}$$

$$sE_{C2} - A_{C2} := \text{block diag}\left\{ \left. \underbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s \end{bmatrix}}_{\sigma_i} \right\} \sigma_i \right\}_{i=1}^{k_\sigma}$$

$$sE_{C3} - A_{C3} := \text{block diag}\left\{ \left. \underbrace{\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}}_{q_i} \right\} q_i + 1 \right\}_{i=1}^{k_q}$$

$$sE_{C4} - A_{C4} := \text{block diag}\left\{ \left. \underbrace{\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}}_{p_i + 1} \right\} p_i + 1 \right\}_{i=1}^{k_p}$$

$$sE_{C5} - A_{C5} := \text{block diag}\left\{ sI_{l_i} - A_{ri} \right\}_{j=1}^{t}$$

with

$$A_{ri} := \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_{i0} & -\alpha_{i1} & \dots & -\alpha_{il_i-2} & -\alpha_{il_i-1} \end{bmatrix} \in \mathbb{R}^{l_i \times l_i}$$

$$sE_{C6} - A_{C6} := \text{block diag}\left\{ \left. \underbrace{\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}}_{\eta_i} \right\} \eta_i + 1 \right\}_{i=1}^{k_\eta}.$$

It should be noted that the integers σ_i and l_i are positive while the integers ϵ_i, p_i, q_i and η_i are just non-negative. These indexes are supposed to be non-increasingly ordered, i.e. $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_{k_\epsilon}$ and so on. If $\epsilon_i = 0$, or $q_i = 0$, or $\eta_i = 0$, then the corresponding columns, or rows, of $sE_C - A_C$ are equal to zero.

The matrix B_C is of the form

$$B_C := \begin{bmatrix} 0 & 0 \\ B_{C2} & 0 \\ 0 & B_{C3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $B_{C2} := \text{block diag}\,\{\mathfrak{e}_{\sigma_i}\}_{i=1}^{k_\sigma}, B_{C3} := \text{block diag}\,\{\mathfrak{e}_{q_i+1}\}_{i=1}^{k_q}, \mathfrak{e}_i := [0, \dots, 0, 1]^T \in \mathbb{R}^i$.

The integers $\{\epsilon_i\}_{i=1}^{k_\epsilon}$, $\{\sigma_i\}_{i=1}^{k_\sigma}$, $\{q_i\}_{i=1}^{k_q}$, $\{p_i\}_{i=1}^{k_p}$ are called, see Loiseau et al. (1991), *the non-proper*, *proper*, *almost proper*, *almost non-proper controllability indices of* (1), respectively, while the integers $\{\eta_i\}_{i=1}^{k_\eta}$ are known as the *row minimal indices* of $[sE_C - A_C \;\; B_C]$. The polynomials $\alpha_i(s) := s^{l_i} + \alpha_{il_i-1}s^{l_i-1} + \cdots + \alpha_{i1}s + \alpha_{i0}$, $i = 1, 2, \dots, t$, which are assumed to satisfy the divisibility conditions $\alpha_1(s) \rhd \alpha_2(s) \rhd \cdots \rhd \alpha_t(s)$, are the invariant polynomials of $[sE_C - A_C \;\; B_C]$, the zeros of which are termed the (finite) input decoupling zeros in Rosenbrock (1970).

There is a clear reason for introducing the above canonical form. This form is the simplest form that enables us to study the effect of state feedback (2) upon (1) and moreover the original system (1) can always be recovered by transformations of the state feedback group.

2.2 *Normal external description*

The next definition concerns the concept of an NED of the controllable system (1), see Malabre, Kučera and Zagalak (1990), which will frequently be used throughout the article.

Let (1) be a controllable system and let $N(s)$ and $D(s)$ be polynomial matrices such that

- $[sE - A \;\; -B]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$,
- $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ forms a minimal polynomial basis for $\text{Ker}\,[sE - A \;\; -B]$,

- $N(s)$ forms a minimal polynomial basis for $\operatorname{Ker}\Pi(sE-A)$ where Π denotes a maximal annihilator of B.

Then the matrices $N(s)$ and $D(s)$ are said to form an NED of the system (1).

It has also been shown by Malabre et al. (1990) that the column degrees $c_i := \deg_{ci}\left[\begin{smallmatrix}N(s)\\D(s)\end{smallmatrix}\right]$, $i = 1, 2, \ldots$, are the controllability indices of (1). When $\deg_{ci}D(s) > \deg_{ci}N(s)$ $(\deg_{ci}D(s) \le \deg_{ci}N(s))$ for some i, the corresponding controllability index is called *proper* (*non-proper*), which coincides with the above definition. The concept of controllability indices is closely tied with the concept of controllability: if the system (1), see Malabre et al. (1990), is regularisable, then it is controllable iff $\sum_i c_i = \operatorname{rank} E$. Notice that the controllability indices of the system (1) are given by the indices ϵ_i and σ_i. The integers ϵ_i define the non-proper controllability indices while σ_i are the proper ones.

2.3 *NED and state feedback action*

If the system (1) is uncontrollable, we can still find matrices $N(s)$ and $D(s)$ having the properties of NED. However, such an NED will not reflect all the information shown by the feedback canonical form. One can see that the only blocks contributing to an NED of (1) are those corresponding to the indices ϵ_i and σ_i. On the other hand, the matrices $N(s)$ and $D(s)$ of any NED do not depend on the integers p_i, q_i, η_i and the polynomials $\alpha_i(s)$, which means that these quantities represent hidden parts of the system. More particularly, the zeros of $\alpha_i(s)$ are the finite uncontrollable (hidden) modes of (1) while p_i and q_i give the orders of uncontrollable (hidden) mode at infinity.

To remedy this situation, the matrix B will be extended in a way resulting in a controllable system – hereafter called the extended system of (1). An NED of that system will then be used for studying the effect of state feedback (2) upon (E_C, A_C, B_C). To that end, consider the system (E_C, A_C, B_C) and define

$$\bar{B}_C := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{B}_{C3} & 0 & 0 & 0 \\ 0 & \bar{B}_{C4} & 0 & 0 \\ 0 & 0 & \bar{B}_{C5} & 0 \\ 0 & 0 & 0 & \bar{B}_{C6} \end{bmatrix}, \tag{5}$$

$$\bar{B}_{C3} := \text{block diag}\left\{\mathfrak{e}^{q_i+1}\right\}_{i=1}^{k_q},$$

$$\bar{B}_{C4} := \text{block diag}\left\{\mathfrak{e}_{p_i+1}\right\}_{i=1}^{k_p}, \quad \bar{B}_{C5} := \text{block diag}\left\{\mathfrak{e}_{l_i}\right\}_{i=1}^{k_l},$$

$$\bar{B}_{C6} := \text{block diag}\left\{\left[\mathfrak{e}^{\eta_i+1} \;\; \mathfrak{e}_{\eta_i+1}\right]\right\}_{i=1}^{k_\eta} \quad \text{and}$$

$$\mathfrak{e}^i := [1, 0, \ldots, 0]^T \in \mathbb{R}^i.$$

It can be verified that the system $(E_C, A_C, [B_C, \bar{B}_C])$ is controllable. Simple calculation also shows that an NED of $(E_C, A_C, [B_C, \bar{B}_C])$ is formed by matrices $N_C(s)$ and $D_C(s)$,

$$N_C(s) := \text{block diag}\left\{N_{Ci}(s)\right\}_{i=1}^{6}, \tag{6}$$

where

$$N_{C1}(s) := \text{block diag}\left\{[1, s, \ldots, s^{\epsilon_i}]^T\right\}_{i=1}^{k_\epsilon},$$

$$N_{C2}(s) := \text{block diag}\left\{\left[1, s, \ldots, s^{\sigma_i-1}\right]^T\right\}_{i=1}^{k_\sigma},$$

$$N_{C3}(s) := \text{block diag}\left\{\left[1, s, \ldots, s^{q_i-1}\right]^T\right\}_{i=1}^{k_q},$$

$$N_{C4}(s) := \text{block diag}\left\{[s^{p_i}, \ldots, s, 1]^T\right\}_{i=1}^{k_p},$$

$$N_{C5}(s) := \text{block diag}\left\{\left[1, s, \ldots, s^{l_i-1}\right]^T\right\}_{i=1}^{t},$$

$$N_{C6}(s) := \text{block diag}\left\{[s^{\eta_{i-1}}, \ldots, s, 1]^T\right\}_{i=1}^{k_\eta}$$

and

$$D_C(s) := \begin{bmatrix} D_{C1}(s) \\ D_{C2}(s) \end{bmatrix} := \begin{bmatrix} 0 & S_\sigma & 0 & 0 & 0 & 0 \\ 0 & 0 & S_q & 0 & 0 & 0 \\ - & - & - & - & - & - \\ 0 & 0 & -I_{k_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{k_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & S_\eta \end{bmatrix} \tag{7}$$

with

$$S_\sigma := \text{diag}\{s^{\sigma_i}\}_{i=1}^{k_\sigma}, \quad S_q := \text{diag}\{s^{q_i}\}_{i=1}^{k_q},$$

$$S_\alpha := \text{diag}\left\{\alpha_i(s)\right\}_{i=1}^{t}, \quad S_\eta := \text{block diag}\left\{[s^{\eta_i} - 1]^T\right\}_{i=1}^{k_\eta}. \tag{8}$$

The matrix $D_C(s)$ has $k_\sigma + 2k_q + k_p + t + 2k_\eta$ rows and $k_\epsilon + k_\sigma + k_q + k_p + t + k_\eta$ columns and is square when $k_\epsilon = k_q + k_\eta$.

Now, using the concept of NED, the action of state feedback (2) upon the system $(E_C, A_C, [B_C \; \bar{B}_C])$ is described as follows:

$$\begin{bmatrix} sE_C - A_C & -[B_C \;\; \bar{B}_C] \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ F & I_m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ -F & I_m & 0 \\ 0 & 0 & I \end{bmatrix} \times \begin{bmatrix} N_C(s) \\ D_{C1}(s) \\ D_{C2}(s) \end{bmatrix} = 0, \tag{9}$$

which gives

$$\begin{bmatrix} sE_C - A_C - B_CF & -[B_C \;\; \bar{B}_C] \end{bmatrix} \begin{bmatrix} N_C(s) \\ D_{C1}(s) - FN_C(s) \\ D_{C2}(s) \end{bmatrix} = 0. \tag{10}$$

The relationships (9) and (10) reveal in fact the main ideas of our approach to the problem. Instead of studying the structure of $sE_C - A_C - B_CF$ that is parametrised by F, the structure of $D_{CF}(s)$ will be investigated. The relationship (10) further shows that the matrices $N_C(s)$ and $D_{CF}(s) := [{}^{D_{C1}(s) - FN_C(s)}_{D_{C2}(s)}]$ form an NED of the system $(E_C, A_C + B_CF, [B_C \;\; \bar{B}_C])$. The matrix $D_{CF}(s)$ is of the form

$$D_{CF}(s) := \begin{bmatrix} D_{11} & S_\sigma + D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & S_q + D_{23} & D_{24} & D_{25} & D_{26} \\ \hline 0 & 0 & -I_{k_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{k_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & S_\eta \end{bmatrix}, \tag{11}$$

where D_{ij} are polynomial matrices of compatible sizes such that

$$\begin{aligned} &\deg_{c_i}\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \le \epsilon_i, \quad i = 1, 2, \ldots, k_\epsilon, \\ &\deg_{c_i}\begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix} \le \sigma_i - 1, \quad i = 1, 2, \ldots, k_\sigma, \\ &\deg_{c_i}\begin{bmatrix} D_{13} \\ D_{23} \end{bmatrix} \le q_i - 1, \quad i = 1, 2, \ldots, k_q, \\ &\deg_{c_i}\begin{bmatrix} D_{14} \\ D_{24} \end{bmatrix} \le p_i, \quad i = 1, 2, \ldots, k_p, \\ &\deg_{c_i}\begin{bmatrix} D_{15} \\ D_{25} \end{bmatrix} \le l_i - 1, \quad i = 1, 2, \ldots, t, \\ &\deg_{c_i}\begin{bmatrix} D_{16} \\ D_{26} \end{bmatrix} \le \eta_i - 1, \quad i = 1, 2, \ldots, k_\eta. \end{aligned} \tag{12}$$

All the above observations are now summarised in the following proposition.

Proposition 1: *The following holds for the extended system* $(E_C, A_C, [B_C \;\; \bar{B}_C])$:

(i1) *The system* $(E_C, A_C, [B_C \;\; \bar{B}_C])$ *is controllable.*
(i2) *The matrices* $N_C(s), D_C(s)$ *and* $N_C(s), D_{CF}(s)$ *form NEDs of* $(E_C, A_C, [B_C \;\; \bar{B}_C])$ *and* $(E_C, A_C + B_CF, [B_C \;\; \bar{B}_C])$, *respectively.*
(i3) *The system* (1) *is regularisable by state feedback* (2) *if and only if* $k_\epsilon = k_q$ & $k_\eta = 0$ (*it is square and without row minimal indices*).
(i4) *The system* (1) *is properisable* (*i.e. there exists* F *such that* $(sE_C - A_C - B_CF)^{-1}$ *exists and is proper*) *if and only if it is regularisable with* $p_i = 0$ *and* $q_i = 0$.
(i5) *The system* (1) *is controllable* $\iff t = k_\eta = 0 \wedge ((q_i = 0 \oplus k_q = 0) \vee (p_i = 0 \oplus k_p = 0))$. ($\oplus$ *means XOR.*)
(i6) *The non-unit invariant polynomials of both* $sE_C - A_C - B_CF$ *and* $D_{CF}(s)$ *coincide for any* F.
(i7) *The infinite zero orders of* $sE_C - A_C - B_CF$ *and* $D_{CF}(s)\mathrm{diag}\{s^{-k_i}\}$, *where* $k_i := \deg_{ci}[{}^{N_C(s)}_{D_C(s)}]$ *and* $F \in \mathbb{R}^{m \times n}$, *are the same.*

The proof is omitted since the above assertions can be found either in Loiseau et al. (1991), or directly follows from the properties of the feedback canonical form and the corresponding matrices $N_C(s), D_C(s)$.

2.4 *Conformal mapping*

The article is devoted to assigning both finite and infinite zero structures to $sE_C - A_C - B_CF$ by choosing F. Therefore we need to handle both finite and infinite zeros of the pencil in a unified way. To that end the conformal mapping

$$s = \frac{1 + aw}{w}, \tag{13}$$

where $a \in \mathbb{R}$, $a \neq 0$, is not a pole of (1), is applied to the extended system of (1). As a result, the poles at infinity are brought to the point 0, while other finite poles are still kept in finite positions. Applying (13) to $(E_C, A_C, [B_C \;\; \bar{B}_C])$, see Zagalak and Kučera (1995) for details, means that (13) is applied to the equation

$$\begin{bmatrix} sE_C - A_C & -[B_C \;\; \bar{B}_C] \end{bmatrix} \begin{bmatrix} N_C(s) \\ D_C(s) \end{bmatrix} = 0, \tag{14}$$

where $N_C(s)$ and $D_C(s)$ form an NED of $(E_C, A_C, [B_C \;\; \bar{B}_C])$. This is done as follows. Perform first the substitution given by (13) and then premultiply

(14) by the matrix $\operatorname{diag}\{w^{\nu_i}\}$, where $\nu_i := \deg_{ri}[sE_C - A_C \;\; -B_C]$. Postmultiply further (14) by $\operatorname{diag}\{w^{\mu_i}\}$, $\mu_i := \deg_{ci}\left[\begin{smallmatrix} N_C(s) \\ D_C(s) \end{smallmatrix}\right]$, to get (14) in the form

$$\begin{bmatrix} w\tilde{E}_C - \tilde{A}_C & -\tilde{B}_C(w) \end{bmatrix} \begin{bmatrix} \tilde{N}_C(w) \\ \tilde{D}_C(w) \end{bmatrix} = 0, \tag{15}$$

where both $[w\tilde{E}_C - \tilde{A}_C \;\; -\tilde{B}_C(w)]$ and $\left[\begin{smallmatrix} \tilde{N}_C(w) \\ \tilde{D}_C(w) \end{smallmatrix}\right]$ are polynomial matrices over $\mathbb{R}[w]$. Moreover, $\deg_{ci}\tilde{N}_C(w) = \deg_{ci}\left[\begin{smallmatrix} N_C(s) \\ D_C(s) \end{smallmatrix}\right]$ and $\tilde{N}_C(w)$ is column reduced. Similarly, as in (i5) of Proposition 1, it holds that the matrices $w\tilde{E}_C - \tilde{A}_C$ and $\tilde{D}_C(w)$ have the same (non-unit) invariant polynomials. These polynomials reflect the finite and infinite pole structures of (1). Next, a w-analogue, $\tilde{D}_{CF}(w)$, of the matrix $D_{CF}(s)$ defined by (11) is of the form

$$\tilde{D}_{CF}(w) := \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} & \tilde{D}_{15} & \tilde{D}_{16} \\ \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} & \tilde{D}_{24} & \tilde{D}_{25} & \tilde{D}_{26} \\ - & - & - & - & - & - \\ 0 & 0 & \operatorname{diag}\{w^{q_i}\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{diag}\{w^{p_i}\} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{\tilde{\alpha}} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{\tilde{\eta}} \end{bmatrix}, \tag{16}$$

where $S_{\tilde{\eta}} := \text{block diag}\{[(1+aw)^{\eta_i} - w^{\eta_i}]^T\}_{i=1}^{k_\eta}$, $S_{\tilde{\alpha}} := \operatorname{diag}\{\tilde{\alpha}_i(w)\}_{i=1}^{t}$, $\tilde{\alpha}_i(w) := w^{l_i}\alpha_i(\frac{1+aw}{w})$ for $i = 1, 2, \ldots, t$, and

$$\begin{bmatrix} \tilde{D}_{12}(0) & \tilde{D}_{13}(0) \\ \tilde{D}_{22}(0) & \tilde{D}_{23}(0) \end{bmatrix} = \begin{bmatrix} I_{k_q} & 0 \\ 0 & I_{k_\sigma} \end{bmatrix}. \tag{17}$$

As the use of the feedback canonical form does not bring any restriction on what will follow, it is supposed, from now on, that the system (1) is already in that form, i.e. the index C will be dropped.

3. Problem formulation

Proposition 1 shows that the NED of $(E, A, [B, \bar{B}])$ is a very useful tool when investigating the ability of state feedback in modifying the zero structures of $sE - A - BF$. To be more precise, the following question is the main problem under consideration in this article:

> Let a system (1) be given and let $\psi_1(s) \triangleright \psi_2(s) \triangleright \cdots \triangleright \psi_r(s)$ be monic polynomials. Let further $d_1 \geq d_2 \geq \cdots \geq d_{k_d}$ be positive integers. The matter in question is the existence of a matrix F in (2) such that the polynomials $\psi_i(s)$ and integers d_i will be the invariant polynomials and infinite zero orders of $sE - A - BF$. This problem will subsequently be called the problem of PSA by state feedback (2).

4. Characteristic polynomial assignment

Let the system (1) be regularisable (i.e. $k_\epsilon = k_q$ and $k_\eta = 0$). We will first consider the problem of regularisation of (1) by state feedback (2) as the problem of characteristic polynomial assignment, which is a simpler case of the invariant factors assignment problem. To that end, let $\psi(s)$ denote the determinant of $sE - A - BF$ and let d stand for the sum of the infinite zero orders of $sE - A - BF$. Then the freedom in choosing $\psi(s)$ and d is described by the following conditions:

$$\deg\psi(s) + d = \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{t} l_i \tag{18}$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)\ldots\alpha_t(s) \tag{19}$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i, \tag{20}$$

where $\alpha_i(s)$, $i = 1, 2, \ldots, t$ are the fixed invariant polynomials of $sE - A$. The necessity of these conditions can directly be deduced from the matrix $\tilde{D}$ introduced in (16) where, due to the condition $k_\eta = 0$, the last blocks of columns and rows are missing. Let $\tilde{D}_F(w)$ denote this matrix.

The matrix $\tilde{D}_F(w)$ is column reduced (for almost all F; if not, then such F assigns some zeros at the point a to $\tilde{D}_F(w)$, which would contradict the assumptions regarding the conformal mapping (13)) with column degrees $\epsilon_1, \ldots, \epsilon_{k_\epsilon}$, $\sigma_1, \ldots, \sigma_{k_\sigma}$, $q_1, \ldots, q_{k_q}$, $p_1, \ldots, p_{k_p}$, $l_1, \ldots, l_t$, which implies the condition (18). The relationship (19) is evident and (20) is a consequence of the block triangular structure of the matrix $\tilde{D}_F(w)$.

The conditions (18)–(20) are also sufficient for there to exist F such that $\psi(s) = \det(sE - A - BF)$. These conditions imply that

$$\deg\big(\alpha_1(s)\ldots\alpha_t(s)\big) \leq \deg\psi(s) \leq \sum \epsilon_i + \sum \sigma_i + \sum l_i,$$

which means that an F can be chosen such that

$$D_F(s) := \begin{bmatrix} D_{11} & S_\sigma + D_{12} & 0 & 0 & 0 \\ D_{21} & D_{22} & S_q & 0 & 0 \\ - & - & - & - & - \\ 0 & 0 & -I_{k_q} & 0 & 0 \\ 0 & 0 & 0 & -I_{k_p} & 0 \\ 0 & 0 & 0 & 0 & S_\alpha \end{bmatrix}$$

with $\det\begin{bmatrix} D_{11} & S_\sigma + D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \dfrac{\psi(s)}{\alpha_1(s)\cdots\alpha_t(s)}$.

Such a matrix F always exists because of the conditions (12) and since $\deg \frac{\psi(s)}{\alpha_1(s)\cdots\alpha_t(s)} \leq \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i$. It is to be noted that if $k_\epsilon = 0$, then the condition

$$\deg\psi(s) = \sum \sigma_i + \sum l_i \tag{21}$$

has to be added to the conditions (18) – (20). In fact we have proved the following.

Theorem 1: *Given a regularisable system* (1), *a monic polynomial* $\psi(s)$, *and an integer* $d \geq 0$, *then there exists a state feedback* (2) *such that* $\det(sE - A - BF) = \psi(s)$ *and the sum of the infinite zero orders of* $sE - A - BF$ *equals d if and only if the conditions* (18)–(20) (*and* (21) if $k_\epsilon = 0$) *are satisfied.*

It has already been mentioned that the matrix S_α represents the finite and uncontrollable poles of the system (1) while the integers q_i and p_i affect the orders of the uncontrollable pole of (1) at infinity. The relationships (19) and (20) then show that these quantities cannot be changed by state feedback (2). On the other hand, the numbers ϵ_i and σ_i are the controllability indices (non-proper and proper) of (E, A, B) and their sum is the number of the controllable poles that can freely be assigned either to finite or infinite locations.

5. Pole structure assignment in regularisable systems

The regularisability of (1) is a reasonable assumption when studying the PSA by state feedback; see Özcaldiran and Lewis (1990) and Ishihara and Terra (2001) for more details. In this section, we recall first the already known results that concern some special cases of PSA, which also enables us to introduce needed mathematical tools, and then we approach the main problem. The achieved results are stated in Theorems 4 and 5, and Corollary 1.

5.1 *Implicit and controllable systems*

When the system (1) is controllable, much more can be said on the structure of the zeros of $sE - A - BF$. The following theorem can be found in Zagalak and Loiseau (1992).

Theorem 2: *Given a controllable system* (1), *monic polynomials* $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_r(s)$, *and positive integers* $d_1 \geq d_2 \geq \cdots \geq d_{k_d}$, *then there exists a matrix F in* (2) *such that the polynomials* $\psi_i(s)$ *and integers* d_i *will determine the zero structures* (*finite and infinite*) *of* $sE - A - BF$ *if and only if*

$$n - k_\epsilon + k_d \leq r \leq n \tag{22}$$

$$\sum_{i=j}^{r} (\deg\psi_i(s) + d_i) \leq \sum_{i=j}^{r} c_i^*, \quad j = 1, 2, \ldots, r \tag{23}$$

where $c_1^* \geq c_2^* \geq \cdots \geq c_r^*$ *is the list consisting of all the indices* σ_i *and* $k_\epsilon - n + r$ *greatest* ϵ_i (*completed by zeros to the number* r, *if necessary*), *and* $d_i := 0$ *for* $i > k_d$.

It is to be noted that equality holds in (23) just in one case, namely for $j = 1$ and $r = n$. If this condition is satisfied, the pencil $sE - A - BF$ becomes non-singular, that is to say, the closed-loop system (3) is regular. Some other particular cases of Theorem 2 are discussed in the remarks below.

Remark 1: From practical point of view the most important version of Theorem 2 is obtained if $r = n$ and $d_i = 0$ for all i, which means that just a finite pole structure is assigned to the closed-loop system (3). The resulting system is proper and the so-called impulsive behaviour is eliminated; see Kučera and Zagalak (1988).

Remark 2: Theorem 2 is a generalisation of the aforementioned result of Rosenbrock (1970). Assuming an explicit and controllable system (1), the inequalities (22) imply that $k_\epsilon = k_d = 0$, which gives $r = n$. Hence, the inequalities (23) are of the form

$$\sum_{i=\nu}^{n} \deg \psi_i(s) \leq \sum_{i=\nu}^{n} c_i, \quad \nu = 1, 2, \ldots, n, \tag{24}$$

where the integers $c_1 \geq c_2 \geq \cdots \geq c_m$ are the controllability indices of the system (i.e. the indexes σ_i), $c_i := 0$ for $i > m$, and equality holds for $\nu = 1$.

Remark 3: It is worth noting that the conditions (24) also hold between the column degrees $c_1 \geq c_2 \geq \cdots \geq c_m$ of any $n \times m$ polynomial matrix $P(s)$ of rank m and the degrees of its invariant polynomials, say $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_m(s)$.

Remark 4: In terms of the 'w-notation' introduced above, the result given by Theorem 2 reads (just the case of regularisation is considered) as follows. Under the assumptions of Theorem 2 there exists an F in (2) such that the $m \times m$ matrix

$$\tilde{D}_F(w) := \begin{bmatrix} \tilde{D}_{11}(w) & \tilde{D}_{12}(w) \\ \tilde{D}_{21}(w) & \tilde{D}_{22}(w) \end{bmatrix}$$

has given polynomials $\psi_1(w)w^{d_1} \rhd \psi_2(w)w^{d_2} \rhd \cdots \rhd \psi_{k_d}(w)w^{d_{k_d}} \rhd \psi_{k_d+1}(w) \rhd \cdots \rhd \psi_m(w)$, $m := k_\epsilon + k_\sigma$, as its invariant polynomials if and only if

(1) $\tilde{D}_F(w)$ is column reduced with column degrees $\epsilon_1, \epsilon_2, \ldots, \epsilon_{k_\epsilon}, \sigma_1, \sigma_2, \ldots, \sigma_{k_\sigma}$,

(2) $\begin{bmatrix} \tilde{D}_{12}(0) \\ \tilde{D}_{22}(0) \end{bmatrix} = \begin{bmatrix} I_{k_\sigma} \\ 0 \end{bmatrix}$,

(3) $k_d \leq k_\epsilon$.

It should be noted that the above formulation will turn to play an important role when establishing the main results of the article in § 5.3.

5.2 *Explicit and uncontrollable systems*

There is another way to generalise Rosenbrock's result (Remark 2). To that end, consider an explicit but uncontrollable system (1), i.e. the system characterised by $k_\epsilon = k_q = k_p = k_\eta = 0$. This leads to investigating the invariant polynomials of the matrix

$$D_F(s) := \begin{bmatrix} S_\sigma + D_{12} & D_{15} \\ 0 & S_\alpha \end{bmatrix}, \tag{25}$$

where S_σ and S_α are given matrices while D_{12} and D_{15} are arbitrary but satisfying the corresponding conditions given in (12).

For the sake of simplifying the subsequent notation, let $m := k_\sigma$, i.e. S_σ and S_α are $m \times m$ and $t \times t$ matrices, respectively. The polynomials $\alpha_1(s) \rhd \alpha_2(s) \rhd \cdots \rhd \alpha_t(s)$ describe the dynamics of the uncontrollable part of (1). Next, by (i6) of Proposition 1, the non-unity invariant polynomials of $D_F(s)$ are the same as those of $sE - A - BF$, which means that the study of the zero structure of $D_F(s)$ is equivalent to the study of the zero structure of $sE - A - BF$. Based on that, the problem of PSA can be reformulated in the following way:

> Given a $t \times t$ matrix S_α defined in (8) and monic polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_{m+t}(s)$, do there exist matrices D_{12} and D_{15} satisfying the degree conditions (12) such that the matrix $D_F(s)$ defined by (25) will have the polynomials $\psi_i(s)$ as its invariant polynomials?

The following lemma is a key technical result on the way to solve the above problem.

Lemma 1 (Sá 1979; Thompson 1979)**:** *Given a $t \times t$ polynomial matrix $A(s) := \operatorname{diag}\{\alpha_i\}_{i=1}^t$ with $\alpha_1(s) \rhd \alpha_2(s) \rhd \cdots \rhd \alpha_t(s)$ and monic polynomials $\varphi_1(s) \rhd \varphi_2(s) \rhd \cdots \rhd \varphi_{t+1}(s)$, there exist polynomials $\delta_i(s)$, $i = 1, 2, \ldots, t+1$ such that the matrix*

$$\left[\begin{array}{c|c} \delta_{t+1}(s) & \delta_t(s), \ldots, \delta_1(s) \\ \hline 0 & A(s) \end{array}\right] \tag{26}$$

has the polynomials $\varphi_i(s)$ as its invariant polynomials if and only if

$$\varphi_{i+1}(s) \lhd \alpha_i(s) \lhd \varphi_i(s), \quad i = 1, 2, \ldots, t. \tag{27}$$

Thus, to find the matrices D_{12} and D_{15} in (25), one recurrently applies Lemma 1, as many times as needed, and constructs polynomials $\varphi_i^j(s)$, $i = 1, 2, \ldots, t+j$, $j = 0, 1, \ldots, m$ such that

$$\begin{aligned} \varphi_i^0(s) &= \alpha_i(s), && i = 1, 2, \ldots, t, \\ \varphi_i^m(s) &= \psi_i(s), && i = 1, 2, \ldots, m+t, \\ \varphi_{i+1}^{j+1}(s) \lhd \varphi_i^j(s) &\lhd \varphi_i^{j+1}(s), && i = 1, 2, \ldots, t+j, \\ & && j = 0, 1, \ldots, m-1. \end{aligned}$$

The polynomials $\varphi_i^{j+1}(s)$ are called *a polynomial path* from the polynomials $\alpha_i(s)$ to the polynomials $\psi_i(s)$ and are evidently far from unique. The point now is whether at least one such a path can be found. To that end let

$$\beta_i^j(s) := \operatorname{lcm}(\alpha_i(s), \psi_{m+i-j}(s)) \tag{28}$$

for $i = 1, 2, \ldots, t+j$ and $j = 0, 1, \ldots, m-1$. Then the polynomials $\beta_i^j(s)$ form a polynomial path from $\{\alpha_i(s)\}$ to $\{\psi_i(s)\}$, see Sá (1979), which is called the *minimal polynomial path* since $\beta_i^j(s) \lhd \varphi_i^j(s)$, $i = 1, 2, \ldots, t+j$ and $j = 1, 2, \ldots, m$, where $\{\varphi_i^j(s)\}$ denotes any other polynomial path.

But this is not all. As the matrix $D_F(s)$ is column reduced with column degrees $\sigma_1, \sigma_2, \ldots, \sigma_m, 1, 1, \ldots, 1$, which are supposed to be non-increasingly ordered, then, in view of Remark 3, the relationship (24) has to hold between the invariant polynomials of $D_F(s)$ and its column degrees. All these facts lead to the subsequent.

Theorem 3 (Zaballa 1987)**:** *Let* (1) *be an explicit system* ($q = n$ *and* $\operatorname{rank} E = n$) *with controllability indices $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ that has an uncontrollable part characterised by the polynomials $\alpha_1(s) \rhd \alpha_2(s) \rhd \cdots \rhd \alpha_t(s)$. Let further $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_{m+t}(s)$ be monic polynomials. Then there exists an F in* (2) *such that the pencil $sE - A - BF$ has the polynomials $\psi_i(s)$* (*completed by* 1*s to the number n*) *as its invariant polynomials if and only if*

$$\psi_{i+m}(s) \lhd \alpha_i(s) \lhd \psi_i(s), \quad i = 1, 2, \ldots, t \tag{29}$$

and

$$\sum_{i=1}^{t+j} \deg \beta_i^j(s) \leq \sum_{i=1}^{t} \deg \alpha_i(s) + \sum_{i=m-j+1}^{m} \sigma_i, \quad j = 1, 2, \ldots, m, \tag{30}$$

where equality holds for $j = m$ and the polynomials $\beta_i^j(s)$ are defined by (28).

5.3 *Implicit and uncontrollable systems*

5.3.1 *Finite PSA*

The result stated in Theorem 3 can also be applied to the case where assigning just a finite pole structure to a regularisable system (1) ($k_\epsilon = k_q$ & $k_\eta = 0$)

is considered. This leads to investigating the finite zero structure of the matrix

$$D_F(s) = \left[\begin{array}{cc|c} D_{11} & S_\sigma + D_{12} & \bar{D}_{15} \\ D_{21} & D_{22} & \bar{D}_{25} \\ \hline 0 & 0 & \bar{S}_\alpha \end{array}\right], \tag{31}$$

which is a matrix resembling that in (25), where $\bar{S}_\alpha$ is a block-diagonal matrix consisting of the blocks $S_\alpha, I_{k_q}, I_{k_p}$, and where the matrices $\bar{D}_{15} := [D_{15}, 0, 0]$, $\bar{D}_{25} := [D_{25}, 0, 0]$ have the parts corresponding to I_{k_q} and I_{k_p} equal to zero since these parts can always be zeroed by unimodular operations.

Let $c_1 \geq c_2 \geq \cdots \geq c_m$ denote a non-increasingly ordered set of indices ϵ_i and σ_i, $m := k_\epsilon + k_\sigma$. Then Theorem 3 implies that the conditions (29) and (30) are necessary in this case, too. Just the equality for $j=m$ in (30) need not hold since the submatrix

$$P(s) := \begin{bmatrix} D_{11} & S_\sigma + D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

need not be column reduced. It will be shown below that the conditions (29) and (30), where equality for $j=m$ need not be satisfied, are also sufficient.

Theorem 4: *Let* (1) *be an implicit and regularisable system* (i.e. $k_\epsilon = k_q$ *and* $k_\eta = 0$) *and let* $\{c_i\}_{i=1}^m$ *stand for a non-increasingly ordered set of indices* ϵ_i *and* σ_i. *Let further* $m := k_\epsilon + k_\sigma$, $k := k_q + k_p + t$ *and let* $\alpha_1(s) \rhd \alpha_2(s) \rhd \cdots \rhd \alpha_t(s)$, $\alpha_{t+1}(s) = \alpha_{t+2}(s) = \cdots = \alpha_k(s) = 1$, $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_{m+k}(s)$ *be monic polynomials. Then there exists an* $m \times n$ *matrix* F *over* $\mathbb{R}$ *such that the matrix* (31) *is non-singular and the polynomials* $\psi_i(s)$ *are its invariant polynomials if and only if the following conditions hold*:

$$\psi_{i+m}(s) \lhd \alpha_i(s) \lhd \psi_i(s), \quad i = 1, 2, \ldots, k, \tag{32}$$

$$\sum_{i=1}^{k+j} \deg\beta_i^j(s) \leq \sum_{i=1}^{k} \deg\alpha_i(s) + \sum_{i=m-j+1}^{m} c_i, \quad j = 1, 2, \ldots, m, \tag{33}$$

$$\deg(\psi_1\psi_2\cdots\psi_{m+k}) \geq \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k} \deg\alpha_i(s). \tag{34}$$

Proof of Theorem 4 (Necessity)**:** The conditions (32) and (33) follow from the conditions (29) and (30) of Theorem 3 while the condition (34) is a simple consequence of the fact that the matrix $P(s)$ is non-singular having the submatrix $S_\sigma + D_{12}$ column reduced. Thus the following inequalities hold:

$$\sum_{i=1}^{k_\sigma} \sigma_i \leq \deg\det P(s) \leq \sum_{i=1}^{m} c_i,$$

which implies (34).

(Sufficiency) Provided that (32)–(34) hold then there exist integers $\bar{c}_1 \geq \bar{c}_2 \geq \cdots \geq \bar{c}_m \geq 0$ such that (33) holds with equality for $j=m$. These integers can be chosen such that $\bar{c}_i \leq c_i$ for $i = 1, 2, \ldots, m$. More precisely, these indices comprise all the indices σ_i and some others, denoted by $\hat{c}_i$, satisfying $\hat{c}_i \leq \epsilon_i$, $i = 1, 2, \ldots, k_\epsilon$. Thus

$$\{\bar{c}_i\}_{i=1}^m = \{\hat{c}_i\}_{i=1}^{k_\epsilon} \cup \{\sigma_i\}_{i=1}^{k_\sigma}.$$

Now the conditions (29) and (30) (with σ_i replaced by $\bar{c}_i$) of Theorem 3 are satisfied, which means that an $m \times m$ column reduced matrix, say $\bar{P}(s)$, with the highest column-degree coefficient matrix equal to I_m can be constructed (see Zagalak, Kučera and Loiseau (1993) and Zaballa (1999) for details) giving rise to the matrix

$$\left[\begin{array}{c|c} \bar{P}(s) & Q(s) \\ \hline 0 & S_\alpha \end{array}\right]$$

with the invariant polynomials $\psi_i(s)$ and $\deg_{ci}Q(s) < \deg_{ci}S_\alpha$, $i = 1, 2, \ldots, k$. This matrix can easily be brought, by permutations of columns and rows, into the form of the matrix $P(s)$.

Once the matrix $D_F(s)$ is constructed, the feedback gain F is obtained from a constant solution (X, Y), with X non-singular, to the matrix linear equation

$$XD(s) + YN(s) = P(s), \tag{35}$$

where $D(s)$ and $N(s)$ form an NED of (1), on putting $F := X^{-1}Y$, see Kučera and Zagalak (1991).

It should be noted, see Kučera and Zagalak (1991), that F calculated from (35) is far from unique. It depends on the way the matrix $P(s)$ is obtained. More specifically, the entries of F depend on a particularly assigned eigenvector structure (which is, in this case, implicitly assigned). □

Remark 5: The result stated in Theorem 4 concerns the problem of regularisation of regularisable systems (1), which is, from the control theoretical point of view, the most important case. A more general form of this result, where just $k + \mu, k_\sigma \leq \mu \leq m$, invariant polynomials are assigned, can also be established (using Theorem 2, see Zaballa (1987)). The conditions (32)–(34) are still necessary and sufficient in this case, too; just m is replaced by μ.

5.3.2 *Finite and infinite PSA*

A trick how to handle both the finite and infinite pole structures in a unified way lies in applying the conformal mapping (13) to the extended system of (1). Keeping the same notation as in §2.4, the transformed system is described by the matrices $\tilde{N}(w)$ and $\tilde{D}(w)$ in (15). It has been shown (see Zagalak and

Kučera (1995) for details) that the invariant polynomials of $\tilde{D}_F(w)$ are

$$\tilde{\psi}_i(w) := w^{d_i+\deg\psi_i(s)}\psi_i\left(\frac{1+aw}{w}\right). \tag{36}$$

Analogously, $\alpha_i(s)$ are transformed to $\tilde{\alpha}_i(w) = w^{h_i+\deg\alpha_i(s)}\alpha_i(\frac{1+aw}{w})$.

Thus, applying the mapping (13) to (1) results in a system that possesses just a finite pole structure (this structure contains – as a substructure – a structure of the pole at the point 0 that defines the structure at infinity of the original system). But this is a situation to which Theorem 4 can be applied. The condition (32) is now of the form

$$\tilde{\psi}_{i+m}(w) \lhd \tilde{\alpha}_i(w) \lhd \tilde{\psi}_i(w), \quad i = 1, 2, \ldots, k,$$

which gives the following two conditions:

$$\psi_{i+m}(s) \lhd \alpha_i(s) \lhd \psi_i(s), \quad i = 1, 2, \ldots, k, \tag{37}$$

$$d_{i+m} \le h_i \le d_i, \quad i = 1, 2, \ldots, k_h, \tag{38}$$

where $\{h_i\}$ denotes the non-increasingly ordered set $\{p_i\} \cup \{q_i\}$ consisting of the almost (non-proper and proper) controllability indices of (1), $k_h := \mathrm{card}\{h_i\}$, $d_1 \ge d_2 \ge \cdots \ge d_{k_d} > 0$ are multiplicities of the pole at $w = 0$ that are to be assigned, $k_d := \mathrm{card}\{d_i\}$, and the polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \cdots \rhd \psi_{k+m}(s)$ define the finite pole structure to be assigned. Notice that the set $\{p_i\} \cup \{q_i\}$ may contain zeros while the set $\{h_i\}$ consists just of positive integers.

The second condition, (33), of Theorem 4 takes the form

$$\sum_{i=1}^{k+j}(\deg\beta_i^j(s) + \max(h_i, d_{i+m-j})) \le \sum_{i=1}^{k}(\deg\alpha_i(s) + h_i) + \sum_{i=m-j+1}^{m} c_i, \quad j = 1, 2, \ldots, m, \tag{39}$$

where $k := t + k_p + k_q$, $\alpha_i(s) := 1$ for $i > t$, $h_i := 0$ for $i > k_h$ and equality holds for $j = m$ if the matrix

$$\tilde{D}_F(w) := \left[\begin{array}{ccccc} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}^*_{13} & 0 & \tilde{D}_{15} \\ \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}^*_{23} & 0 & \tilde{D}_{25} \\ \hline 0 & 0 & \mathrm{diag}\{w^{q_i^*}\}_{i-1}^{k_q^*} & 0 & 0 \\ 0 & 0 & 0 & I_{(k_q-k_q^*)} & 0 \\ 0 & 0 & 0 & 0 & S_\alpha \end{array}\right] \tag{40}$$

with $k_q^* := \mathrm{card}\{q_i > 0\}$, is non-singular – see also proof of Corollary 1.

The condition (34) remains unchanged. However, another necessary condition comes from the fact that just the ϵ-, q-, and p-blocks of $[sE - A \;\; B]$ determine a possible infinite pole structure of the closed-loop system. Inspecting the rank of (40) at $w = 0$ reveals that

$$k_d \le k_\epsilon + k_p^* \quad \text{(column rank)} \tag{41}$$

$$k_d \le k_q + k_p^* \quad \text{(row rank)}, \tag{42}$$

where k_p^* denotes the number of p_i greater than zero.

These conditions are clearly equivalent in the case of regularisable systems ($k_\epsilon = k_q$ and $k_\eta = 0$). Moreover, the conditions (38), (41) and (42) also describe the freedom in choosing the number of multiplicities of the pole at $s = \infty$ since they imply the following inequality for k_d:

$$k_h \le k_d \le k_q + k_p. \tag{43}$$

At this moment one could wonder whether the conditions (37)–(41) are also sufficient. Unfortunately the following example shows that these conditions are just necessary.

Example 1: Consider the system given by $\epsilon_1 = 1$, $q_1 = 2$, and $p_1 = 1$, i.e. $\{\tilde{\alpha}_i(w)\} := \{w^2, w\}$ and $\{h_i\} := \{2, 1\}$. Then $\bar{D}_F(w)$ is equivalent to

$$\begin{bmatrix} \alpha_2 w + \alpha_1 & \beta w + 1 & \gamma \\ 0 & w^2 & 0 \\ 0 & 0 & w \end{bmatrix}, \quad \alpha_i, \beta, \gamma \in \mathbb{R}, \; \alpha_2 \ne 0,$$

and has the invariant factors $\{1, w, w^2(\alpha_2 w + \alpha_1)\}$. One can notice that there are no coefficients $\alpha_1, \alpha_2, \beta$ and γ such that the polynomials $\{1, w^2, w^2\}$ would be the invariant polynomials of $\bar{D}_F(w)$ or, using the above notation, the list $\{d_1, d_2\} = \{2, 2\}$ cannot be assigned. However, the conditions (37)–(41) are satisfied.

Remark 6: To enlighten the above example a bit more, consider the matrix (16) again. As the matrices $\tilde{D}_{23}(w)$ and $\mathrm{diag}\{w^{q_i}\}$ are right coprime, the matrix $\tilde{D}_F(w) := \tilde{D}_{CF}(w)$ has the same non-unit invariant polynomials as the matrix

$$\tilde{D}'_F(w) = \begin{bmatrix} I & -\tilde{D}_{13}X & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \times \left[\begin{array}{cccc} \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{14} & \tilde{D}_{15} \\ \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{24} & \tilde{D}_{25} \\ \hline 0 & 0 & \mathrm{diag}\{w^{p_i}\} & 0 \\ 0 & 0 & 0 & S_{\tilde{\alpha}} \end{array}\right],$$

where $\left[\begin{smallmatrix} X & Y \\ A & B \end{smallmatrix}\right]$ is a unimodular matrix satisfying the Bezout identity

$$\begin{bmatrix} X & Y \\ A & B \end{bmatrix}\begin{bmatrix} \tilde{D}_{23} \\ \operatorname{diag}\{w^{q_i}\} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Considering now the matrix $\tilde{D}_F(w)$ from Example 1, we obtain

$$\tilde{D}'_F(w) = \begin{bmatrix} w^2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha_2 w + \alpha_1 & 0 \\ 0 & w \end{bmatrix}.$$

Such a matrix cannot have $\{w^2, w^2\}$ as its invariant polynomials.

The following theorem summarises the above observations.

Theorem 5: *Let* (1) *be an implicit and regularisable system* ($k_\epsilon = k_q$ & $k_\eta = 0$) *and let* $\{h_i\}_{i=1}^m$ *stand for the non-increasingly ordered set of the indices* p_i *and* q_i. *Let* $k := t + k_p + k_q$ *and let the polynomials* $\alpha_i(s)$ *and* $\psi_i(s)$ *be as in Theorem* 4. *Let further* c_i, $i = 1, 2, \ldots, m$, $m := k_\epsilon + k_\sigma$, *denote the non-increasingly ordered set of the indices* ϵ_i *and* σ_i, $d_1 \geq d_2 \geq \cdots \geq d_{k_d} > 0$ *be given integers, and* F *be an* $m \times n$ *matrix over* $\mathbb{R}$ *such that the matrix* (40) *is non-singular and its finite and infinite zero structures are given by the polynomials* $\psi_i(s)$ *and the integers* d_i, *respectively. Then the conditions* (37)–(41) *hold.*

As the conditions (37)–(41) are just necessary, one could be interested in finding special cases in which these conditions would also be sufficient. Such special cases clearly exist and have already been mentioned. The conditions (37)–(41) are also sufficient if the system (1) is controllable (Theorem 2), or explicit (Theorem 3), or just a finite pole structure is to be assigned (Theorem 4). Some other cases are described in the following corollary.

Corollary 1: *The conditions* (37)–(41) *of Theorem* 5 *are also sufficient if in addition the following holds*:

(i) $\{p_i\} = \emptyset$,
(ii) $p_i = 0, i = 1, 2, \ldots, k_p$,
(iii) $\{p_i\}$ *is a subset of* $\{d_i\}$, *i.e.* $\{d_i\} = \{p_i\} \cup \{\bar{n}_i\}$.

Proof of Corollary 1: First, if $k_p = 0$, then the matrix (16) reduces to the matrix (40) where all the entries above the term $I_{(k_q - k_q^*)}$ have been eliminated by unimodular operations.

Next, it follows from (41) and (43) that $k_q^* \leq k_d \leq k_q$. So, to prove the sufficiency of the conditions (37)–(41), a way to construct a matrix $\tilde{D}'_F(w)$ satisfying (17) should be shown.

Let $\alpha_i(s)$, $\psi_i(s)$ and d_i be given polynomials and integers. Then the polynomials $\tilde{\alpha}_i(w)$ and $\tilde{\psi}_i(w)$, where

$$\tilde{\psi}_i(w) := \begin{cases} \psi_i w^{d_i} & i = 1, 2, \ldots, k_d \\ \psi_i(w), & i = k_d,\ k_d + 1, \ldots, m + k \end{cases} \tag{44}$$

$$\tilde{\alpha}_i(w) := \begin{cases} \alpha_i w^{q_i} & i = 1, 2, \ldots, k_q \\ \alpha_i(w), & i = k_q,\ k_q + 1, \ldots, k, \end{cases} \tag{45}$$

are obtained first. Now, as (37) and (38) hold, there exists a polynomial path (particularly the minimal polynomial path $\{\beta_i^j(w)\}$ (28) can be chosen) from $\tilde{\alpha}_i(w)$ to $\tilde{\psi}_i(w)$, which means that – similarly as in the proof of Theorem (4) – an $(m + k) \times (m + k)$ matrix

$$\left[\begin{array}{c|c} \tilde{P}(w) & \tilde{Q}(w) \\ \hline 0 & S_{\tilde{\alpha}} \end{array}\right], \tag{46}$$

with $S_{\tilde{\alpha}} := \operatorname{diag}\{\tilde{\alpha}_i(w)\}$ and with $\tilde{\psi}_i(w)$ as its invariant polynomials, can be constructed. The matrix $S_{\tilde{\alpha}}$ can further be brought by unimodular row and column operations into the form (40), i.e.

$$S_{\tilde{\alpha}} \sim \begin{bmatrix} \operatorname{diag}\{w^{q_i^*}\}_{i-1}^{k_q^*} & 0 & 0 \\ 0 & I_{(k_q - k_q^*)} & 0 \\ 0 & 0 & S_\alpha \end{bmatrix}. \tag{47}$$

It is assumed, without any loss of generality, that the matrix (46) is of a column reduced form.

Next, investigating the rank deficiency of the matrix (46) at $w = 0$, it can be seen that it equals k_d. Then, as the rank deficiency of $S_{\tilde{\alpha}}$ is k_q^*, it implies that the rank deficiency of the matrix $[\tilde{P}(0), \tilde{Q}(0)]$ is $k_d - k_q^*$, which is lower or equal to $k_q - k_q^*$, and hence, the matrix $[\tilde{P}(w), \tilde{Q}(w)]$ can be modified by unimodular row operations such that $\operatorname{rank}[\tilde{P}(0), \tilde{Q}(0)] = k_\sigma + k_\epsilon$. One can notice that in fact $\operatorname{rank}[\tilde{P}(0), \tilde{Q}_{k_q^*}(0)] = k_\sigma + k_\epsilon$, where the matrix $\tilde{Q}_{k_q^*}(s)$ is formed by the first k_{q^*} columns of $\tilde{Q}(s)$ (rank S_α is full). This implies that $\operatorname{rank}\tilde{P}(0) \geq k_\sigma$ and consequently the matrix $\tilde{P}(s)$ can be brought to the form

$$\tilde{P}'(s) := \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix}$$

which is column reduced with column indices $\epsilon_1, \ldots, \sigma_1, \ldots, \sigma_{k_\sigma}$ with $\tilde{D}_{12}(0) = I_{k_\sigma}$.

The matrix $\tilde{Q}_{k_q^*}(0)$ need not be of full column rank; however, this feature can be achieved by adding certain columns of $\tilde{P}'(s)$ to $\tilde{Q}_{k_q^*}(s)$. Then, after dividing $\tilde{Q}_{k_q^*}(s)$ by the matrix $\operatorname{diag}\{w^{q_i}\}_{i=1}^{k_q^*}$, we can combine row and column operations performed on $\tilde{Q}_{k_q^*}(s)$ to obtain the matrix $\tilde{Q}_{k_q^*}(0)$ in the form $[0, I_{k_q^*}, 0]^T$.

In the last step, the rows containing the submatrix $I_{(k_q - k_q^*)}$ are added to the last $k_q - k_q^*$ rows of

$[\tilde{P}(w), \tilde{Q}(w)]$ such that the matrix $\tilde{Q}_{k_q^*}(0)$ is of the form $[0, I_{k_q}]^T$.

Such a constructed matrix $D_F(w)$ has all the features described in § 2.4 and can be used (together with the corresponding matrix $N(w)$) for calculating a feedback gain F. This is done as follows. The inverse conformal mapping is applied first to the matrices $N(w)$ and $D_F(w)$, which results in an NED of the closed-loop system (3). Then an F is obtained in a similar way as in the proof of Theorem 4. This proves (i).

A proof of the second claim is the same as that above. Just, the matrix diag $\{w^{p_i}\}_{i=1}^{k_p^*} = I_{k_p}$ is considered as a submatrix of $S_{\tilde{\alpha}}$.

To prove the third claim, i.e. when $\{d_i\} = \{p_i\} \cup \{\bar{n}_i\}$, it is sufficient to choose $D_{14} = D_{24} = 0$ and proceed as in the previous case.

6. Conclusions

The question of pole structure assignment by state feedback in implicit systems is discussed in the article. First, in a brief discussion, it is shown that the system (1) should have some 'reasonable' properties under which the PSA problem is solvable. Then, provided the system is regularisable, necessary and sufficient conditions for placing the poles of (3) at desired locations (the characteristic polynomial assignment to the matrix $sE - A - BF$) are established.

As far as the problem of pole structure assignment is concerned, just necessary conditions of its solvability have been established. However, these conditions are also sufficient if the system does not have almost non-proper controllability indices. Obtaining a more complete solution seems to be more involved and remains thus as a topic for a future work.

This work is mainly of theoretical interest but there is also motivation coming from applications. One can find systems to which the achieved results might be applied. Such systems occur for instance when modelling a constrained movement of a manipulator (or robot) arm, as briefly shown in the example below. We deliberately say 'might be applied' since the issue of robustness have not been considered here; this remains as a challenge for some future studies. On the other hand Example 2 also shows that certain structural features of (1) may play a key role in control design (whatever the system parameters are, the impulsive behaviour of (49) cannot be eliminated).

It should be noted that the design methods based on the pole structure (or eigenstructure) assignment has progressed significantly in the last two decades – see for instance Liu and Patton (1998) and the references therein – and we hope the presented results could stimulate further development in that field.

Example 2: The equation of motion of a manipulator arm, which is constrained by one point in contact with a rigid frictionless surface, is of the form (see for instance Mills and Goldenberg (1989))

$$\Phi(q)\ddot{q} + H(\dot{q}, q) + G(q) = \tau + J^T(q)D^T(p)\lambda, \qquad (48)$$

where $q \in \mathbb{R}^{m\times 1}$ denotes the manipulator generalised coordinates, p and $\lambda \in \mathbb{R}^k$ are the position vector ($p = L(q)$) and Lagrange multiplier, $\Phi(q) \in \mathbb{R}^{m\times m}$ is the manipulator inertia matrix, $H(\dot{q}, q) \in \mathbb{R}^{m\times 1}$ denotes the vector of Coriolis and centripetal forces, $G(q) \in \mathbb{R}^{m\times 1}$ stands for the vector of gravity forces, $\tau := \tau(q, \dot{q}, \ddot{q}, \lambda) \in \mathbb{R}^{m\times 1}$ is the vector of generalised forces applied at each joint, $J(q) \in \mathbb{R}^{3\times m}$ stands for the manipulator Jacobian matrix ($J(q) := \frac{\partial}{\partial q} L(q)$), and $D(p) := \frac{\partial}{\partial p}\varphi(p)$ where $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given scalar function with continuous gradient defining the constraint surface.

After linearising the non-linear system (48) about a nominal state, say $[q_0, \dot{q}_0, \lambda_0]^T$, in which the manipulator is at rest (i.e. $\dot{q}_0 = 0$ and $\ddot{q}_0 = 0$), and denoting $\delta q := q - q_0$, $\delta\tau := \tau - \tau_0$, $\delta\lambda := \lambda - \lambda_0$, the equation of the manipulator motion has the form

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & \Phi(q_0) & \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\dot{q} \\ \delta\ddot{q} \\ \delta\dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 & I_n & 0 \\ -\dfrac{\partial}{\partial q}(G - J^T D^T \lambda)\big|_0 & 0 & J^T D^T\big|_0 \\ DJ\big|_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta q \\ \delta\dot{q} \\ \delta\lambda \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} \delta\tau, \qquad (49)$$

where $X\big|_0$ stands for the value of X at the nominal point. Modifying the notation slightly, this system is feedback equivalent to the system

$$\begin{bmatrix} 0 & I_{m-k} & 0 & 0 & 0 \\ 0 & 0 & I_{m-k} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & 0 & I_k \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 & I_{m-k} & 0 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & 0 & I_k \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ I_{m-k} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & I_k \end{bmatrix} u,$$

the feedback invariants of which are

- the non-proper controllability indices $\epsilon_i = 0$, $i = 1, 2, \ldots, k$,
- the proper controllability indices $\sigma_i = 2$, $i = 1, 2, \ldots, m-k$,
- and the almost non-proper controllability indices $q_i = 2$, $i = 1, 2, \ldots, k$.

This is the case to which Theorem 5 and Corollary 1 can now be applied. Thus, analysing the relationships (37)–(41) of Theorem 5, it readily follows that the integers d_i and the polynomials $\psi_i(s)$ have to satisfy the following conditions:

$$k_d = k_q = k \tag{50}$$

$$d_i \geq 2 \quad \text{for } i = 1, 2, \ldots, k \tag{51}$$

$$\begin{aligned} &\sum_{i=1}^{k+j} \left(\deg\psi_{i+m-j}(s) + \max\{q_i, d_{i+m-j}\} \right) \\ &\qquad \leq \sum_{i=1}^{k} q_i + \sum_{i=m-j+1}^{m} c_i, \quad j = 1, 2, \ldots, m \end{aligned} \tag{52}$$

where c_i, $i = 1, 2, \ldots, m$ are defined in Theorem 5.

For example, let $k = 2$ and $m = 3$. Then $\epsilon_1 = \epsilon_2 = 0$, $\sigma_1 = 2$, $c_1 = 2$, $c_2 = c_3 = 0$, and $h_1 = q_1 = h_2 = q_2 = 2$. It can also be verified that such a system is regularisable. Suppose now that the system is regular, $\psi_1(s) \rhd \psi_2(s) \rhd \psi_3(s) \rhd \psi_4(s) \rhd \psi_5(s)$ are the invariant polynomials of $sE - A$ and $d_1 \geq d_2$ its infinite zero multiplicities. Then, the relationship (52) gives, for $j = 2$,

$$\begin{aligned} &\deg\psi_2(s) + \cdots + \deg\psi_5(s) + \max\{2, d_2\} \\ &\quad + \max\{2, 0\} + 0 + 0 \leq 2 + 2, \end{aligned}$$

which implies that $\psi_2(s) = \psi_3(s) = \psi_4(s) = \psi_5(s) = 1$ and $d_2 = 2$. If $j = 3$, we obtain that $\deg\psi_1(s) + \max\{2, d_1\} + 2 \leq 6$, which reveals that just the following three cases are possible:

(a) $\deg\psi_1(s) = 2$ and $d_1 = 2$
(b) $\deg\psi_1(s) = 1$ and $d_1 = 3$
(c) $\deg\psi_1(s) = 0$ and $d_1 = 4$.

As the impulsive behaviour of the closed-loop system cannot be eliminated by state feedback, see (51), an obvious control design strategy lies in minimising the impulses (i.e. the numbers d_i) of (3). Hence, to obtain a minimal-impulse closed-loop system, one should choose the quantities $\psi_i(s)$ and d_i as in the case (a). So, let $\psi_1(s) = s^2 + \alpha s + \beta$ and $d_1 = d_2 = 2$. Then one of the possible matrices $D_F(s)$ is of the form

$$\begin{bmatrix} 1 & 0 & s^2 & 0 & 0 \\ 0 & 1 & 0 & s^2 & 0 \\ -1 & 0 & \alpha s + \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which gives a feedback gain F of the following form:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\beta & -\alpha & 0 & 0 & 0 & 0 \end{bmatrix}.$$

More generally, to obtain a minimal-impulse closed-loop system, one should choose $d_i = q_i$, $i = 1, 2, \ldots, k$, which is always possible. For $j = k$, the inequalities (52) imply

$$\begin{aligned} &\deg\psi_{m-k+1}(s) + \cdots + \deg\psi_{m+k}(s) + \max\{2, d_{m-k+1}\} \\ &\quad + \cdots + \max\{2, 0\} = 2k, \end{aligned}$$

which means that

$$\begin{aligned} \psi_{m-k+i} &= 1 \quad \text{for } i = 1, 2, \ldots \\ d_{m-k+i} &= \begin{cases} 2 & \text{for } m - k + i \leq k \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{53}$$

and the inequalities (52) can be rewritten in the form

$$\begin{aligned} &\sum_{i=j}^{m-k} \deg\psi_i(s) + \sum_{i=1}^{k} \max\{2, d_{i+j-1}\} \\ &\qquad \leq \sum_{i=1}^{k} q_i + \sum_{i=j}^{m-k} \sigma_i, \quad j = 1, 2, \ldots, m. \end{aligned}$$

The inequalities (51), or even better the relationships (53), show that the so-called impulsive modes of (49) cannot be removed by state feedback. It should be noted that the question of impulsive modes elimination was originally posed by Mills and Goldenberg (1989, p. 42) who also called for further work in that area.

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