Preprint, Institute of Mathematics, AS CR, Prague. 2008-5-12

On EP elements in a C^*-algebra

D. S. Djordjević, J. J. Koliha and and I. Straškraba*

Abstract

We give several characterizations of EP elements in C^*-algebras. The motivation for the factorization results comes in part from a recent paper by Drivaliaris, Karanasios and Pappas on Hilbert space operators.

MSC: 46L05, 47A05
Keywords: EP elements, Moore-Penrose inverse, factorizations

1 Results in a C^*-algebra

Several authors in [1, 6, 7] have recently addressed themselves to the question of characterizing EP operators on Hilbert spaces or characterizing EP elements in C^*-algebras, and Boasso [3] considered this question in Banach spaces and algebras. Basic facts about EP matrices and operators can be found in [2, 4, 5].

Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is *regular* if $a \in a\mathcal{A}a$; any element $x \in \mathcal{A}$ satisfying $axa = a$ is called a *generalized inverse* of a. By $\mathcal{A}^{\mathsf{inv}}$ and $\mathcal{A}^{\mathsf{reg}}$ we denote the set of all invertible and regular elements of $\mathcal{A}$, respectively. We note that a is regular if and only if a^* is regular. A special case of a generalized inverse of $a \in \mathcal{A}$ is the *Moore–Penrose inverse*, written $a^\dagger$, which satisfies three additional conditions

$$a^\dagger a a^\dagger = a^\dagger, \qquad (a^\dagger a)^* = a^\dagger a, \qquad (aa^\dagger)^* = aa^\dagger.$$

The paper [9] gives a good account of the Moore-Penrose inverse in C^*-algebras. In particular it proves that

$$a \text{ is regular} \iff a\mathcal{A} \text{ is closed} \iff a \text{ is Moore–Penrose invertible.}$$

*Partially supported by the Institutional Research Plan AV OZ 10190503 of the Czech Academy of Sciences and by the grant of the Czech Grant Agency no. 201/108/0012.

For $a \in \mathcal{A}$ we define two annihilators

$$a^\circ = \{x \in \mathcal{A} : ax = 0\}, \quad {}^\circ a = \{x \in \mathcal{A} : xa = 0\}.$$

In this paper we state the results for the annihilators of the type a°, and for the cosets of the type $a\mathcal{A}$. The results for the other types of annihilators and cosets can be obtained from the symmetry relations

$$(a^*)^\circ = a^\circ \iff {}^\circ(a^*) = {}^\circ a, \qquad a\mathcal{A} = a^*\mathcal{A} \iff \mathcal{A}a = \mathcal{A}a^*.$$

In the following lemma we summarize some well known facts for a future reference. A proof is given for completeness.

Lemma 1.1. *The following are true for $a \in \mathcal{A}$.*

(i) $a \in \mathcal{A}^{\mathsf{inv}} \iff a\mathcal{A} = \mathcal{A}$ *and* $a^\circ = \{0\}$.

(ii) $a \in \mathcal{A}^{\mathsf{reg}} \implies \mathcal{A} = (a^*\mathcal{A}) \oplus a^\circ$.

(iii) $a^*\mathcal{A} = \mathcal{A} \iff a \in \mathcal{A}^{\mathsf{reg}}$ *and* $a^\circ = \{0\}$.

Proof. (i) If a is invertible, then clearly $\mathcal{A} = a\mathcal{A}$ and $a^\circ = \{0\}$. Conversely, let $\mathcal{A} = a\mathcal{A}$ and $a^\circ = \{0\}$. Define the left regular representation $L_a \colon \mathcal{A} \to \mathcal{A}$ by $L_a(x) = ax$ for all $x \in \mathcal{A}$. Then L_a is a bijective bounded linear operator on the Banach space $\mathcal{A}$, and thus invertible. Let $b = L_a^{-1}(1)$. Then $ab = L_a L_a^{-1}(1) = 1$. Further, $L_a L_b = L_{ab} = L_1 = I$, that is, $L_b = L_a^{-1}$. Hence $ab = 1 = ba$.

(ii) If a is regular, the Moore–Penrose inverse $a^\dagger$ exists, and

$$a^\dagger a\mathcal{A} = a^*\mathcal{A}, \qquad (1 - a^\dagger a)\mathcal{A} = a^\circ. \tag{1.1}$$

The result follows from $\mathcal{A} = a^\dagger a\mathcal{A} \oplus (1 - a^\dagger a)\mathcal{A}$. The converse need not hold (the left and right shift on ℓ^2).

(iii) If $a^*\mathcal{A} = \mathcal{A}$, then a^* is regular, and so is a. There is $u \in \mathcal{A}$ with $1 = a^*u$. If $ax = 0$, then $x^* = x^*a^*u = (ax)^*u = 0$, and $x = 0$. The converse follows from part (ii). $\square$

Definition 1.2. An element $a \in \mathcal{A}$ is said to be *EP* if $a \in \mathcal{A}^{\mathsf{reg}}$ and $a\mathcal{A} = a^*\mathcal{A}$ (or, equivalently, if $a \in \mathcal{A}^{\mathsf{reg}}$ and $a^\circ = (a^*)^\circ$).

The condition $a\mathcal{A} = a^*\mathcal{A}$ gave the EP elements their name for *equal projections* onto the range of a and a^* in the case of matrices and closed range Hilbert space operators. The set of all EP elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{\mathsf{ep}}$. There are many equivalent characterizations of EP elements in a C^*-algebra (see, for instance, [9, 10, 11]), many more still for Hilbert space operators and matrices. We mention only one well known characterization relevant to our present enquiry (see, for instance, [11]).

Lemma 1.3. *An element a of a C^*-algebra is EP if and only if it is regular and commutes with its Moore–Penrose inverse.*

Proof. Let a be regular. If $a^\dagger$ commutes with a, then $a^*\mathcal{A} = a^\dagger a\mathcal{A} = aa^\dagger\mathcal{A} = (a^*)^\dagger a^*\mathcal{A} = a\mathcal{A}$. Conversely, if a is EP, then $a\mathcal{A} = a^*\mathcal{A}$, and so $\mathcal{A}a = \mathcal{A}a^*$. Thus $a \in a^\dagger\mathcal{A} \cap \mathcal{A}a^\dagger$ in view of (1.1). Let $a = ua^\dagger$. Then $a - a^2a^\dagger = u(a^\dagger - a^\dagger aa^\dagger) = 0$. Let $a = a^\dagger v$. Then $a - a^\dagger a^2 = (a^\dagger - a^\dagger aa^\dagger)v = 0$. Thus $a^2a^\dagger = a = a^\dagger a^2$, and $a^\dagger a = a^\dagger(a^2a^\dagger) = (a^\dagger a^2)a^\dagger = aa^\dagger$. □

Our first task is to characterize EP elements in terms of the existence of projections; a *projection* in a C^*-algebra is an element $p \in \mathcal{A}$ satisfying $p^2 = p = p^*$.

Theorem 1.4. *An element $a \in \mathcal{A}$ is EP if and only if there exists a projection $p \in \mathcal{A}$ such that*

$$pa = a = ap, \qquad a \in (p\mathcal{A}p)^{\mathsf{inv}}. \tag{1.2}$$

Proof. Suppose that p is such a projection. Let q be the inverse of a in the C^*-algebra $p\mathcal{A}p$; then $aq = p = qa$, and q is a generalized inverse of a as $aqa = ap = a$. In fact, q is the Moore–Penrose inverse of a: We have $qaq = qp = q$ as $q \in p\mathcal{A}p$, and $(aq)^* = p^* = p = aq$; since a, q commute, $(qa)^* = qa$. Hence $q = a^\dagger$, and $aa^\dagger = a^\dagger a$. By Lemma 1.3, a is EP.

The converse follows on setting $p = a^\dagger a$. □

In the preceding theorem we can express $a^\dagger$ in terms of the projection p and ordinary inverse in $\mathcal{A}$:

$$a^\dagger = q = (a + 1 - p)^{-1}p \tag{1.3}$$

using the relation between the ordinary and $p\mathcal{A}p$ inverses; it is known that pap is invertible in $p\mathcal{A}p$ if and only if $pap + 1 - p$ is invertible in $\mathcal{A}$.

We now turn our attention to characterizing EP elements in terms of factorizations. The motivation for this part of the present paper is provided by an interesting paper by Drivaliaris, Karanasios and Pappas [8] who studied such characterizations for EP operators in a Hilbert space.

1.1 Factorization $a = ba^*$

In view of Definition 1.2, the simplest factorization of an EP element of $\mathcal{A}$ is of the form $a = ba^*$ with $b^\circ = \{0\}$, which implies the equality $a^\circ = (a^*)^\circ$ of annihilators. Then we have the following slightly more general result which again follows from a direct verification of the equality of the annihilators.

Theorem 1.5. *Let $a \in \mathcal{A}^{\mathsf{reg}}$. Then the following conditions are equivalent:*

(i) *a is EP.*

(ii) *$a = ba^* = a^*c$ for some $a, c \in \mathcal{A}$.*

(iii) *$a^*a = b_1a^*$ and $aa^* = c_1a$ for some $b_1, c_1 \in \mathcal{A}$.*

(iv) *$a^*a = b_2a^\dagger$ and $a^\dagger = c_2a$ for some $b_2, c_2 \in \mathcal{A}$.*

Proof. The equivalence of (i) and (ii) is a well known result, see for instance [11, Theorem 3.1]. Taking into account the equalities

$$(a^*a)^\circ = a^\circ \text{ and } (aa^*)^\circ = (a^*)^\circ = (a^\dagger)^\circ, \tag{1.4}$$

we deduce the equivalence of the remaining two conditions to (i). □

1.2 Factorization $a = ucw$

First an auxiliary result.

Lemma 1.6. *An element $a \in \mathcal{A}^{\mathsf{reg}}$ is EP if and only if $a = ucu^*$ for some $c, u \in \mathcal{A}$ satisfying $c^\circ = (c^*)^\circ$ and $u^\circ = \{0\}$.*

Proof. Let a have the specified factorization. We show that $(ucu^*)^\circ = (uc^*u^*)^\circ$. Let $ucu^*x = 0$. Then $u^*x \in c^\circ = (c^*)^\circ$, that is, $c^*u^*x = 0$, and $x \in (uc^*u^*)^\circ = (a^*)^\circ$. The reverse inclusion is obtained by interchanging c and c^*.

The converse follows on choosing $c = a$ and $u = 1$. □

An interesting question arises whether c in the preceding result needs to be EP.

Theorem 1.7. *Let $a \in \mathcal{A}^{\mathsf{reg}}$. Then the following conditions are equivalent:*

(i) *a is EP.*

(ii) *$a = ucw = w^*d^*v^*$ for some $c, d, u, v, w \in \mathcal{A}$ satisfying $c^\circ = d^\circ$ and $u^\circ = v^\circ = \{0\}$.*

(iii) *$a^*a = u_1c_1w_1$ and $aa^* = v_1d_1w_1$ for some $c_1, d_1, u_1, v_1, w_1 \in \mathcal{A}$ satisfying $c_1^\circ = d_1^\circ$ and $u_1^\circ = v_1^\circ = \{0\}$.*

(iv) *$a = u_2c_2w_2$ and $a^\dagger = v_2d_2w_2$ for some $c_2, d_2, u_2, v_2, w_2 \in \mathcal{A}$ satisfying $c_2^\circ = d_2^\circ$ and $u_2^\circ = v_2^\circ = \{0\}$.*

Proof. Suppose (ii) holds with c, d, u, v, w as specified. We show that $a^\circ = (a^*)^\circ$. Suppose that $x \in a^\circ$. Then $ucwx = 0$, and $cwx = 0$ as $u^\circ = \{0\}$. Since $d^\circ = c^\circ$, we have $dwx = 0$, and $a^*x = vdwx = 0$. Thus $a^\circ \subset (a^*)^\circ$ and (i) holds. The reverse imclusion follows by interchanging a and a^*.

Conversely, if (i) holds, then by Lemma 1.6, $a = ucu^*$ with $c \in \mathcal{A}^{\mathsf{ep}}$ and $u^\circ = \{0\}$. Hence $a^* = uc^*u^*$, where $c^\circ = (c^*)^\circ$, and (ii) is proved.

Applying the identities (2.1) we deduce the equivalence of the remaining two conditions to (i). ☐

Theorem 1.8. *An element $a \in \mathcal{A}^{\mathsf{reg}}$ is EP if and only if*

$$a = ucw = w^*d^*u^* \tag{1.5}$$

for some $c, d, u, w \in \mathcal{A}$ with $c^\circ \subset d^\circ$, $(c^*)^\circ \subset (d^*)^\circ$ and $u^\circ = (w^*)^\circ = \{0\}$.

Proof. First we show that $a^\circ \subset (a^*)^\circ$. Let $ax = ucwx = 0$. Then $cwx = 0$, and $dwx = 0$ as $c^\circ \subset d^\circ$. Hence $a^*x = udwx = 0$.

To prove the reverse inclusion let $a^*x = w^*c^*u^*x = 0$. Then $c^*u^*x = 0$, and $d^*u^*x = 0$ as $(c^*)^\circ \subset (d^*)^\circ$. Hence $w^*d^*u^*x = 0$, that is, $ax = 0$.

The converse follows by the choice $u = w = 1$, $c = a$ and $d = a^*$. ☐

1.3 Factorization $a = bc$

We now consider a factorization of $a \in \mathcal{A}$ of the form

$$a = bc, \quad b^*\mathcal{A} = \mathcal{A} = c\mathcal{A}. \tag{1.6}$$

By Lemma 1.1, $b^*\mathcal{A} = \mathcal{A}$ is equivalent to b being regular and $b^\circ = \{0\}$. Likewise, c is regular and $(c^*)^\circ = \{0\}$. Hence the elements b^*b and cc^* are invertible in $\mathcal{A}$, again by Lamma 1.1, as $(b^*b)^\circ = b^\circ = \{0\}$ and $b^*b\mathcal{A} = b^*\mathcal{A} = \mathcal{A}$, and $(cc^*)^\circ = (c^*)^\circ = \{0\}$ and $cc^*\mathcal{A} = c\mathcal{A} = \mathcal{A}$. It then follows that

$$b^\dagger b = (b^*b)^{-1}b^*b = 1, \qquad cc^\dagger = cc^*(cc^*)^{-1} = 1. \tag{1.7}$$

Lemma 1.9. *If a has a factorization* (1.6)*, then a is regular with $a^\dagger = c^\dagger b^\dagger$.*

Proof. Using (1.7) we directly verify that $x = c^\dagger b^\dagger$ satisfies the definition of the Moore–Penrose inverse for a. ☐

Theorem 1.10. *Let $a \in \mathcal{A}$ have the factorization* (1.6)*. Then a is regular and the following conditions are equivalent:*

(i) $a \in \mathcal{A}^{\mathsf{ep}}$.

(ii) $bb^\dagger = c^\dagger c$.

(iii) $b(b^*b)^{-1}b^* = c^*(cc^*)^{-1}c$.

(iv) $(b^*)^\circ = c^\circ$.

Proof. (i) implies (ii). If $a \in \mathcal{A}^{\mathsf{ep}}$, then $a^\dagger a = aa^\dagger$. Substituting into this equation from Lemma 1.9 and (1.7), we get the result.

(ii) is equivalent to (iii) as $b^\dagger = (b^*b)^{-1}b^*$ and $c^\dagger = c^*(cc^*)^{-1}$.

(iii) implies (iv). Let $x \in (b^*)^\circ$. By (iii), $c^*(cc^*)^{-1}cx = 0$. But $(c^*)^\circ = \{0\}$, and so $cx = 0$ and $x \in c^\circ$. The converse follows by symmetry.

(iv) implies (i). We observe that $a^\circ = c^\circ$ and $(a^*)^\circ = (b^*)^\circ$. □

2 Applications to Hilbert space operators

The results of the preceding section apply to Hilbert space operators, but unlike in Drivaliaris, Karanasios and Pappas [8], a direct application would cover only operators acting on the same space. In this section we develop theory of EP operators from that of elements of a C^*-algebra $\mathcal{B}(H)$, and describe a transcription of C^*-algebra results to results for operators between Hilbert spaces. By $\mathcal{B}(H, K)$ we denote the set of all bounded linear operators on H to K; we write $\mathcal{B}(H) = \mathcal{B}(H, H)$. The direct sum $H \oplus K$ of unrelated Hilbert space H, K is always treated as an orthogonal sum.

We shall see that by using Theorem 1.4 we can bypass the necessity of relating the algebra annihilator $A^\circ = \{S \in \mathcal{B}(H) : AS = 0\}$ to the spatial nullspace $N(A) = \{x \in H : Ax = 0\}$ of an operator $A \in \mathcal{B}(H)$. It is well known that an operator A is regular in $\mathcal{B}(H)$ (and Moore–Penrose invertible) if and only if it has closed range.

First we give a canonical form of an EP operator on a Hilbert space H.

Theorem 2.1. *An operator $A \in \mathcal{B}(H)$ is EP (relative to the C^*-algebra $\mathcal{B}(H)$) if and only if it is of the form*

$$A = A_1 \oplus^\perp 0 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where A_1 is invertible.

Proof. Suppose A has the specified decomposition relative to the orthogonal space decomposition $H = K \oplus^\perp L$, and suppose $P = I \oplus^\perp 0$ is the orthogonal projection of H onto K. We can then check that P satisfies the conditions of Theorem 1.4: A and P clearly commute, and $PA = (I \oplus^\perp 0)(A_1 \oplus^\perp 0) =$

$A_1 \oplus^\perp 0 = A$. To show that A is invertible in the C^*-algebra $\mathcal{D} = P\mathcal{B}(H)P$, set $Q = A_1^{-1} \oplus^\perp 0$. Then $A, Q \in \mathcal{D}$ commute, and

$$AQ = (A_1 \oplus^\perp 0)(A_1^{-1} \oplus^\perp 0) = I \oplus^\perp 0 = P.$$

By Theorem 1.4, A is EP in the C^*-algebra $\mathcal{B}(H)$.

Conversely, assume that A is EP. By Theorem 1.4 there exists an orthogonal projection P such that $AP = A = PA$ and A is invertible in the C^*-algebra $P\mathcal{B}(H)P$. Let $A = A_1 \oplus^\perp 0$ be the decomposition of A, and $Q = Q_1 \oplus^\perp 0$ the decomposition of the $P\mathcal{B}(H)P$ inverse Q of A relative to $H = R(P) \oplus^\perp N(P)$. From $AQ = P$ we get $A_1Q_1 \oplus^\perp 0 = I \oplus^\perp 0$, that is, $A_1Q_1 = I$. Since A and Q commute, also $Q_1A_1 = I$, and A_1 is invertible. $\square$

Remark 2.2. The canonical form of an EP operator given in Theorem 2.1 features prominently in [8, Section 3] in a slightly different form. In [8] the authors show that an operator $A \in \mathcal{B}(H)$ is EP if and only there exist Hilbert spaces K and L, a unitary operator $U \in \mathcal{B}(K \oplus L, H)$, and an invertible operator $A_1 \in \mathcal{B}(K)$ such that $A = U(A_1 \oplus^\perp 0)U^*$. We observe that the existence of a unitary operator $U \in \mathcal{B}(K \oplus L, H)$ means that the spaces $K \oplus L$ and H are isometrically $*$-isomorphic to each other.

Theorem 2.3. *An operator $A \in \mathcal{B}(H)$ is EP (relative to the C^*-algebra $\mathcal{B}(H)$) if and only if A has closed range and $N(A) = N(A^*)$.*

Proof. This follows from Theorem 2.1. If A is EP, then A has the decomposition $A = A_1 \oplus^\perp 0$ described in Theorem 2.1. Then $R(A) = R(A_1)$ is closed. Further, $A^* = A_1^* \oplus^\perp 0$, and $N(A) = N(A_1) = N(A_1^*) = N(A^*)$. Conversely, let A have closed range and let $N(A) = N(A^*)$. Then $H = R(A) \oplus^\perp N(A)$, and A has a decomposition $A = A_1 \oplus^\perp 0$ with the properties described in Theorem 2.1 relative to this space decomposition. $\square$

The preceding theorem is the key used to transcribe the C^*-algebra results of the preceding section in terms of operators between Hilbert spaces.

2.1 Factorization $A = BA^*$

An application of Theorem 1.5 together with

$$N(A^*A) = N(A) \text{ and } N(AA^*) = N(A^*) = N(A^\dagger) \tag{2.1}$$

yields the following result:

Theorem 2.4. *Let $A \in \mathcal{B}(H)$ be a closed range operator. Then the following conditions are equivalent:*

(i) A *is EP.*
(ii) $A = BA^* = A^*C$ *for some* $B, C \in \mathcal{B}(H)$.
(iii) $A^*A = B_1A^*$ *and* $AA^* = C_1A$ *for some* $B_1, C_1 \in \mathcal{B}(H)$.
(iv) $A^*A = B_2A^\dagger$ *and* $A^\dagger = C_2A$ *for some* $B_2, C_2 \in \mathcal{B}(H)$.

2.2 Factorization $A = UCW$

We can now give an operator version of Lemma 1.6 followed by the operator version of Theorems 1.7 and 1.8.

Lemma 2.5. *A closed range operator* $A \in \mathcal{B}(H)$ *is EP if and only if there exists a Hilbert space* K *and operators* $B \in \mathcal{B}(K)$, $U \in \mathcal{B}(K, H)$ *such that* $N(B) = N(B^*)$, $N(U) = \{0\}$, *and* $A = UBU^*$.

Proof. Suppose that a closed range operator A has the specified factorization. It is not difficult to prove that $N(A) = N(UBU^*) = N(UB^*U^*) = N(A^*)$. By Theorem 2.3, A is EP. The converse follows on choosing $K = H$, $B = A$ and $U = I$. □

Observe that we merely assume that $N(B) = N(B^*)$ without requiring B to be a closed range operator.

Theorem 2.6. *A closed range operator* $A \in \mathcal{B}(H)$ *is EP if and only if there exist Hilbert spaces* K, L *and operators* $U, V \in \mathcal{B}(L, H)$, $C, D \in \mathcal{B}(K, L)$ *and* $W \in \mathcal{B}(H, K)$ *such that*

$$A = UCW = W^*D^*V^*, \tag{2.2}$$

where $N(C) = N(D)$ *and* $N(U) = N(V) = \{0\}$.

Proof. Suppose that the factorization of a closed range operator A with the specified properties exists. We can then verify that $N(A) = N(UCW) = N(VDW) = N(A^*)$. By Theorem 2.3, A is EP. The converse follows on choosing $K = L = H$, $U = V = W = I$ and $C = A$, $D = A^*$. □

Theorem 2.7. *A closed range operator* $A \in \mathcal{B}(H)$ *is EP if and only if there exist Hilbert spaces* K, L *and operators* $U \in \mathcal{B}(L, H)$, $C, D \in \mathcal{B}(K, L)$ *and* $W \in \mathcal{B}(H, K)$ *such that*

$$A = UCW = W^*D^*U^*, \tag{2.3}$$

where $N(C) \subset N(D)$, $N(C^*) \subset N(D^*)$ *and* $N(U) = N(W^*) = \{0\}$.

Proof. Proceeding as in the proof of Theorem 1.8 we show that under the decomposition (2.3), $N(A) = N(A^*)$. The converse follows on choosing $K = L = H$, $U = W = I$ and $C = A$, $D = A^*$. $\square$

Corollary 2.8. *A closed range operator $A \in \mathcal{B}(H)$ is EP if and only if there exist Hilbert spaces K, L and operators $U \in \mathcal{B}(L, H)$, $C, D \in \mathcal{B}(K, L)$ and $W \in \mathcal{B}(H, K)$ such that*

$$A = UCW = W^* D^* U^*, \tag{2.4}$$

where $C = C_1 \oplus^\perp 0$ and $D = D_1 \oplus^\perp 0$ relative to the same space decomposition, C_1 is injective, and $N(U) = N(W^*) = \{0\}$.

Proof. This follows from the preceding theorem when we observe that the decompositions for C and D imply $N(C) \subset N(D)$ and $N(C^*) \subset N(D^*)$. $\square$

Theorem 2.7 can be extended by the inclusion of conditions equivalent to (2.3) corresponding to conditions (iii) and (iv) of Theorem 1.7.

2.3 Factorization $A = BC$

Following our C^*-algebra investigation in the preceding section, we consider an operator factorization of $A \in \mathcal{B}(H)$ of the form

$$A = BC, \quad B^* \text{ and } C \text{ are surjective,} \tag{2.5}$$

where $B \in \mathcal{B}(K, H)$ and $C \in \mathcal{B}(H, K)$. From the conditions on B it follows that B is injective and has closed range. Further, B^*B and CC^* are invertible in $\mathcal{B}(K)$. Applying Theorem 1.10, we have:

Theorem 2.9. *Let an operator $A \in \mathcal{B}(H)$ have the factorization* (2.5)*. Then A has closed range, and the following conditions are equivalent:*

(i) *A is EP.*
(ii) *$BB^\dagger = C^\dagger C$.*
(iii) *$B(B^*B)^{-1}B^* = B^*(CC^*)^{-1}C$.*
(iv) *$N(B^*) = N(C)$.*

Remark 2.10. The preceding theorem is stronger than [8, Theorem 5.1] as we do not assume—as it is done in [8]— that A has closed range.

References

[1] O. M. Baksalary and G. Trenkler. Characterizations of EP, normal and Hermitian matrices. *Linear and Multilinear Algebra*, 56:299–304, 2006.

[2] A. Ben-Israel and T. N. E. Greville. *Generalized Inverses: Theory and Applications, 2nd ed.* Springer, New York, 2003.

[3] E. Boasso. On the Moore–Penrose inverse, EP Banach space operators, and EP Banach algebra elements. *J. Math. Anal. Appl.*, 339:1003–1014, 2008.

[4] S. L. Campbell and C. D. Meyer Jr. EP operators and generalized inverse. *Canad. Math. Bull.*, 18:327–333, 1975.

[5] S. L. Campbell and C. D. Meyer Jr. *Generalized Inverses of Linear Transformations*. Dover, New York, 1991.

[6] D. S. Djordjević. Characterizations of normal, hyponormal and EP operators. *J. Math. Anal. Appl.*, 329:1181–1190, 2007.

[7] D. S. Djordjević and J. J. Koliha. Characterizing Hermitian, normal and EP operators. *Filomat*, 2:39–54, 2007.

[8] D. Drivaliaris, S. Karanasios, and D. Pappas. Factorizations of EP operators. *Linear Algebra Appl.*, to appear.

[9] R. E. Harte and M. Mbekhta. On generalized inverses in C^*-algebras. *Studia Math.*, 103:71–77, 1992.

[10] R. E. Harte and M. Mbekhta. On generalized inverses in C^*-algebras II. *Studia Math.*, 106:129–138, 1993.

[11] J. J. Koliha. Elements of C^*-algebras commuting with their Moore–Penrose inverse. *Studia Math.*, 139:81–90, 2000.

D. S. Djordjević, Department of Mathematics and Informatics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia, e-mail: dragan@pmf.ni.ac.yu

J. J. Koliha, Department of Mathematics and Statistics, The University of Melbourne, Melbourne VIC 3010, Australia, e-mail: j.koliha@ms.unimelb.edu.au

I. Straškraba, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: strask@math.cas.cz