Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-10

Stability for retarded functional differential equations

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Dedicated to A. M. Samoĭlenko on the occasion of his 70th birthday

Abstract

It is known that retarded functional differential equations can be regarded as Banach-space valued generalized ordinary differential equations (GODEs). See [1]. In this paper some stability concepts for retarded functional differential equations are introduced and they are discussed using known stability results for GODEs (see [9]). Then the equivalence of the different concepts of stabilities considered here are proved and converse Lyapunov theorems for a very wide class of retarded functional differential equations are obtained by means of the correspondence of this class of equations with GODEs.

Notations

Let X be a Banach space and $I \subset \mathbb{R}$ be an interval of the real line.

We denote by $G(I,X)$ be the space of locally regulated functions $f : I \to X$, that is, for each compact interval $[a,b] \subset I$, the lateral limits $f(t+) = \lim_{\rho\to 0+} f(t+\rho)$, $t \in [a,b)$, and $f(t-) = \lim_{\rho\to 0-} f(t+\rho)$, $t \in (a,b]$, exist and are finite. When $I = [a,b]$ we write $G([a,b],X)$ which is a Banach space when endowed with the usual supremum norm. In $G(I,X)$ we consider the topology of locally uniform convergence. By $G^-(I,X)$, we mean the subspace of $G(I,X)$ of left continuous functions for which $f(t-) = \lim_{\rho\to 0-} f(t+\rho) = f(t)$, $t \in I$, except for the left endpoint of I.

We denote by $BV(I,X)$ the space of functions $f : I \to X$ which are locally of bounded variation, that is, for each compact interval $[a,b] \subset I$, the restriction of f to $[a,b]$, $f\big|_{[a,b]}$, is of bounded variation. In $BV([a,b],X)$, we consider the variation norm given by $\|f\| = \|f(a)\| + \mathrm{var}_a^b f$, where $\mathrm{var}_a^b f$ stands for the variation of f in the interval $[a,b]$. Then $BV([a,b],X)$ is a Banach space and $BV([a,b],X) \subset G([a,b],X)$. When $f \in BV(I,X)$ is also left continuous ($f \in BV(I,X) \cap G^-(I,X)$), we write $f \in BV^-(I,X)$.

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We write $C(I,X)$ to denote the space of continuous functions $f : I \to X$. We consider the Banach space $C([a,b],X)$ equipped with the usual supremum norm and in $C(I,X)$ we consider the topology of locally uniform convergence.

It is clear that $C(I,X) \subset G^-(I,X)$ and $BV^-(I,X) \subset G^-(I,X)$.

For simplifying our considerations we restrict ourselves to the case of left continuous functions everywhere, when some discontinuities can occur.

1 Retarded functional differential equations

Let us consider the initial value problem for a retarded functional differential equation:

$$\begin{cases} \dot{y}(t) = f(y_t, t), \\ y_{t_0} = \phi, \end{cases} \tag{1.1}$$

where $\phi \in G^-([-r,0],\mathbb{R}^n)$, $r \geq 0$, and $f(\phi,t)$ maps an open subset Ω of $G^-([-r,0],\mathbb{R}^n) \times [t_0,+\infty)$ to $\mathbb{R}^n$. Given a function $y : [t_0 - r, +\infty) \to \mathbb{R}^n$, we consider $y_t : [-r,0] \to \mathbb{R}^n$ defined, as usual, by

$$y_t(\theta) = y(t+\theta), \qquad \theta \in [-r,0],\ t \in [t_0,+\infty).$$

Let us recall the concept of a solution of problem (1.1).

Definition 1.1. Let $\sigma > 0$. A function $y \in G^-([t_0 - r, t_0 + \sigma],\mathbb{R}^n)$ such that $(y_t,t) \in G^-([-r,0],\mathbb{R}^n) \times [t_0,t_0+\sigma]$ for all $t \in [t_0,t_0+\sigma]$, $y_{t_0} = \phi$ and

$$\dot{y}(t) = f(y_t,t)$$

for almost all $t \in [t_0,t_0+\sigma]$ is called a (local) *solution of* (1.1) *in* $[t_0,t_0+\sigma]$ (or sometimes also in $[t_0 - r, t_0 + \sigma]$) *with initial condition* (ϕ,t_0).

The system (1.1) is known to be equivalent to the "integral" equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s,s)\,ds, \quad t \in [t_0,+\infty), \\ y_{t_0} = \phi, \end{cases} \tag{1.2}$$

when the integral exists in the Lebesgue sense (cf. [4]). In fact we will use (1.2) for the concept of the initial value problem (1.1). This makes it clear that if a solution y is defined on some interval $[t_0,t_0+\sigma]$ with $\sigma > 0$ then y, being an indefinite integral of a Lebesgue integrable function, is necessarily absolutely continuous on $[t_0,t_0+\sigma]$ (we write $y \in AC([t_0,t_0+\sigma],\mathbb{R}^n)$).

Let $G_1 \subset G^-([t_0 - r, +\infty), \mathbb{R}^n)$ with the following property: if $y = y(t)$, $t \in [t_0 - r, +\infty)$, is an element of G_1 and $\bar{t} \in [t_0 - r, +\infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), \ t_0 - r \leq t \leq \bar{t} \\ y(\bar{t}), \ \bar{t} < t < +\infty \end{cases}$$

also belongs to G_1.

Let $L_1(I, X)$ denote the space of locally Bochner integrable functions $f : I \to X$, integrable in each compact of I, where $I \subset \mathbb{R}$ is an interval and X is a Banach space. If X is finite-dimensional then we have in mind the Lebesgue integral.

Let $|\cdot|$ be a norm in $\mathbb{R}^n$.

We consider $f(\phi, t) : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$, the righthand side of the differential equation in (1.1), such that the mapping $t \mapsto f(y_t, t)$ belongs to $L_1([t_0, +\infty), \mathbb{R}^n)$ for $y \in G_1$ and the following conditions are fulfilled:

(A) there is $M \in L_1([t_0, +\infty), \mathbb{R})$ such that for all $x \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} f(x_s, s)\, ds \right| \leq \int_{u_1}^{u_2} M(s)\, ds;$$

(B) there is $L \in L_1([t_0, +\infty), \mathbb{R})$ such that for all $x, y \in G_1$ and all $u_1, u_2 \in [t_0, +\infty)$,

$$\left| \int_{u_1}^{u_2} [f(x_s, s) - f(y_s, s)]\, ds \right| \leq \int_{u_1}^{u_2} L(s)\, \|x_s - y_s\|\, ds,$$

the norm on the righthand side is the norm in $G^-([-r, 0], \mathbb{R}^n)$ given by $\|\phi\| = \sup_{t \in [-r,0]} |\phi(t)|$ for $\phi \in G^-([-r, 0], \mathbb{R}^n)$.

Of course the functions M and L above depend on the choice of t_0.

If $f(0, t) = 0$ for every $t \in \mathbb{R}$, then $y \equiv 0$ is a solution of (1.1). The next definitions concern stability concepts for the solution $y \equiv 0$ of (1.1). The following three definitions are the classical definitions for Lyapunov stability, uniform (Lyapunov) stability and uniform asymptotic stability of the trivial solution of (1.1). See [4], for instance.

Definition 1.2. The trivial solution of system (1.1) is called *(Lyapunov) stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $\phi \in G^-([-r, 0], \mathbb{R}^n)$ and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0 - r, +\infty)$ and $[\gamma, v] \ni t_0$, is a solution of (1.1) such that $\overline{y}_{t_0} = \phi$ and

$$\|\phi\| < \delta,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [t_0, v].$$

Definition 1.3. The trivial solution of system (1.1) is called *uniformly stable* if the number δ in Definition 1.2 is independent of t_0.

Definition 1.4. The solution $y \equiv 0$ of (1.1) is called *uniformly asymptotically stable* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exists a $T = T(\varepsilon, \delta_0) \geq 0$ such that if $\phi \in G^-([-r, 0], \mathbb{R}^n)$, and $\overline{y} : [\gamma, v] \to \mathbb{R}^n$, with $[\gamma, v] \subset [t_0 - r, +\infty)$ and $[\gamma, v] \ni t_0$, is solution of (1.1) such that $\overline{y}_{t_0} = \phi$ and

$$\|\phi\| < \delta_0,$$

then

$$\|\overline{y}_t(t_0, \phi)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty).$$

The next definition of stability of the solution $y \equiv 0$ of (1.1) is borrowed from [3].

Definition 1.5. The solution $y \equiv 0$ of (1.1) is said to be *integrally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\phi \in G^-([-r, 0], \mathbb{R}^n)$, $\|\phi\| < \delta$ and $p \in L_1([t_0, t_1], \mathbb{R}^n)$ with $\int_{t_0}^{t_1} |p(s)| ds < \delta$, then

$$|y(t; t_0, \phi)| < \varepsilon \ \text{ for every } \ t \in [t_0, t_1],$$

where $y(t; t_0, \phi)$ is a solution of the perturbed equation

$$\begin{cases} \dot{y}(t) = f(y_t, t) + p(t), \\ y_{t_0} = \phi. \end{cases} \tag{1.3}$$

The solution of equation (1.3) has to be interpreted as a solution of the "integral" equation

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\, ds + \int_{t_0}^{t} p(s) ds \\ y_{t_0} = \phi, \end{cases} \tag{1.4}$$

where the integral is considered in the Lebesgue sense. The solution of (1.3), when it exists, is absolutely continuous on $[t_0, t_1]$ (i.e., $y(\cdot; t_0, \phi) \in AC([t_0, t_1], \mathbb{R}^n)$).

Now we introduce a concept of stability of the trivial solution of (1.1) which generalizes Definition 1.5 and will be essential to our purposes.

Definition 1.6. The solution $y \equiv 0$ of (1.1) is said to be *variationally stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $\phi \in G^-([-r, 0], \mathbb{R}^n)$, $\|\phi\| < \delta$ and $P \in BV^-([t_0, t_1], \mathbb{R}^n)$ with $\text{var}_{t_0}^{t_1} P < \delta$, then

$$|y(t; t_0, \phi)| < \varepsilon \ \text{ for every } \ t \in [t_0, t_1],$$

where $y(t; t_0, \phi)$ is a solution of

$$\begin{cases} y(t) = y(t_0) + \displaystyle\int_{t_0}^{t} f(y_s, s)\, ds + P(t) - P(t_0), \quad t \in [t_0, t_1] \\ y_{t_0} = \phi. \end{cases} \tag{1.5}$$

It can be seen immediately that the solution y of (1.5) is of bounded variation and left continuous, that is, $y \in BV^-([t_0,t_1],\mathbb{R}^n) \subset G^-([t_0,t_1],\mathbb{R}^n)$.

Note that (1.4) is a particular case of (1.5) for $P(t) = \int_{t_0}^t p(s)ds,\ t \geq t_0$. If $p \in L_1([t_0,t_1],\mathbb{R}^n)$, then we have $P \in AC([t_0,t_1],\mathbb{R}^n) \subset BV^-([t_0,t_1],\mathbb{R}^n)$, the derivative $\dot{P}(s) = \frac{dP}{ds}$ exists almost everywhere in $[t_0,t_1]$ and

$$\mathrm{var}_{t_0}^{t_1} P = \int_{t_0}^{t_1} |\dot{P}(s)|ds = \int_{t_0}^{t_1} |p(s)|ds.$$

Having this in mind we can easily see that the variational stability of the trivial solution of (1.1) is a concept which is more general than that of integral stability. Therefore we consider the variational stability only.

Definition 1.7. The solution $y \equiv 0$ of (1.1) is called *variationally attracting* if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|\phi\| < \widetilde{\delta} \qquad \text{and} \qquad \mathrm{var}_{t_0}^{t_1} P < \rho$$

with $P \in BV^-([t_0,t_1],\mathbb{R}^n)$, then

$$|y(t;t_0,\phi)| < \varepsilon \ \text{ for all } \ t \geq t_0 + T,\ t \in [t_0,t_1]$$

where $y(t;t_0,\phi)$ is a solution of the equation (1.5) satisfying $y_{t_0} = \phi$.

Definition 1.8. The solution $y \equiv 0$ of (1.1) is called *variationally asymptotically stable* if it is variationally stable and variationally attracting.

It is clear by the definitions that if the solution $y \equiv 0$ of (1.1) is variationally stable then it is also Lyapunov stable. Similarly also for the asymptotic stabilities.

Maybe at this moment the reader is wondering why Definitions 1.6 to 1.8 are presented for RFDEs. One reason is that stability with respect to permanently acting perturbations is of interest for technology. The second is a pragmatic one, since we have results on stability for GODEs at our disposal which can be used in this context. See [2] and [9] and the development of the theory in the next section.

To the first reason we add that the perturbation in the case of integral stability can be large enough as long as its integral is small. One could also consider perturbations of the form $p(t,y,y_t)$ and the same technique would apply. However, the theory around would be more complicated technically. In the case of variational stability, we can think about the possibility of perturbing the original equation (1.1) by an integrable function plus a Dirac sum acting on a countable set and then interpret the solution appropriately. In this case, the solution is a left continuous function. It is clear that (1.5) can be interpreted as an equation with impulses acting at points of discontinuity of the function P and described in the form given e.g. in the book [7] of A. M. Samoĭlenko and N. A. Perestyuk and, of course, in numerous papers of the Kiev ODE group concentrated around this two personalities.

2 The GODE corresponding to (1.5)

Let X be a Banach space and consider $\Omega \subset X \times \mathbb{R}$. Assume that $G : \Omega \to X$ is a given X-valued function with $G(x,t)$ defined for each $(x,t) \in \Omega$.

Having the concept of Kurzweil integrability in mind (see for instance [1], [2], [8] or [9]), we now present the concept of generalized differential equation.

Definition 2.1. A function $x : [\alpha, \beta] \to X$ is called *a solution of the generalized ordinary differential equation*

$$\frac{dx}{d\tau} = DG(x,t) \tag{2.1}$$

on the interval $[\alpha, \beta] \subset \mathbb{R}$ if $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - x(\gamma) = \int_{\gamma}^{v} DG(x(\tau), t) \tag{2.2}$$

holds for every γ, $v \in [\alpha, \beta]$, where the integral is considered in the sense of Kurzweil.

Let us mention that the theory of generalized ordinary differential equations presented e.g. in [8] is for the case when $X = \mathbb{R}^n$, but it is easy to check that all the basic results hold also for the case of a Banach space.

Given an initial condition $(z_0, t_0) \in \Omega$ the following definition of the solution of the initial value problem for the equation (2.1) will be used.

Definition 2.2. A function $x : [\alpha, \beta] \to X$ is a *solution of the generalized ordinary differential equation* (2.1) *with the initial condition* $x(t_0) = z_0$ *on the interval* $[\alpha, \beta] \subset \mathbb{R}$ if $t_0 \in [\alpha, \beta]$, $(x(t), t) \in \Omega$ for all $t \in [\alpha, \beta]$ and if the equality

$$x(v) - z_0 = \int_{t_0}^{v} DG(x(\tau), t) \tag{2.3}$$

holds for every $v \in [\alpha, \beta]$.

Now we consider $\Omega = G_1 \times [t_0, +\infty)$ and we define a special class of functions $F : \Omega \to X$.

Definition 2.3. We say that a function $G : \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, if there exists a nondecreasing, left continuous function $h : [t_0, +\infty) \to \mathbb{R}$ such that

$$\|G(x, s_2) - G(x, s_1)\| \leq |h(s_2) - h(s_1)| \tag{2.4}$$

for all (x, s_2), $(x, s_1) \in \Omega$ and

$$\|G(x, s_2) - G(x, s_1) - G(y, s_2) + G(y, s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)| \tag{2.5}$$

for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$.

Suppose $f(\phi,t) : G_1 \times [t_0,+\infty) \to \mathbb{R}^n$ such that for each $y \in G_1$ the mapping $t \mapsto f(y_t,t)$ belongs to $L_1([t_0,+\infty),\mathbb{R}^n)$ and f satisfies conditions (A) and (B).

Assume further that $P \in BV^-([t_0,+\infty),\mathbb{R}^n)$.

For $y \in G_1$ and $t \in [t_0-r,+\infty)$, define

$$F(y,t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \le \vartheta \le t_0 \text{ or } t_0 - r \le t \le t_0 \\ \int_{t_0}^{\vartheta} f(y_s,s)\,ds, \quad t_0 \le \vartheta \le t < +\infty; \\ \int_{t_0}^{t} f(y_s,s)\,ds, \quad t_0 \le t \le \vartheta < +\infty. \end{cases} \tag{2.6}$$

and, for $t \in [t_0-r,+\infty)$, put

$$\overline{P}(t)(\vartheta) = \begin{cases} 0, \quad t_0 - r \le \vartheta \le t_0 \text{ or } t_0 - r \le t \le t_0 \\ P(\vartheta) - P(t_0), \quad t_0 \le \vartheta \le t < +\infty; \\ P(t) - P(t_0), \quad t_0 \le t \le \vartheta < +\infty. \end{cases} \tag{2.7}$$

Then

$$G(y,t) = F(y,t) + \overline{P}(t) \tag{2.8}$$

defines an element $G(y,t)$ of $G^-([t_0-r,+\infty),\mathbb{R}^n)$ and $G(y,t)(\vartheta) \in \mathbb{R}^n$ is the value of $G(y,t)$ at a point $\vartheta \in [t_0-r,+\infty)$, that is,

$$G : G_1 \times [t_0-r,+\infty) \to G^-([t_0-r,+\infty),\mathbb{R}^n).$$

The idea to construct the righthand side of a GODE which corresponds to a functional differential equation of the form (1.1) is due to C. Imaz, F. Oliva and Z. Vorel from their papers [5] and [6].

Let $h : [t_0,+\infty) \to \mathbb{R}$ be defined by

$$h(t) = \int_{t_0}^{t} [M(s) + L(s)]ds + \operatorname{var}_{t_0}^{t} P, \quad t \in [t_0,+\infty).$$

Then the function h is left continuous and nondecreasing, since $M, L : [t_0,+\infty) \to \mathbb{R}$ are nonnegative a.e. and $P \in BV^-([t_0,+\infty),\mathbb{R}^n)$.

Under the given assumptions, it is a matter of routine to prove that the function G given by (2.8) belongs to the class $\mathcal{F}(\Omega,h)$, where $\Omega = G_1 \times [t_0,+\infty)$ (see e.g. [1]).

Consider G given by (2.8). If $[\alpha,\beta] \subset [t_0,+\infty)$ and $x : [\alpha,\beta] \to G^-([t_0-r,+\infty),\mathbb{R}^n)$ is a solution of (2.1) in $[\alpha,\beta]$, then x is of bounded variation in $[\alpha,\beta]$ and

$$\operatorname{var}_{\alpha}^{\beta} x \le h(\beta) - h(\alpha) < +\infty.$$

Moreover, every point in $[\alpha,\beta]$ at which the function h is continuous is a point of continuity of the solution $x : [\alpha,\beta] \to G^-([t_0-r,+\infty),\mathbb{R}^n)$ and we have

$$x(\sigma+) - x(\sigma) = \lim_{s\to\sigma+} x(s) - x(\sigma) = G(x(\sigma),\sigma+) - G(x(\sigma),\sigma)$$

for $\sigma \in [\alpha, \beta)$ and

$$x(\sigma) - x(\sigma-) = x(\sigma) - \lim_{s \to \sigma-} x(s) = G(x(\sigma), \sigma) - G(x(\sigma), \sigma-)$$

for $\sigma \in (\alpha, \beta]$, where $G(x, \sigma+) = \lim_{s \to \sigma+} G(x, s)$, for $\sigma \in [\alpha, \beta)$ and $G(x, \sigma-) = \lim_{s \to \sigma-} G(x, s)$, for $\sigma \in (\alpha, \beta]$. For a proof of these facts, the reader may want to consult [8], for instance.

Now we present a result on the existence of the integral involved in the definition of the solution of the generalized equation (2.1). This result is a particular case of Corollary 3.16 and Proposition 3.6, both from [8].

Lemma 2.1. *Let $G \in \mathcal{F}(\Omega, h)$. Suppose $x : [\alpha, \beta] \to X$, $[\alpha, \beta] \subset [t_0, +\infty)$, is of bounded variation in $[\alpha, \beta]$ and $(x(s), s) \in \Omega$ for every $s \in [\alpha, \beta]$. Then the integral $\int_\alpha^\beta DG(x(\tau), t)$ exists and the function $s \mapsto \int_\alpha^s DG(x(\tau), t) \in X$ is of bounded variation for all $s \in [\alpha, \beta]$.*

The next result concerns the existence of a solution of (2.1) (see [1], Theorem 2.15).

Theorem 2.4. *Let $G : \Omega \to X$ be an element of the class $\mathcal{F}(\Omega, h)$, where the function h is left continuous (i.e. $h(t-) = h(t)$, $t \in (a, +\infty)$). Then for every $(\widetilde{x}, t_0) \in \Omega$ such that for $\widetilde{x}_+ = \widetilde{x} + G(\widetilde{x}, t_0+) - G(\widetilde{x}, t_0)$ we have $(\widetilde{x}_+, t_0) \in \Omega$ and there exists a $\Delta > 0$ such that on the interval $[t_0, t_0 + \Delta]$ there exists a unique solution $x : [t_0, t_0 + \Delta] \to X$ of the generalized ordinary differential equation (2.1) for which $x(t_0) = \widetilde{x}$.*

Consider the generalized equation (2.1). We will work now with a specific initial value problem for equation (2.3) with G given by (2.8).

Let $\phi \in G^-([-r, 0], \mathbb{R}^n)$ and $\sigma > 0$ be given. A function $x(t)$ defined on the interval $[t_0 - r, t_0 + \sigma]$ and taking values in $G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ is a (local) *solution of the generalized ordinary differential equation* (2.1) in the interval $[t_0, t_0 + \sigma]$ (or in $[t_0 - r, t_0 + \sigma]$), with *initial condition* $x(t_0) \in G_1$ given for $\phi \in G^-([-r, 0], \mathbb{R}^n)$ by

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0) & \text{for } \vartheta \in [t_0 - r, t_0], \\ x(t_0)(t_0) & \text{for } \vartheta \in [t_0, t_0 + \sigma] \end{cases}$$

if

$$x(v) = x(t_0) + \int_{t_0}^{v} DG(x(\tau), t)$$

for every $v \in [t_0, t_0 + \sigma]$.

For a proof of the next result, see [1], Lemma 3.3.

Proposition 2.1. *If $x(t)$ is a solution of (2.1) in the interval $[t_0, t_0 + \sigma]$, then for $v \in [t_0, t_0 + \sigma]$ we have*

$$x(v)(\vartheta) = x(v)(v), \quad \vartheta \geq v,\ \vartheta \in [t_0 - r, t_0 + \sigma]$$

and

$$x(v)(\vartheta) = x(\vartheta)(\vartheta), \quad v \geq \vartheta,\ \vartheta \in [t_0 - r, t_0 + \sigma].$$

Left continuous regulated functions with the properties of Proposition 2.1 are candidates for considering them as solutions of the initial value problem described above for (2.1).

The next result is the key to our approach to retarded functional differential equations by the theory of generalized differential equations. It states the correspondence between these equations by relating their solutions in a one-to-one manner. For a proof of it see [1].

Proposition 2.2.

(i) *Consider equation* (1.5)*, where* $f : G_1 \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$*,* $t \mapsto f(y_t, t)$ *is Lebesgue integrable on* $[t_0, t_0+\sigma]$*,* $P \in BV^-([t_0, t_0+\sigma], \mathbb{R}^n)$ *and* (A)*,* (B) *are fulfilled. Let* $y(t)$ *be a solution of problem* (1.5) *in the interval* $[t_0, t_0 + \sigma]$*. Given* $t \in [t_0 - r, t_0 + \sigma]$*, let*

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), \ \vartheta \in [t_0 - r, t] \\ y(t), \ \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then $x(t) \in G^-([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *is a solution of* (2.1) *in* $[t_0 - r, t_0 + \sigma]$ *where the right hand side of* (2.1) *is given by* (2.8) *.*

(ii) *Reciprocally, let* G *be given by* (2.8) *and* $x(t)$ *be a solution of* (2.1) *in the interval* $[t_0 - r, t_0 + \sigma]$ *satisfying the initial condition*

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), \ t_0 - r \leq \vartheta \leq t_0, \\ x(t_0)(t_0), \ t_0 \leq \vartheta \leq t_0 + \sigma \end{cases} \tag{2.9}$$

For every $\vartheta \in [t_0 - r, t_0 + \sigma]$*, define*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), \quad t_0 - r \leq \vartheta \leq t_0 \\ x(\vartheta)(\vartheta), \quad t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

Then $y(\vartheta)$ *is a solution of* (1.5) *in* $[t_0 - r, t_0 + \sigma]$ *and* $y(\vartheta) = x(t_0 + \sigma)(\vartheta)$ *for all* $\vartheta \in [t_0 - r, t_0 + \sigma]$*.*

Proposition 2.2 gives a one-to-one correspondence between the solutions y of (1.5) and the solutions x of (2.1). Thus given a solution y of (1.5), we have an x given by Proposition 2.2, (i), which satisfies equation (2.1). Therefore taking $t_0 \leq t_1 \leq t_2 \leq t_0 + \sigma$ we get

$$\begin{aligned} \|x(t_2) - x(t_1)\| &= \sup_{\vartheta \in [t_0 - r, t_0 + \sigma]} |x(t_2)(\vartheta) - x(t_1)(\vartheta)| \\ &= \sup_{\vartheta \in [t_0, t_0 + \sigma]} |x(t_2)(\vartheta) - x(t_1)(\vartheta)| \\ &= \sup_{\vartheta \in [t_1, t_2]} |y(\vartheta) - y(t_1)| \leq \mathrm{var}_{t_1}^{t_2} y \end{aligned}$$

and taking $t_0 < t_1 < t_2 < \ldots < t_k = t_0 + \sigma$ we get

$$\sum_{i=1}^{k} \|x(t_i) - x(t_{i-1})\| \leq \sum_{i=1}^{k} \mathrm{var}_{t_{i-1}}^{t_i} y = \mathrm{var}_{t_0}^{t_0+\sigma} y.$$

Hence

$$\mathrm{var}_{t_0}^{t_0+\sigma} x \leq \mathrm{var}_{t_0}^{t_0+\sigma} y.$$

It has to be noted that a solution y of (1.5) is a function of bounded variation and therefore the corresponding x is also of bounded variation.

Reciprocally, if G is given by (2.8) and $x\,(t)$ is a solution of (2.1) with the initial condition (2.9), then it can be shown by the procedure above that y given by Proposition 2.2, (ii), satisfies

$$\mathrm{var}_{t_0}^{t_0+\sigma} y \leq \mathrm{var}_{t_0}^{t_0+\sigma} x < +\infty.$$

In this manner, we have the situation of a one-to-one correspondence between the solutions of (1.5) and (2.1) and their variations (in different spaces) are the same and finite.

Remark 2.5. Let us note that in our paper [1] a similar approach to impulsive retarded functional equations was presented. Of course the definition in this case is slightly more complicated by an additional term. The complication is technical only, the reasoning of this note can be used similarly for this case, too. Again, the link between GODE's and classical systems with impulses as they are described in the book [7] of A.M. Samoĭlenko and N.A. Perestyuk is given in [8].

3 Concepts of stability for GODE's

In this section, $\Omega = B_c \times [t_0 - r, \infty)$, where $B_c = \{y \in X; \|y\| < c\}$, $c > 0$, and X is any Banach space. Let $r \geq 0$. In the sequel, we assume that for $F : \Omega \to X$ we have $F \in \mathcal{F}(\Omega, h)$ and $F(0,t) - F(0,s) = 0$, for $t, s \in [t_0 - r, +\infty)$. Then for every $[\gamma, v] \subset [t_0 - r, +\infty)$, we have

$$\int_{\gamma}^{v} DF(0,t) = F(0,v) - F(0,\gamma) = 0$$

and, therefore, $x \equiv 0$ is a solution of the generalized equation

$$\frac{dx}{d\tau} = DF(x,t) \tag{3.1}$$

on $[t_0 - r, +\infty)$.

If $F \in \mathcal{F}(\Omega, h)$ and $x : [\gamma, v] \to X$ is a solution of (3.1), where $[\gamma, v] \subset [t_0 - r, +\infty)$, then x is of bounded variation in $[\gamma, v]$. Thus it is natural to measure the distance between two solutions by the variation norm.

The next stability concepts are based on the variation of the solutions around $x \equiv 0$.

Definition 3.1. The solution $x \equiv 0$ of (3.1) is called *variationally stable* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 - r \le \gamma < v < +\infty$ is a function of bounded variation on $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta$$

and

$$\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t) \right) < \delta,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v].$$

Definition 3.2. The solution $x \equiv 0$ of (3.1) is called *variationally attracting* if there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \ge 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $t_0 - r \le \gamma < v < +\infty$, is a function of bounded variation in $[\gamma, v]$ such that

$$\|\overline{x}(\gamma)\| < \delta_0$$

and

$$\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t) \right) < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad \text{for } t \in [\gamma, v] \cap [\gamma + T, +\infty) \text{ and } \gamma \ge t_0 - r.$$

Definition 3.3. The solution $x \equiv 0$ of (3.1) is called *variationally asymptotically stable* if it is variationally stable and variationally attracting.

To Definitions 3.1-3.3, it should be noted that if $\overline{x} : [\gamma, v] \to X$ is a solution of (3.1) then:

(a) $\overline{x}$ is of bounded variation on $[\gamma, v]$ and

(b) $\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t) \right) = 0.$

Also, the conditions in Definition 3.1 mean that the function $\overline{x}$ of bounded variation is close (in the variation norm: $\|\overline{x}(\gamma)\| + \mathrm{var}(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t)))$ to the solution $x \equiv 0$ of (3.1).

Besides the generalized differential equation (3.1), let us consider the perturbed generalized equation

$$\frac{dx}{d\tau} = D[F(x, t) + \overline{P}(t)] \tag{3.2}$$

where $\overline{P} \in BV^-([t_0 - r, \infty), X)$. It is easy to verify that for the function $G(x, t) = F(x, t) + \overline{P}(t)$ we have $G \in \mathcal{F}(\Omega, h_{\overline{P}})$, where $h_{\overline{P}}(t) = h(t) + \mathrm{var}_{-r}^t \overline{P}$. Therefore the solutions of (3.2) have good properties (existence, uniqueness, etc., see Theorem 2.4, for instance).

Let us present now some other definitions.

Definition 3.4. The solution $x \equiv 0$ of (3.1) is called *stable with respect to perturbations* if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta$ and $\overline{P} \in BV^-([\gamma, v], X)$ is continuous from the left with $\text{var}_\gamma^v \overline{P} < \delta$ then

$$\|x(t, \gamma, x_0)\| < \varepsilon \ \text{ for every } \ t \in [\gamma, v]$$

where $x(t, \gamma, x_0)$ is a solution of the perturbed generalized equation (3.2) with $x(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0 - r, +\infty)$.

Definition 3.5. The solution $x \equiv 0$ of (3.1) is called *attracting with respect to perturbations* if there is a $\widetilde{\delta} > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if

$$\|x_0\| < \widetilde{\delta} \qquad \text{and} \qquad \text{var}_\gamma^v \overline{P} < \rho$$

with $\overline{P} \in BV^-([\gamma, v], X)$, then

$$\|x(t, \gamma, x_0)\| < \varepsilon \ \text{ for all } \ t \geq \gamma + T, \ t \in [\gamma, v]$$

where $x(t, \gamma, x_0)$ is a solution of the perturbed generalized equation (3.2) with $x(\gamma, \gamma, x_0) = x_0$ and $[\gamma, v] \subset [t_0 - r, +\infty)$.

Definition 3.6. The solution $x \equiv 0$ of (3.1) is called *asymptotically stable with respect to perturbations* if it is both stable and attracting with respect to perturbations.

It turns out that the respective definitions presented above are equivalent. Indeed, we have the following result.

Proposition 3.1. *The following statements hold.*

(i) The solution $x \equiv 0$ of (3.1) *is variationally stable if and only if it is stable with respect to perturbations.*

(ii) The solution $x \equiv 0$ of (3.1) *is variationally attracting if and only if it is attracting with respect to perturbations.*

(iii) The solution $x \equiv 0$ of (3.1) *is variationally asymptotically stable if and only if it is asymptotically stable with respect to perturbations.*

Proof. Let us prove (i). Assume that the solution $x \equiv 0$ of (3.1) is variationally stable. Let for $\varepsilon > 0$ the quantity $\delta > 0$ be given according to Definition 3.4. Then for the solution $x(t) = x(t, \gamma, x_0)$ of the perturbed generalized equation (3.2) on $[\gamma, v]$, we have $\|x(\gamma)\| = \|x(\gamma, \gamma, x_0)\| < \delta$ and for any $s_1, s_2 \in [\gamma, v]$ we get

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t) + \overline{P}(s_2) - \overline{P}(s_1),$$

that is,

$$x(s_2) - \int_{\gamma}^{s_2} DF(x(\tau), t) - \left(x(s_1) - \int_{\gamma}^{s_1} DF(x(\tau), t) \right) = \overline{P}(s_2) - \overline{P}(s_1)$$

and this implies

$$\mathrm{var}_{\gamma}^{v} \left(x(s) - \int_{\gamma}^{s} DF(x(\tau), t) \right) = \mathrm{var}_{\gamma}^{v} \overline{P} < \delta.$$

Therefore the variational stability implies

$$\|x(t)\| = \|x(t, \gamma, x_0)\| < \varepsilon \;\text{ for }\; t \in [\gamma, v]$$

and the trivial solution of (3.1) is stable with respect to perturbations.

Reciprocally, if the solution $x \equiv 0$ of (3.1) is stable with respect to perturbations, take $\overline{x} : [\gamma, v] \to B_c$, $-r \leq \gamma < v < +\infty$, a function of bounded variation on $[\gamma, v]$ such that $\|\overline{x}(\gamma)\| < \delta$ and

$$\mathrm{var}_{\gamma}^{v} \left(\overline{x}(s) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right) < \delta,$$

where $\delta > 0$ corresponds to some $\varepsilon > 0$ from Definition 3.4.

For $s \in [\gamma, v]$, let $\overline{P}(s) = \overline{x}(s) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t)$. Then for $s_1, s_2 \in [\gamma, v]$, we have

$$\overline{P}(s_2) - \overline{P}(s_1) = \overline{x}(s_2) - \overline{x}(s_1) - \int_{s_1}^{s_2} DF(\overline{x}(\tau), t).$$

Hence

$$\overline{x}(s_2) - \overline{x}(s_1) = \int_{s_1}^{s_2} DF(\overline{x}(\tau), t) + \overline{P}(s_2) - \overline{P}(s_1), \quad s_1, s_2 \in [\gamma, v],$$

which means that $\overline{x}$ is a solution of (3.2) in $[\gamma, v]$. Besides, $\mathrm{var}_{\gamma}^{v} \overline{P} < \delta$, $\overline{P}$ is left continuous and $\|\overline{x}(\gamma)\| = \|\overline{x}(\gamma, \gamma, x_0)\| = \|\overline{P}(\gamma)\| < \delta$. Therefore the stability with respect to perturbations implies $\|\overline{x}(t)\| = \|\overline{x}(t, \gamma, x_0)\| < \varepsilon$, for all $t \in [\gamma, v]$, and this means that the solution $x \equiv 0$ of (3.1) is variationally stable.

Coming to the attractive part in item (ii), assume first that the solution $x \equiv 0$ of (3.1) is variationally attracting. Then there is a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that if $\overline{x} : [\gamma, v] \to B_c$, $-r \leq \gamma < v < +\infty$, is a function of bounded variation in $[\gamma, v]$ such that $\|\overline{x}(\gamma)\| < \delta_0$ and

$$\mathrm{var}_{\gamma}^{v} \left(\overline{x}(s) - \int_{\gamma}^{s} DF(\overline{x}(\tau), t) \right) < \rho,$$

then

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty),\ \gamma \geq -r.$$

If $\|x_0\| < \widetilde{\delta}$ and $\overline{P} \in BV^-([\gamma, v], X)$ is such that $\mathrm{var}_\gamma^v \overline{P} < \rho$, then denote $x(t) = x(t, \gamma, x_0)$ the solution of the perturbed generalized equation (3.2) satisfying $x(\gamma, \gamma, x_0) = x_0$. It follows that $\|x(\gamma)\| < \widetilde{\delta}$ and we have

$$\mathrm{var}_\gamma^v \left(x(s) - \int_\gamma^s DF(x(\tau), t) \right) = \mathrm{var}_\gamma^v \overline{P} < \delta.$$

Hence by Definition 3.2, we get

$$\|x(t, \gamma, x_0)\| = \|x(t)\| < \varepsilon \ \text{ for all } \ t \geq \gamma + T,\ t \in [\gamma, v],$$

that is, the solution $x \equiv 0$ of (3.1) is attracting with respect to perturbations.

Reciprocally, if the solution $x \equiv 0$ of (3.1) is attracting with respect to perturbations, suppose $\overline{x} : [\gamma, v] \to B_c$, $-r \leq \gamma < v < +\infty$, is a left continuous function of bounded variation in $[\gamma, v]$ and such that $\|\overline{x}(\gamma)\| < \delta_0$ and

$$\mathrm{var}_\gamma^v \left(\overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t) \right) < \rho.$$

As in the previous part of the proof, it is easy to see that $\overline{x}(t)$ is a solution of (3.2) on $[\gamma, v]$, where $\overline{P}(s) = \overline{x}(s) - \int_\gamma^s DF(\overline{x}(\tau), t)$ for $s \in [\gamma, v]$. This function $\overline{P}$ belongs to $BV^-([\gamma, v], X)$ and there exists a $\delta_0 > 0$ and for every $\varepsilon > 0$, there exist a $T = T(\varepsilon) \geq 0$ and a $\rho = \rho(\varepsilon) > 0$ such that $\mathrm{var}_\gamma^v \overline{P} < \rho$. Definition 3.5 now yields

$$\|\overline{x}(t)\| < \varepsilon, \quad t \in [\gamma, v] \cap [\gamma + T, +\infty),\ \gamma \geq -r,$$

which means that we have the variational attractivity of the trivial solution of (3.1).

Item (iii) follows from (i) and (ii) and we finished the proof. □

4 Stability relations between the equations

Consider the retarded system (1.1). Let $G_1 \subset G([t_0 - r, +\infty), \mathbb{R}^n)$ be defined as in the beginning of the paper.

We assume that $f(\phi, t) : G_1 \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for each $y \in G_1$ the mapping $t \mapsto f(y_t, t)$ belongs to $L_1([t_0 - r, +\infty), \mathbb{R}^n)$ and conditions (A) and (B) are fulfilled. Suppose in addition that $f(0, t) = 0$ for every $t \in [t_0, +\infty)$. Thus $y \equiv 0$ is a solution of (1.1) in $[t_0 - r, +\infty)$.

For $y \in G_1$ and $t \in [t_0 - r, +\infty)$, define $F(y, t)$ as in (2.6). Then

$$F : G_1 \times [t_0 - r, +\infty) \to C([t_0 - r, +\infty), \mathbb{R}^n)$$

and by definition we have $F(0,t)=0$, for all $t\in[t_0-r,+\infty)$. Then $x\equiv 0$ is a solution of the generalized differential equation

$$\frac{dx}{d\tau}=DF(x,t) \tag{4.1}$$

in $[t_0-r,+\infty)$.

By the results from Proposition 2.2, there is a well described one-to-one correspondence between solutions of equations (1.1) and (4.1) with F given by (2.6).

We will also consider the perturbed retarded equation (1.5) and, again by Proposition 2.2, its corresponding perturbed generalized equation

$$\frac{dx}{d\tau}=DG(x,t)=D[F(x,t)+\overline{P}(t)], \tag{4.2}$$

where F is given by (2.6) and $\overline{P}$ given by (2.7).

We have

$$\overline{P}:[t_0-r,+\infty)\to G^-([t_0-r,+\infty),\mathbb{R}^n).$$

and then

$$G: G_1\times[t_0-r,+\infty)\to G^-([t_0-r,+\infty),\mathbb{R}^n).$$

We are now able to present a result which relates the respective concepts of variational stability and variational attractivity of the trivial solution of the retarded equation (1.1) and the trivial solution of its corresponding generalized equation (4.1).

Theorem 4.1. *The following statements hold.*

(i) *The solution $y\equiv 0$ of* (1.1) *is variationally stable if and only if the solution $x\equiv 0$ of* (4.1) *is variationally stable.*

(ii) *The solution $y\equiv 0$ of* (1.1) *is variationally attracting if and only if the solution $x\equiv 0$ of* (4.1) *is variationally attracting.*

(iii) *The solution $y\equiv 0$ of* (1.1) *is variationally asymptotically stable if and only if the solution $x\equiv 0$ of* (4.1) *is variationally asymptotically stable.*

Proof. We start by proving (i). Suppose the trivial solution of (1.1) in $[t_0-r,+\infty)$ is variationally stable. Then given $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that if $\phi\in G^-([-r,0],\mathbb{R}^n)$ is such that $\|\phi\|<\delta$ and $P(t)$ belongs to $BV^-([t_0,t_1],\mathbb{R}^n)$ with $\mathrm{var}_{t_0}^{t_1}P<\delta$, then

$$|y(t;t_0,\phi)|<\frac{\varepsilon}{2},\quad t\in[t_0,t_1],$$

where $y(t;t_0,\phi)$ is a solution of (1.5).

We want to prove that the trivial solution of generalized equation (4.1), with F given by (2.6), is stable with respect to perturbations. Then the result will follow by Proposition 3.1.

Suppose $\delta = \delta(\varepsilon) > 0$ from Definition 1.6 is such that $\delta < \varepsilon/2$. Let $x(t; t_0, x_0)$ be a solution of the perturbed generalized equation (4.2) with F given by (2.6), $\overline{P}$ given by (2.7) and $x(t_0; t_0, x_0) = x_0$ and assume that $\|x_0\| < \delta$, where $x_0 \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, and $\overline{P} \in BV^-([t_0, t_1], \mathbb{R}^n)$ with $\mathrm{var}_{t_0}^{t_1} \overline{P} < \delta$.

We have $\|x(t_0)\| = \|x(t_0; t_0, x_0)\| = \|x_0\| < \delta$ which means that $\sup_{\theta \in [t_0 - r, +\infty)} |x(t_0)(\theta)| < \delta$ and therefore $\sup_{\theta \in [t_0 - r, t_0]} |\phi(\theta - t_0)| < \delta$. Thus

$$\|\phi\| < \delta.$$

Since x is a solution of the perturbed generalized equation we have

$$\begin{aligned} x(s_2) - x(s_1) &= \int_{s_1}^{s_2} D\left[F(x(\tau), t) + \overline{P}(t)\right] \\ &= \int_{s_1}^{s_2} DF(x(\tau), t) + \overline{P}(s_2) - \overline{P}(s_1) \end{aligned}$$

for $s_1, s_2 \in [t_0, t_1]$.

Therefore

$$x(s_2) - \int_{t_0}^{s_2} DF(x(\tau), t) - x(s_1) + \int_{t_0}^{s_1} DF(x(\tau), t) = \overline{P}(s_2) - \overline{P}(s_1).$$

Hence

$$\mathrm{var}_{t_0}^{t_1} \left(x(s) - \int_{t_0}^{s} DF(x(\tau), t) \right) = \mathrm{var}_{t_0}^{t_1} \overline{P} < \delta.$$

Thus by the variational stability of the trivial solution of (1.1), $|y(t)| < \varepsilon/2$, for all $t \in [t_0, t_1]$.

Finally, we have

$$\begin{aligned} \|x(t)\| &= \sup_{\theta \in [t_0 - r, +\infty)} |x(t)(\theta)| = \sup_{\theta \in [t_0 - r, t]} |y(\theta)| \\ &\leq \|\phi\| + \sup_{\theta \in [t_0, t]} |y(\theta)| \leq \delta + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

and we have the sufficiency of item (i).

Now, using (i) from Proposition 3.1, we assume that the trivial solution of (4.1) is stable with respect to perturbations. Thus given $\varepsilon > 0$, let $\delta = \delta(\varepsilon) > 0$ be the quantity from Definition 3.4.

Let $y(t; t_0, \phi)$ be a solution of the perturbed retarded equation (1.5). Suppose $\|\phi\| < \delta$ and $P \in BV^-([t_0, t_1], \mathbb{R}^n)$ with $\mathrm{var}_{t_0}^{t_1} P < \delta$. We want to prove that $y \equiv 0$ is variationally stable, that is, $|y(t; t_0, \phi)| < \varepsilon$, $t \in [t_0, t_1]$. Then the reciprocal of item (i) (necessity) will follow by Proposition 3.1.

Let $x(t; t_0, x_0)$ be the solution of the perturbed generalized equation (4.2) with F given by (2.6) and $\overline{P}$ given by (2.7), that is, x is the solution corresponding to y obtained according to Proposition 2.2. We have $\mathrm{var}_{t_0}^{t_1} x \le \mathrm{var}_{t_0}^{t_1} y$ (see the comments after Proposition 2.2) and, analogously, $\mathrm{var}_{t_0}^{t_1} \overline{P} \le \mathrm{var}_{t_0}^{t_1} P < \delta$. Thus, from the stability with respect to perturbations of the trivial solution of (4.1), $\|x(t)\| < \varepsilon$, that is,

$$\sup_{\theta \in [t_0 - r, +\infty)} |x(t)(\theta)| < \varepsilon.$$

Therefore the relation in Proposition 2.2 implies

$$\sup_{\theta \in [t_0 - r, t]} |y(\theta)| < \varepsilon, \quad t \in [t_0, t_1].$$

In particular,

$$\sup_{\theta \in [t_0, t_1]} |y(\theta)| \le \sup_{\theta \in [t_0 - r, t_1]} |y(\theta)| < \varepsilon.$$

Now we will prove (ii).

At first, suppose the trivial solution of the retarded equation (1.1) is variationally attracting. Thus there exists $\delta_0 > 0$ and for every $\varepsilon > 0$, let $T = T(\varepsilon) \ge 0$ and $\rho = \rho(\varepsilon) > 0$ be from Definition 1.7.

Let $x(t; t_0, x_0)$ be a solution of the perturbed generalized equation (4.2) with F given by (2.6) and $\overline{P}$ given by (2.7) and let $y(t; t_0, \phi)$ be the solution of the perturbed retarded equation (1.5) obtained from x according to Proposition 2.2.

Let $\delta > 0$ be such that $\|x_0\| < \delta$ and suppose $\overline{P} \in BV^{-}([t_0, t_1], \mathbb{R}^n)$ with $\mathrm{var}_{t_0}^{t_1} \overline{P} < \rho$. We can suppose, without loss of generality, that $\delta < \min\{\delta_0, \rho, \varepsilon/2\}$. Then

$$\|x(t_0)\| = \|x_0\| < \delta_0$$

and

$$\mathrm{var}_{t_0}^{t_1} \left(x(s) - \int_{t_0}^{s} DF(x(\tau), t) \right) = \mathrm{var}_{t_0}^{t_1} \overline{P} < \rho.$$

But the variational attractivity of the trivial solution of (1.1) implies

$$|y(t; t_0, \phi)| = |y(t)| < \varepsilon/2, \quad t \ge t_0 + T,\ t \in [t_0, t_1].$$

Then, taking $\delta < \varepsilon/2$, we obtain

$$\begin{aligned} \|x(t)\| &= \sup_{\theta \in [t_0 - r, +\infty)} |x(t)(\theta)| = \sup_{\theta \in [t_0 - r, t]} |y(\theta)| \\ &\le \|\phi\| + \sup\nolimits_{\theta \in [t_0, t]} |y(\theta)| < \|x_0\| + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

for $t \ge t_0 + T$, $t \in [t_0, t_1]$, where we applied the relations of Proposition 2.2 to get the second equality and $\|\phi\| = \|x_0\|$, since

$$\begin{aligned} \|x_0\| &= \|x(t_0)\| = \sup_{\theta \in [t_0 - r, +\infty)} |x(t)(\theta)| \\ &= \sup_{\theta \in [t_0 - r, t_0]} |\phi(\theta - t_0)| = \|\phi\|. \end{aligned}$$

Thus

$$\|x(t;t_0,x_o)\| = \|x(t)\| < \varepsilon, \quad t \geq t_0+T,\ t \in [t_0,t_1],$$

and hence x is attracting with respect to perturbations. The sufficiency of (ii) follows then by Proposition 3.1.

Now we will prove the reciprocal of item (ii). Suppose then that the trivial solution of generalized equation (4.1) is attracting with respect to perturbations. Then there exists $\delta_0 > 0$ and given $\varepsilon > 0$ let $T = T(\varepsilon) \geq 0$ and $\rho = \rho(\varepsilon) > 0$ be from Definition 3.5.

Let $y(t;t_0,\phi)$ be a solution of the perturbed retarded equation (1.3), or equivalently, of equation (1.5) with $P(t) = \int_{t_0}^t p(s)ds$, $t \geq t_0$. Suppose $\|\phi\| < \delta_0$ and $P \in BV^-([t_0,t_1],\mathbb{R}^n)$ with $\text{var}_{t_0}^{t_1} P < \rho$.

By Proposition 2.2, it follows that $\|x_0\| = \|\phi\| < \delta_0$. Also, for $\overline{P}$ given by (2.7), we have $\text{var}_{t_0}^{t_1}\overline{P} \leq \text{var}_{t_0}^{t_1} P < \rho$ (see the comments after Proposition 2.2. Therefore the attractivity with respect to perturbations of the trivial solution of (4.1) implies

$$\|x(t)\| = \|x(t;t_0,x_0)\| < \varepsilon, \quad t \geq t_0+T,\ t \in [t_0,t_1].$$

Therefore, for $t \geq t_0+T$, $t \in [t_0,t_1]$, we have by Proposition 2.2,

$$|y(t)| = |y(t;t_0,\phi)| = |x(t)(t)| \leq \|x(t)\| < \varepsilon.$$

Assertion (iii) follows from (i) and (ii) and from Proposition 3.1. □

5 Converse Lyapunov theorems

In the book [8] and in [9] direct Lyapunov-type theorems for stability of a solution of a GODE are given. In [2] they are used for equation (1.1).

Converse Lyapunov theorems are an interesting topic, we present them shortly in this concluding section of the paper.

In order to obtain converse Lyapunov theorems for equation (1.1), we need the following results, borrowed from [8] or [9], for the generalized differential equation (4.1).

Let us consider the general case where $\Omega = B_c \times [t_0 - r, \infty)$, with $B_c = \{y \in X; \|y\| < c\}$, $c > 0$, and X is a Banach space. Suppose $F : \Omega \to X$ is such that $F \in \mathcal{F}(\Omega,h)$ and $F(0,t) - F(0,s) = 0$, for $t,s \in [t_0 - r, +\infty)$ and consider the generalized differential equation

$$\frac{dx}{d\tau} = DF(x,t). \tag{5.1}$$

The following two results are respectively Theorems 10.23 and 10.24 from [8]. They can also be found in [9].

Theorem 5.1. *If the trivial solution $x \equiv 0$ of the generalized differential equation* (5.1) *is variationally stable, then for every $0 < a < c$, there exists a function $V : [t_0 - r, +\infty) \times B_a \to \mathbb{R}$, where $\overline{B_a} = \{y \in X; \|y\| < a\}$, such that for every $x \in B_a$, the function $V(\cdot,x)$ belongs to $BV^-([t_0 - r, +\infty),\mathbb{R})$ and the following conditions hold:*

(i) $V(t,0)=0$, $t\in[t_0-r,+\infty)$;

(ii) $|V(t,z)-V(t,y)|\le\|z-y\|$, $t\in[t_0-r,+\infty)$, $z,y\in B_a$.

(iii) V *is positive definite along every solution* $x(t)$ *of the generalized equation* (5.1), *that is, there is a function* $b:[0,+\infty)\to\mathbb{R}$ *of Hahn class such that*

$$V(t,x(t))\ge b(\|x(t)\|),\quad (t,x(t))\in[t_0-r,+\infty)\times B_a;$$

(iv) *for all solutions* $x(t)$ *of* (5.1),

$$\dot V(t,x(t))=\limsup_{\eta\to 0_+}\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\le 0,$$

that is, the right derivative of V *along every solution* $x(t)$ *of* (5.1) *is non-positive.*

Theorem 5.2. *If the trivial solution* $x\equiv 0$ *of the generalized differential equation* (5.1) *is variationally asymptotically stable, then for every* $0<a<c$, *there exists a function* $V:[t_0-r,+\infty)\times B_a\to\mathbb{R}$ *such that for every* $x\in B_a$, *the function* $V(\cdot,x)$ *belongs to* $BV^-([t_0-r,+\infty),\mathbb{R})$ *and the following conditions hold:*

(i) $V(t,0)=0$, $t\in[t_0-r,+\infty)$;

(ii) $|V(t,z)-V(t,y)|\le\|z-y\|$, $t\in[t_0-r,+\infty)$, $z,y\in B_a$.

(iii) V *is positive definite along every solution* $x(t)$ *of the generalized equation* (5.1), *that is, there is a function* $b:[0,+\infty)\to\mathbb{R}$ *of Hahn class such that*

$$V(t,x(t))\ge b(\|x(t)\|),\quad (t,x(t))\in[t_0-r,+\infty)\times B_a;$$

(iv) *for all solutions* $x(s)$ *of* (5.1) *defined for* $s\ge t$, *where* $x(t)=z\in B_a$, *the relation*

$$\dot V(t,x(t))=\limsup_{\eta\to 0_+}\frac{V(t+\eta,x(t+\eta))-V(t,x(t))}{\eta}\le V(t,z)$$

holds.

Now let us consider the more specialized equation (4.1), with F given by (2.6), corresponding to the retarded system (1.1). We consider $X=G^-([t_0-r,+\infty),\mathbb{R}^n)$. As in [2], we need to relate a Lyapunov functional for (4.1) to a Lyapunov functional for (1.1)

Let $y:[\gamma,v]\to\mathbb{R}^n$ be a solution of equation (1.1) on $[\gamma,v]\subset[t_0-r,+\infty)$, $[\gamma,v]\ni t_0$, such that $y_t=\psi$ for a given $t\ge t_0$, that is, $\psi\in G^-([-r,0],\mathbb{R}^n)$ and

$$\psi(\theta)=y_t(\theta)=y(t+\theta)=y(t)-\int_{[t+\theta,t]}f(y_s,s)\,ds$$

for almost every $\theta \in [-r, 0]$. In this case, we write $y_{t+\eta} = y_{t+\eta}(t, \psi)$ for every $\eta \geq 0$. Then if $U : [t_0 - r, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$, we define

$$D^+U(t, \psi) = \limsup_{\eta \to 0_+} \frac{U(t + \eta, y_{t+\eta}(t, \psi)) - U(t, y_t(t, \psi))}{\eta},$$

for $t \geq t_0$.

Let x be a solution of the generalized equation (4.1) on the interval $[\gamma, v] \subset [t_0 - r, +\infty)$, $[\gamma, v] \ni t_0$, with initial condition $x(t_0) = x_0$, where

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0) & \text{for } \vartheta \in [t_0 - r, t_0], \\ x(t_0)(t_0) & \text{for } \vartheta \in [t_0, +\infty). \end{cases} \tag{5.2}$$

Then $x(t)(t+\theta) = y(t+\theta)$, for all $t \in [t_0 - r, +\infty)$ and all $\theta \in [-r, 0]$ and hence $(x(t))_t = y_t$ for all t.

On the other hand, given $x(t) \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, since x is locally of bounded variation, we can consider $x(t)$ as a solution on $[\gamma, v] \subset [t_0 - r, +\infty)$, $[\gamma, v] \ni t_0$, of the generalized equation (4.1), with initial condition $x(t_0) = x_0$ given by (5.2). Then Proposition 2.2 implies we can find a solution $y(t; t_0, \phi)$ of (1.1) by means of the solution $x(t; t_0, x_0)$ of (4.1). Suppose $(x(t))_t = \psi$. In this case, we write $x_\psi(t)$ instead of $x(t)$ and we have $y_t = \psi$.

Therefore $(t, x_\psi(t)) \mapsto (t, y_t(t, \psi))$ is a one-to-one mapping and we can define $V : [t_0 - r, +\infty) \times G^-([t_0 - r, +\infty), \mathbb{R}^n) \to \mathbb{R}$ by

$$V(t, x_\psi(t)) = U(t, y_t(t, \psi)), \quad t \geq t_0. \tag{5.3}$$

Then we have

$$D^+U(t, \psi) = \limsup_{\eta \to 0_+} \frac{V(t + \eta, x_\psi(t + \eta)) - V(t, x_\psi(t))}{\eta}$$

for all $t \geq t_0$. We write $\dot{U}(t, y_t) = D^+U(t, y_t)$.

With the notation above, we now are able to present converse Lyapunov results for equation (1.1).

Theorem 5.3. *If the trivial solution $y \equiv 0$ of the retarded differential equation* (1.1) *is variationally stable, then for every $0 < a < c$, there exists a function $U : [t_0 - r, +\infty) \times E_a \to \mathbb{R}$, where $\overline{E_a} = \{\psi \in G^-([-r, 0], \mathbb{R}^n); \|\psi\| < a\}$, such that for every $x \in E_a$, the function $U(\cdot, \psi)$ belongs to $BV^-([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:*

(i) $U(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;

(ii) $|U(t, \psi) - U(t, \overline{\psi})| \leq \|\psi - \overline{\psi}\|$, $t \in [t_0 - r, +\infty)$, $\psi, \overline{\psi} \in E_a$.

(iii) *U is positive definite along every solution $y(t)$ of the retarded equation (1.1), that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that*

$$U(t, y_t) \geq b(\|y_t\|), \quad (t, y_t) \in [t_0 - r, +\infty) \times E_a;$$

(iv) *for all solutions $y(t)$ of (1.1),*

$$\dot{U}(t, y_t) = \limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}) - V(t, y_t))}{\eta} \leq 0,$$

that is, the right derivative of U along every solution $y(t)$ of (1.1) is non-positive.

Proof. If the trivial solution of (1.1) is variationally stable, then by Theorem 4.1 the trivial solution of the generalized equation (4.1) with F given by (2.6) and $\overline{P}$ is given by (2.7) is also variationally stable. Then by Theorem 5.1, there exists a function V satisfying all conditions in that theorem. Define $U : [t_0 - r, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ by the relation in (5.3). Then as in the proof of [2], Theorem 4.3, U has the properties above and the proof is complete. □

The proof of the next result follows as in the proof of Theorem 5.3, but applying [2], Theorem 4.5 instead of [2], Theorem 4.3.

Theorem 5.4. *If the trivial solution $y \equiv 0$ of the retarded differential equation (1.1) is variationally asymptotically stable, then for every $0 < a < c$, there exists a function $U : [t_0 - r, +\infty) \times E_a \to \mathbb{R}$ such that for every $x \in B_a$, the function $V(\cdot, x)$ belongs to $BV^-([t_0 - r, +\infty), \mathbb{R})$ and the following conditions hold:*

(i) *$U(t, 0) = 0$, $t \in [t_0 - r, +\infty)$;*

(ii) *$|U(t, \psi) - U(t, \overline{\psi})| \leq \|\psi - \overline{\psi}\|$, $t \in [t_0 - r, +\infty)$, $\psi, \overline{\psi} \in E_a$.*

(iii) *U is positive definite along every solution $y(t)$ of the retarded equation (1.1), that is, there is a function $b : [0, +\infty) \to \mathbb{R}$ of Hahn class such that*

$$U(t, y_t) \geq b(\|y_t\|), \quad (t, y_t) \in [t_0 - r, +\infty) \times E_a;$$

(iv) *for all solutions $y(s)$ of (1.1) defined for $s \geq t$, where $y(t) = \psi \in E_a$, the relation*

$$\dot{U}(t, y_t) = \limsup_{\eta \to 0_+} \frac{U(t+\eta, y_{t+\eta}) - U(t, y_t)}{\eta} \leq U(t, \psi)$$

holds.

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