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# Incompressible Maxwell-Boussinesq approximation: Existence, uniqueness and shape sensitivity

Luisa Consiglieri [1,4], Šárka Nečasová [2,4],
Jan Sokolowski [3,4]

**Abstract**

We prove the existence and uniqueness of weak solutions to the variational formulation of the Maxwell-Boussinesq approximation problem. Some further regularity in $W^{1,2+\delta}$, $\delta > 0$, is obtained for the weak solutions. The shape sensitivity analysis by the boundary variations technique is performed for the weak solutions. As a result, the existence of the strong material derivatives for the weak solutions of the problem is shown. The result can be used to establish the shape differentiability for a broad class of shape functionals for the models of Fourier-Navier-Stokes flows under the electromagnetic field.

**Keyword** magnetohydrodynamic flows, shape sensivity.

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[1]Mathematics Dep/FCUL and CMAF, University of Lisbon, 1749-016 Lisboa, Portugal
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[2]Mathematical Institute of the Academy of Sciences, Žitná 25, 11567 Prague 1, Czech Republic
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[3]Institut Elie Cartan, Laboratoire de Mathématiques,Université Henri Poincaré Nancy I, B.P. 239, 54506 Vandouevre les Nancy Cedex, France

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# 1 Introduction

The problem of magnetohydrodynamics flows have been studied by several authors and it goes back to the work of Ladyzhenskay and Solonnikov. After that a lot of scientists investigated the problem see [4, 8, 14, 15, 17, 18, 19, 24]. The full complete problem including the heat conductivity seems to be more realistic and not many authors were dealt with it. Concerning the shape sensitivity analysis we can mention work of Zolesio and his collaborators see [2, 3, 6, 9, 10, 11], who were investigated case of Navier-Stokes problem and also with heat conductivity. The full Navier-Fourier-Maxwell problem was only partially study in the work of Alekseev [1].

Let $\Omega$ be an open bounded subset of $\mathbb{R}^3$ with the boundary $\partial\Omega \in C^{1,1}$ which is splitted into two parts $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, where $\Gamma_D$ is an open nonempty subset of $\partial\Omega$ and $\Gamma_N = \partial\Omega \setminus \bar{\Gamma}_D$. The thermoelectromagnetoflow problem reads in $\Omega$:

$$-\nabla \cdot (\nu(T)D\mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mu(T)\mathrm{rot}\mathbf{H} \times \mathbf{H} + \mathbf{f} - \mathbf{G}T; \tag{1}$$

$$\nabla \times (\sigma^{-1}(T)\nabla \times \mathbf{H}) = \nabla \times (\sigma^{-1}(T)\mathbf{J}_0 + \mu(T)\mathbf{u} \times \mathbf{H}); \tag{2}$$

$$\mathrm{div}\mathbf{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \mathrm{div}\mathbf{H} = 0; \tag{3}$$

$$-\nabla \cdot (k(T)\nabla T) + \mathbf{u} \cdot \nabla T = f. \tag{4}$$

Here $\mathbf{u}$ is the fluid velocity vector, $T$ is the temperature, $D\mathbf{u} = (D_{ij}) = (\partial_i u_j + \partial_j u_i)/2$ $(i, j = 1, 2, 3)$ is the symmetrized gradient of the velocity, $\mu$ the viscosity, $p$ denotes the pressure, $\mathbf{f}$ and $f$ denote the external forces and heat sources, respectively. The buoyancy force as in the Boussinesq approximation is described by $\mathbf{G} = \beta(T)(0, 0, g)^\top$, where $\beta$ denotes the coefficient of thermal dilatation and $g$ is the constant of gravity. The density is assumed to be constant, we set $\rho = 1$. The existence of two body forces in the fluid, the Lorentz force $\mathbf{J} \times \mathbf{B} = (\nabla \times \mathbf{H}) \times (\mu\mathbf{H})$ and the buoyancy force, results from the presence of the magnetic field $\mathbf{H}$. Moreover (2) results if we take the rotational in the second equation of the steady-state Maxwell equations:

$$\nabla \times \mathbf{E} = 0; \qquad \mathbf{J} = \nabla \times \mathbf{H},$$

where $\mathbf{E}$ is the electric intensity field and $\mathbf{J}$ is the current density given by the Ohm's law

$$\mathbf{J} = \mathbf{J}_0 + \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}),$$

where $\sigma$ is the electric conductivity.

Finally, the thermoelectromagnetoflow problem under study has the following boundary conditions

$$\mathbf{u} = \mathbf{g}, \qquad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega; \tag{5}$$

$$T = 0 \text{ on } \Gamma_D, \quad k(T)\frac{\partial T}{\partial \mathbf{n}} + \alpha T = h \quad \text{on} \quad \Gamma_N. \tag{6}$$

## 2 Assumptions and main existence results

We need some assumptions on the model, which are listed below.

Let us assume that

(H1) $\nu, \mu, \sigma, k : \Omega \times \mathbb{R} \to \mathbb{R}$ are Caratheodory functions such that

$$\exists \nu^{\#}, \nu_{\#} > 0 : \quad \nu_{\#} \leq \nu(\cdot, \xi) \leq \nu^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{7}$$

$$\exists \mu^{\#}, \mu_{\#} > 0 : \quad \mu_{\#} \leq \mu(\cdot, \xi) \leq \mu^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{8}$$

$$\exists \sigma^{\#}, \sigma_{\#} > 0 : \quad \sigma_{\#} \leq \sigma(\cdot, \xi) \leq \sigma^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{9}$$

$$\exists k^{\#}, k_{\#} > 0 : \quad k_{\#} \leq k(\cdot, \xi) \leq k^{\#}, \text{ a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}; \tag{10}$$

(H2) $\mathbf{G} = (0, 0, G)$ where $G$ is a real, continuous, and bounded function and we denote by $G^{\#}$ the upper bound for the function $G$;

(H3) $\alpha \in L^q_+(\Gamma_N) = \{\alpha \in L^q(\Gamma_N) : \quad \alpha \geq 0\}$ for $q$ such that $q > 3/2$, which means that its conjugate $q' = q/(q-1)$ verifies $q' < 3$;

(H4) and

$$\mathbf{f} \in \mathbf{L}^2(\Omega), \quad \mathbf{J}_0 \in \mathbf{L}^2(\Omega), \quad f \in L^2(\Omega) \quad \text{ and } h \in L^2(\Gamma_N). \tag{11}$$

(H5) In the variable domain setting the function $\beta_T$ is given by the restriction to $\Omega_\tau$ of a given $H^1$-function defined in $\mathbb{R}^3$.

(H6) In the variable domain setting, the elements

$$\mathbf{f}^\tau \in \mathbf{L}^2(\Omega_\tau), \quad \mathbf{J}_0^\tau \in \mathbf{L}^2(\Omega_\tau), \quad f^\tau \in L^2(\Omega_\tau) \quad \text{ and } h^\tau \in L^2(\Gamma_N^\tau),$$

stand for the data in boundary value problems in $\Omega_\tau$, are simply given by restrictions to $\Omega_\tau$ of some functions

$$\mathbf{f} \in \mathbf{H}^1(\mathbb{R}^3), \quad \mathbf{J}_0 \in \mathbf{H}^1(\mathbb{R}^3), \quad f \in H^1(\mathbb{R}^3) \quad \text{ and } h \in H^1(\mathbb{R}^3). \tag{12}$$

defined in all space. In this way the shape derivatives of all the data vanish, except for $h$, and the material derivatives are just given by the scalar products of the gradients of the data with respect to spatial variables with the velocity vector field, e.g., $\dot{f} = \nabla f \cdot V$, provided that all data are given in the Sobolev spaces $H^1(\mathbb{R}^3)$.

To simplify the presentation it is assumed that $\mathbf{g} = \mathbf{0}$ (cf. Remark 3.5).

In the framework of function spaces of the Lebesgue and Sobolev type, the norms are denoted by the symbols $\|\cdot\|, \|\cdot\|_1, \|\cdot\|_{\Gamma_N}$ in spaces $L^2(\Omega), H^1(\Omega)$, $L^2(\Gamma_N)$, respectively, and there scalar and vector function spaces are not distinguished in our notations. Providing that the meaning remains clear, the canonical norm in $L^p(\Omega)$ for $p \neq 1, 2$ is denoted by $\|\cdot\|_p$. We introduce the Hilbert spaces

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \quad \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\} , \\ \mathbf{V}(rot) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \text{ rot } \mathbf{v} \in \mathbf{L}^2(\Omega), \ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} , \\ Z &= \{\xi \in H^1(\Omega) : \quad \xi = 0 \text{ on } \Gamma_D\} , \end{aligned}$$

equipped with their standard scalar products. We recall that the norms $\|\cdot\|_{\mathbf{V}(rot)}$ and $\|\cdot\|_Z$ are equivalent to the usual seminorms $\|\nabla \times \cdot\|$ and $\|\nabla \cdot\|$ and also to the norms $\|\cdot\|_1$ on spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$, respectively (cf. [8]).

We state the main results of the paper.

**Theorem 2.1.** *Under the above assumptions (7)-(11), and, in addition, under the following assumptions*

$$\begin{aligned} b > 0 \quad \textit{and} \quad & \mu^{\#} a^2 < b^3 , \\ & a = \frac{\nu_{\#}}{\mu^{\#}\sigma_{\#}} \|\mathbf{J}_0\| , \\ & b = \frac{\nu_{\#}}{\mu^{\#}\sigma^{\#}} - \left( \|\mathbf{f}\| + \frac{G^{\#}}{k_{\#}} (\|f\| + \|h\|_{\Gamma_N}) \right) , \end{aligned} \tag{13}$$

*the problem (1)-(6) has a weak solution in the following sense:*

*The triplet* $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ *satisfies the following integral* *identities*

$$\int_\Omega \nu(T)D\mathbf{u} : D\mathbf{v}dx + \int_\Omega (\mathbf{v}\otimes\mathbf{u}) : \nabla\mathbf{u}dx =$$

$$= \int_\Omega \Big(\mu(T)(\nabla\times\mathbf{H})\times\mathbf{H} + \mathbf{f} - \mathbf{G}(T)T\Big)\cdot\mathbf{v}dx, \quad \forall\mathbf{v}\in\mathbf{V}; \tag{14}$$

$$\int_\Omega \frac{1}{\sigma(T)}(\nabla\times\mathbf{H})\cdot(\nabla\times\mathbf{v})dx = \int_\Omega \mu(T)(\mathbf{u}\times\mathbf{H})\cdot(\nabla\times\mathbf{v})dx +$$

$$+ \int_\Omega \frac{1}{\sigma(T)}\mathbf{J}_0\cdot(\nabla\times\mathbf{v})dx, \quad \forall\mathbf{v}\in\mathbf{V}(rot); \tag{15}$$

$$\int_\Omega k(T)\nabla T\cdot\nabla\eta dx + \int_\Omega \mathbf{u}\cdot\nabla T\eta dx + \int_{\Gamma_N}\alpha T\eta ds =$$

$$= \int_\Omega f\eta dx + \int_{\Gamma_N} h\eta ds, \quad \forall\eta\in Z. \tag{16}$$

*Moreover, the pair* $(\mathbf{H}, T)$ *enjoys the additional regularity, actually belongs to* $\mathbf{W}^{1,2+\epsilon}(\Omega)\times W^{1,2+\varepsilon}(\Omega)$ *for some* $\epsilon$, $\varepsilon > 0$.

**Remark 2.2.** If $\epsilon > 2/5$ we can deduce the additional regularity on $\mathbf{u}$ as in [5]. Otherwise, since the operators in the above elliptic equations of the second order have discontinuous coefficients, we can obtain Hölder continuity on $\bar{\Omega}$ of the weak solution $T$ due to the De Giorgi-Nash Theorem if $f, h \in L^q(\Omega)$ for $q > 3$. If $\sigma$ is taken as a continuous function, then the main operator in (15) has continuous coefficient and the regularity theory can be applied to the weak solution $\mathbf{H}$. Or simply if we suppose that the electric conductivity $\sigma$ is constant, it will be sufficient to our purposes in the study of the shape sensivity. However, in the sequel the data assumptions are kept as general as possible.

**Theorem 2.3.** *Let* $\epsilon_0 < \epsilon < 1$ *and* $2 < q < 3$ *be such that*

$$\frac{3q}{3-q} = \frac{(2+\epsilon_0)(2+\epsilon)}{\epsilon-\epsilon_0}. \tag{17}$$

*If* $\mathbf{J}_0\in\mathbf{L}^q(\Omega)$, *then* $\mathbf{H}\in\mathbf{L}^{3q/(3-q)}(\Omega)$. *Under the assumption* $\mathbf{f}\in\mathbf{L}^{2+\delta_1}(\Omega)$, *where* $\delta_1 > 0$, *the weak solution* $\mathbf{u}$ *given by Theorem 2.1 enjoys the additional regularity, actually belongs to* $\mathbf{W}^{1,2+\delta}(\Omega)$ *for some* $\delta > 0$. *Furthermore, under the following Lipschitz-type continuity assumption on the temperature* *dependent function parameters of the model*

$$\exists\bar{\nu} > 0: \qquad |\nu(T^2) - \nu(T^1)| \leq \bar{\nu}|T^2 - T^1|^{3\delta/(2+\delta)}, \tag{18}$$

$$\exists\bar{\mu} > 0: \qquad |\mu(T^2) - \mu(T^1)| \leq \bar{\mu}|T^2 - T^1|, \tag{19}$$

$$\exists\bar{G} > 0: \qquad |G(T^2) - G(T^1)| \leq \bar{G}|T^2 - T^1|, \tag{20}$$

$$\exists\bar{\sigma} > 0: \qquad |\sigma(T^2) - \sigma(T^1)| \leq \bar{\sigma}|T^2 - T^1|^{3\epsilon/(2+\epsilon)}, \tag{21}$$

$$\exists\bar{k} > 0: \qquad |k(T^2) - k(T^1)| \leq \bar{k}|T^2 - T^1|^{3\varepsilon/(2+\varepsilon)}, \quad \forall T^2, T^1 \in \mathbb{R}, \tag{22}$$

*the weak solution* $(\mathbf{u}, \mathbf{H}, T)$ *is unique for small data.*

The existence of the pressure $p$ in the space of distributions follows from the well-known results by using the divergence-free test functions $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$ in (14). Moreover, the pressure is unique up to a constant.

# 3 Proof of Theorem 2.1

First, we recall the Tychonoff extension to weak topologies of the Schauder fixed point theorem [7, pp. 453-456 and 470].

**Theorem 3.1.** *Let $K$ be a nonempty weakly sequentially compact convex subset of a locally convex linear topological vector space $V$. Let $\mathcal{L} : K \to K$ be a weakly sequentially continuous operator. Then $\mathcal{L}$ has at least one fixed point.*

Let $\mathcal{L}$ be the mapping of the form

$$\mathcal{L} : (\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(rot) \times Z \mapsto (\mathbf{H}, T) \mapsto (\mathbf{u}, \mathbf{H}, T) \ ,$$

where the functions $\mathbf{u}$, $\mathbf{H}$ and $T$ are the solutions for the following elliptic boundary value problems. The proofs of such existence results are the straightforward application of the classical existence theory, hence are omitted here.

**Proposition 3.2.** *Assume that conditions (10)-(11) are fulfilled. Then there exists a unique $T \in Z$ such that*

$$\int_\Omega k(\xi)\nabla T \cdot \nabla\eta dx + \int_\Omega \mathbf{w} \cdot \nabla T\eta dx + \int_{\Gamma_N} \alpha T\eta ds =$$
$$= \int_\Omega f\eta dx + \int_{\Gamma_N} h\eta ds, \quad \forall \eta \in Z. \tag{23}$$

*Moreover, the energy estimate holds*

$$k_\#\|T\|_1 \leq \|f\| + \|h\|_{\Gamma_N}. \tag{24}$$

**Proposition 3.3.** *Assume that conditions (8)-(9) and (11) are fulfilled. Then there exists a unique* $\mathbf{H} \in \mathbf{V}(rot)$ *such that*

$$\int_\Omega \frac{1}{\sigma(\xi)}(\nabla \times \mathbf{H}) \cdot (\nabla \times \mathbf{v})dx = -\int_\Omega \mu(\xi)(\mathbf{h} \times \mathbf{w}) \cdot (\nabla \times \mathbf{v})dx + \\ + \int_\Omega \frac{1}{\sigma(\xi)}\mathbf{J}_0 \cdot (\nabla \times \mathbf{v})dx, \quad \forall \mathbf{v} \in \mathbf{V}(rot). \tag{25}$$

*Moreover, the energy estimate holds*

$$\frac{1}{\sigma^\#}\|\mathbf{H}\|_1 \leq \mu^\#\|\mathbf{h} \times \mathbf{w}\| + \frac{1}{\sigma_\#}\|\mathbf{J}_0\|. \tag{26}$$

**Proposition 3.4.** *Assume that conditions (7) and (11) are fulfilled. Then there exists a unique* $\mathbf{u} \in \mathbf{V}$ *such that*

$$\int_\Omega \nu(\xi)D\mathbf{u} : D\mathbf{v}dx + \int_\Omega (\mathbf{v} \otimes \mathbf{w}) : \nabla \mathbf{u}dx = \\ = \int_\Omega \Big(\mu(\xi)(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}(T)T\Big) \cdot \mathbf{v}dx, \quad \forall \mathbf{v} \in \mathbf{V}. \tag{27}$$

*Moreover, the energy estimate holds*

$$\nu_\#\|\mathbf{u}\|_1 \leq \mu^\#\|\nabla \times \mathbf{H}\|\|\mathbf{H}\|_{L^3} + \|\mathbf{f}\| + G^\#\|T\|_{L^{6/5}}. \tag{28}$$

**Remark 3.5.** For given $\mathbf{g} \in \mathbf{H}^{-1/2}(\partial\Omega)$, there exists a lifting $\mathbf{u}_\mathbf{g} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{u}_\mathbf{g} = \mathbf{g}$ on $\partial\Omega$ and $\mathbf{u}_\mathbf{g}$ verifies

$$-\nabla \cdot (\nu(\xi)D\mathbf{u}_\mathbf{g}) + (\mathbf{w} \cdot \nabla)\mathbf{u}_\mathbf{g} = -\nabla p_\mathbf{g}; \quad \nabla \cdot \mathbf{u}_\mathbf{g} = 0 \text{ in } \Omega.$$

If the element $\mathbf{U} = \mathbf{u} - \mathbf{u}_\mathbf{g} \in \mathbf{V}$ is determined by a solution to the problem

$$-\nabla \cdot (\nu(\xi)D\mathbf{U}) + (\mathbf{w} \cdot \nabla)\mathbf{U} = -\nabla p_\mathbf{U} + \mu(\xi)(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}\xi \text{ in } \Omega,$$

then $\mathbf{u} = \mathbf{U} + \mathbf{u}_\mathbf{g}$ is the solution to the problem

$$-\nabla \cdot (\nu(\xi)D\mathbf{u}) + (\mathbf{w} \cdot \nabla)\mathbf{u} = -\nabla p + \mu(\xi)(\nabla \times \mathbf{H}) \times \mathbf{H} + \mathbf{f} - \mathbf{G}\xi \text{ in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega.$$

Therefore, without any loss of generality, it is assumed that $\mathbf{g} = 0$.

In view of Propositions 3.2, 3.3 and 3.4, the operator $\mathcal{L}$ is well defined. Moreover, $\mathcal{L}$ maps the ball

$$K = \{(\mathbf{w}, \mathbf{h}, \xi) \in \mathbf{V} \times \mathbf{V}(rot) \times Z : \quad \|\mathbf{w}\|_1 \leq R_1, \ \|\mathbf{h}\|_1 \leq R_2, \\ \|\xi\|_1 \leq \frac{1}{k_\#}(\|f\| + \|h\|_{\Gamma_N})\}$$

into itself, since by (24), (26) and (28) it follows that

$$\|\mathbf{H}\|_1 \leq \sigma^\# \left( \mu^\# R_1 R_2 + \frac{1}{\sigma_\#}\|\mathbf{J}_0\| \right) \leq R_2 \ , \tag{29}$$

$$\|\mathbf{u}\|_1 \leq \frac{1}{\nu_\#} \left( \mu^\# R_2^2 + \|\mathbf{f}\| + \frac{G^\#}{k_\#}(\|f\| + \|h\|_{\Gamma_N}) \right) = R_1 \ , \tag{30}$$

where $R_2 > 0$ is such that

$$\frac{\mu^\# \sigma^\#}{\nu_\#} R_2 \left( \mu^\# R_2^2 + \|\mathbf{f}\| + \frac{G^\#}{k_\#}(\|f\| + \|h\|_{\Gamma_N}) \right) + \frac{\sigma^\#}{\sigma_\#}\|\mathbf{J}_0\| \leq R_2$$

or equivalently

$$a \leq R_2(b - \mu^\# R_2^2)$$

if $b > 0$ and $(\frac{a}{b})^2 < \frac{b}{\mu^\#}$ which is assured by (13).

In order to apply Theorem 3.1 it remains to prove the weak continuity of $\mathcal{L}$. Since we have the compact embeddings

$$\mathbf{V}, \mathbf{V}(rot) \hookrightarrow\hookrightarrow \{\mathbf{w} \in \mathbf{L}^4(\Omega) : \quad \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega, \ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$
$$Z \hookrightarrow\hookrightarrow L^1(\Omega),$$

let $\{(\mathbf{w}_m, \mathbf{h}_m, \xi_m)\}$ be a sequence such that $\mathbf{w}_m \to \mathbf{w}$ and $\mathbf{h}_m \to \mathbf{h}$ in $\mathbf{L}^4(\Omega)$ and $\xi_m \to \xi$ in $L^1(\Omega)$. Let $(\mathbf{u}_m, \mathbf{H}_m, T_m)$ be the corresponding weak solutions given by Propositions 3.2, 3.3 and 3.4, for each $m \in \mathbb{N}$.

From the estimates (28), (26) and (24), the sequence $\{(\mathbf{u}_m, \mathbf{H}_m, T_m)\}$ is bounded in $\mathbf{V} \times \mathbf{V}(rot) \times Z$. Then there exists the weak limit $(\mathbf{u}, \mathbf{H}, T) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } \mathbf{V}; \qquad \mathbf{H}_m \rightharpoonup \mathbf{H} \text{ in } \mathbf{V}(rot); \qquad T_m \rightharpoonup T \text{ in } Z,$$

possibly for a subsequence, still denoted by $(\mathbf{u}_m, \mathbf{H}_m, T_m)$. Passing to the limit as $m \to +\infty$ in the integral identities (27), (25) and (23), in which replacing $\mathbf{w}, \mathbf{h}, \xi, \mathbf{u}, \mathbf{H}$ and $T$ by the sequences $\mathbf{w}_m, \mathbf{h}_m, \xi_m, \mathbf{u}_m, \mathbf{H}_m$ and $T_m$, respectively, using the continuity properties of the Niemytskii operators in the coefficients combined with the standard arguments, we conclude that the limit $(\mathbf{u}, \mathbf{H}, T)$ is a solution corresponding to $(\mathbf{w}, \mathbf{h}, \xi)$ of the required problem (27), (25) and (23).

Then Theorem 3.1 guarantees the existence of at least one fixed point which is the required weak solution.

The regularity $(\mathbf{H}, T) \in \mathbf{W}^{1,2+\epsilon}(\Omega) \times W^{1,2+\varepsilon}(\Omega)$ for some $\epsilon$, $\varepsilon > 0$ is a consequence of the following regularity results.

**Proposition 3.6.** *If $\mathbf{J}_0 \in \mathbf{L}^2(\Omega)$ then there exists a constant $\epsilon > 0$ such that the weak solution $\mathbf{H} \in \mathbf{V}(rot)$ of (15) belongs to $\mathbf{W}^{1,2+\epsilon}(\Omega)$, i.e.*

$$\|\nabla \mathbf{H}\|_{2+\epsilon} \leq K_1,$$

*with a constant $K_1 > 0$ only dependent on the data.*

**Proof.** Adapting the regularity theory for elliptic equations of the second order [16], we obtain $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega)$ with $2 + \epsilon < 6$ since

$$\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u} \in \mathbf{L}^2(\Omega) \hookrightarrow (\mathbf{W}^{1,6/5}(\Omega))'.$$

The following result is consequence of the regularity of solutions to the mixed boundary value problems for elliptic equations (cf. [16]).

**Proposition 3.7.** *If $f \in L^2(\Omega)$ and $h \in L^2(\Gamma_N)$ then there exists a constant $\varepsilon > 0$ such that the weak solution $T \in Z$ of (16) belongs to $W^{1,2+\varepsilon}(\Omega)$, i.e.*

$$\|\nabla T\|_{2+\varepsilon} \leq K_2,$$

*with a constant $K_2 > 0$ only dependent on the data.*

**Proof.** According to [16] we obtain $T \in W^{1,2+\varepsilon}(\Omega)$ with $2 + \varepsilon < 3$ since $f, h \in (W^{1,3/2}(\Omega))'$.

# 4 Proof of Theorem 2.3

The regularity $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $\delta > 0$ is a consequence of the following regularity results.

**Proposition 4.1.** *For every $2 < q < 3$, if $\mathbf{J}_0 \in \mathbf{L}^q(\Omega)$ then the weak solution $\mathbf{H} \in \mathbf{V}(rot)$ of (15) belongs to $\mathbf{L}^{3q/(3-q)}(\Omega)$.*

**Proof.** Adapting the regularity theory for elliptic equations of the second order [16], the desired result is obtained provided by

$$\mathbf{J}_0 - \sigma(T)\mu(T)\mathbf{H} \times \mathbf{u} \in \mathbf{L}^q(\Omega).$$

**Proposition 4.2.** *If $q$ is given as in (17) and $\mathbf{f} \in \mathbf{L}^{2+\delta_1}(\Omega)$ for some $\delta_1 > 0$, then there exists a constant $\delta > 0$ such that the weak solution $\mathbf{u} \in \mathbf{V}$ of (14) belongs to $\mathbf{W}^{1,2+\delta}(\Omega)$, i.e.*

$$\|\nabla \mathbf{u}\|_{2+\delta} \leq K_3,$$

*with a constant $K_3 > 0$ only dependent on the data.*

**Proof.** For every $x_0 \in \bar{\Omega}$, $0 < r < R$ small enough, $\Omega(x_0, R) := \Omega \cap B(x_0, R)$, $\theta \in ]0, 1[$ and some positive constants $B_1, B_2$, independent of $\mathbf{u}, \mathbf{H}$ and $T$, we have the following reverse estimate (cf. [4, Lemma 3.2])

$$\begin{aligned}\left(\int_{\Omega(x_0,r)} |\nabla \mathbf{u}|^2 dx\right)^{1/2} \leq\ & \theta \left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^2 dx\right)^{1/2} \\ & + \frac{B_1}{R-r}\left(\int_{\Omega(x_0,R)} |\nabla \mathbf{u}|^{6/5} dx\right)^{5/6} \\ & + \frac{B_2}{R-r}\left(\int_{\Omega(x_0,R)} (|\mathbf{u} \otimes \mathbf{u}|^2 + |\mathbf{F}|^2 + |\mathbf{f}|^2 + 1) dx\right)^{1/2}\end{aligned}$$

where $\mathbf{F} = \mu(T)\mathrm{rot}\mathbf{H} \times \mathbf{H} - \mathbf{G}(T)T$. By Propositions 3.6 and 4.1, we have $\mathbf{H} \in \mathbf{W}^{1,2+\epsilon}(\Omega) \cap \mathbf{L}^{(2+\epsilon_0)(2+\epsilon)/(\epsilon-\epsilon_0)}(\Omega)$. Thus it follows that $\mathrm{rot}\mathbf{H} \times \mathbf{H} \in \mathbf{L}^{2+\epsilon_0}(\Omega)$ and $\mathbf{F} = \mu(T)\mathrm{rot}\mathbf{H} \times \mathbf{H} - \mathbf{G}(T)T \in \mathbf{L}^{2+\epsilon_0}(\Omega)$. Since $\mathbf{u} \otimes \mathbf{u} \in \mathbf{L}^3(\Omega)$ then the Gehring inequality [13] guarantees the higher integrability $\mathbf{u} \in \mathbf{W}^{1,2+\delta}(\Omega)$ for some $0 < \delta < \min\{\epsilon_0, \delta_1\}$.

Now, we return to the proof of uniqueness. To this end, let $(\mathbf{u}^1, \mathbf{H}^1, T^1)$ and $(\mathbf{u}^2, \mathbf{H}^2, T^2)$ be two weak solutions to problem (14), (16), and (15). Arguing as in [5], the respective differences $\bar{\mathbf{u}} = \mathbf{u}^1 - \mathbf{u}^2$, $\bar{\mathbf{H}} = \mathbf{H}^1 - \mathbf{H}^2$ and

$\bar{T} = T^1 - T^2$ satisfy

$$\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_\#}\|\bar{T}\|_6^{6\delta/(2+\delta)}\|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2\|D\bar{\mathbf{u}}\|^2\|\nabla\mathbf{u}^2\| +$$
$$+\frac{C_1}{\nu_\#}\Big(\|\mu(T^1)(\nabla\times\mathbf{H}^1)\times\mathbf{H}^1 - \mu(T^2)(\nabla\times\mathbf{H}^2)\times\mathbf{H}^2\|_{6/5} +$$
$$+G^\#\|\bar{T}\|_{6/5} + \bar{G}\|\bar{T}\|_6\|T^2\|_{3/2}\Big)^2;$$
$$\frac{1}{4\sigma^\#}\|\nabla\times\bar{\mathbf{H}}\|^2 \leq \sigma^\#\left\|\left(\frac{1}{\sigma(T^2)} - \frac{1}{\sigma(T^1)}\right)\nabla\times\mathbf{H}^2\right\|^2 +$$
$$+\sigma^\#\|\mu(T^1)(\mathbf{u}^1\times\mathbf{H}^1) - \mu(T^2)(\mathbf{u}^2\times\mathbf{H}^2)\|^2 + \sigma^\#\left\|\left(\frac{1}{\sigma(T^1)} - \frac{1}{\sigma(T^2)}\right)\mathbf{J}_0\right\|^2;$$
$$\frac{k_\#}{2}\|\nabla\bar{T}\|^2 \leq \frac{\bar{k}}{k_\#}\|\bar{T}\|_6^{6\varepsilon/(2+\varepsilon)}\|\nabla T^2\|_{2+\varepsilon}^2 + \frac{C_1}{k_\#}\|\bar{\mathbf{u}}\|_6^2\|\nabla T^2\|_{3/2}^2, \qquad (31)$$

where $C_1, C_2$ are the Sobolev constants of the embeddings $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, respectively. Using the Lipschitz continuity assumptions (18)-(22), and applying Hölder and Young inequalities leads to

$$\frac{\nu_\#}{2}\|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_\#}\|\bar{T}\|_6^{6\delta/(2+\delta)}\|D\mathbf{u}^2\|_{2+\delta}^2 + C_2^2\|D\bar{\mathbf{u}}\|^2\|\nabla\mathbf{u}^2\| +$$
$$+\frac{C_1}{\nu_\#}\Big(\bar{\mu}\|\bar{T}\|_6\|\nabla\times\mathbf{H}^1\|\|\mathbf{H}^1\|_6 + \mu^\#\|\nabla\times\bar{\mathbf{H}}\|\|\mathbf{H}^1\|_3 + \mu^\#\|\nabla\times\mathbf{H}^2\|\|\bar{\mathbf{H}}\|_3$$
$$+G^\#\|\bar{T}\|_{6/5} + \bar{G}\|\bar{T}\|_6\|T^2\|_{3/2}\Big)^2;$$
$$\frac{1}{4(\sigma^\#)^2}\|\nabla\times\bar{\mathbf{H}}\|^2 \leq \frac{\bar{\sigma}}{(\sigma_\#)^2}\|\bar{T}\|_6^{6\epsilon/(2+\epsilon)}(\|\nabla\mathbf{H}^2\|_{2+\epsilon}^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2)$$
$$+\bar{\mu}\|\bar{T}\|_6^2\|\mathbf{u}^1\|_6^2\|\mathbf{H}^1\|_6^2 + \mu^\#(\|\bar{\mathbf{u}}\|_4^2\|\mathbf{H}^1\|_4^2 + \|\mathbf{u}^2\|_4^2\|\bar{\mathbf{H}}\|_4^2).$$

Let $K_1$, $K_2$ and $K_3$ be the upper bounds derived in Propositions 3.6, 3.7 and 4.2, respectively, and $K_4$ stand for the upper bound in estimate (24), namely,

$$K_4 = \frac{1}{k_\#}(\|f\| + \|h\|_{\Gamma_N}).$$

Next, in view of (29)-(30), we set

$$R_1 = \frac{1}{\nu_\#}\left(\mu^\# R_2^2 + \|\mathbf{f}\| + G^\# K_4\right),$$

where $R_2$ is chosen such that

$$\Big(1 - \frac{\mu^{\#}\sigma^{\#}}{\nu_{\#}}\left(\mu^{\#}R_2^2 + \|\mathbf{f}\| + G^{\#}K_4\right)\Big)R_2 = \frac{\sigma^{\#}}{\sigma_{\#}}\|\mathbf{J}_0\|,$$

we have

$$\begin{aligned} \frac{\nu_{\#}}{2}\|D\bar{\mathbf{u}}\|^2 \leq \frac{\bar{\nu}}{\nu_{\#}}\|\bar{T}\|_6^{6\delta/(2+\delta)}K_3^2 + C_2^2\|D\bar{\mathbf{u}}\|^2 R_1 + \frac{C_1}{\nu_{\#}}\Big(\bar{\mu}\|\bar{T}\|_6 R_2^2 + \\ +\mu^{\#}R_2(\|\nabla\times\bar{\mathbf{H}}\| + \|\bar{\mathbf{H}}\|_3) + G^{\#}\|\bar{T}\|_{6/5} + \bar{G}\|\bar{T}\|_6 K_4\Big)^2; \\ \frac{1}{4(\sigma^{\#})^2}\|\nabla\times\bar{\mathbf{H}}\|^2 \leq \frac{\bar{\sigma}}{(\sigma_{\#})^2}\|\bar{T}\|_6^{6\epsilon/(2+\epsilon)}(K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) \\ +\bar{\mu}\|\bar{T}\|_6^2 R_1^2 R_2^2 + \mu^{\#}(\|\bar{\mathbf{u}}\|_4^2 R_2^2 + R_1^2\|\bar{\mathbf{H}}\|_4^2). \end{aligned}$$

Now, sum the above two inequalities with (31) rewritten as follows as

$$\frac{k_{\#}}{2}\|\nabla\bar{T}\|^2 \leq \frac{\bar{k}}{k_{\#}}\|\bar{T}\|_6^{6\varepsilon/(2+\varepsilon)}K_2^2 + \frac{C_1}{k_{\#}}\|\bar{\mathbf{u}}\|_6^2 K_4^2.$$

As a result,

$$\begin{aligned} \left(\frac{\nu_{\#}}{2} - C^2R_1 - C\mu^{\#}R_2^2 - \frac{C}{k_{\#}}K_4^2\right)\|D\bar{\mathbf{u}}\|^2 + \\ + \left(\frac{1}{4(\sigma^{\#})^2} - \frac{2C(\mu^{\#})^2}{\nu_{\#}}R_2^2 - C\mu^{\#}R_1^2\right)\|\nabla\bar{\mathbf{H}}\|^2 + \\ + \Bigg(\frac{k_{\#}}{2} - \frac{C}{\nu_{\#}}(\bar{\nu}K_3^2 + (\bar{\mu}R_2^2 + G^{\#} + \bar{G}K_4)^2) - \\ - \frac{C\bar{\sigma}}{(\sigma_{\#})^2}(K_1^2 + \|\mathbf{J}_0\|_{2+\epsilon}^2) - \bar{\mu}R_1^2R_2^2 - \frac{C\bar{k}}{k_{\#}}K_2^2\Bigg)\|\nabla\bar{T}\|^2 \leq 0, \end{aligned}$$

with $C$ standing for different Sobolev constants, and the uniqueness of solution holds under smallness assumption on the data.

# 5 Shape sensivity analysis

In this section we deal with the shape sensivity analysis to the model correspondent to Theorem 2.1, when the coefficients $\nu$, $\mu$, $k$, $\sigma$ and $\alpha$ are assumed constants. First, a family of mappings $\mathcal{T}_\tau : \mathbb{R}^3 \to \mathbb{R}^3$ associated with a given velocity field $V(\tau, x)$ is constructed. The evolution of geometrical domains, if the vector field $V$ is chosen, is governed by the real parameter $\tau$, so we denote by $\Omega_\tau = \mathcal{T}_\tau(\Omega)$ the variable domain depending on *two parameters*, a vector field $V$ and the real variable $\tau$, therefore, the variable $\tau$ has the meaning of the time in our setting. The field $V$ is compactly supported with respect to the spatial variable $x$, i.e.,

$$V \in C(-\tau_1, \tau_1; \mathcal{D}^2(\Omega; \mathbb{R}^3)),\ \mathrm{supp} V \subset \Omega,$$

for some positive constant $\tau_1$. The mapping is given by the system of differential equations

$$\frac{d}{d\tau} x(\tau) = V(\tau, x(\tau)),\ x(0) = X,$$

with the solution denoted by $x(\tau) = x(\tau, X), \tau \in (-\tau_1, \tau_1),\ X \in \mathbb{R}^3$.
We define the family of perturbations of a given initial configuration $\Omega$ by $\Omega_\tau = \mathcal{T}_\tau(\Omega)$, each specific family parametrized by $\tau$ is defined in the direction of a given vector field $V$, and the variable domains are defined by the images of the mapping, and denoted $\Omega_\tau = \{x \in \mathbb{R}^3 |\ x = x(\tau, X), X \in \Omega\}$.

In our setting all equations defined in variable domain $\Omega_\tau$ can be transported to the reference domain which is also called the fixed domain $\Omega$, using the inverse transformation $\mathcal{T}_\tau^{-1} : \Omega_\tau \to \Omega$.

## 5.1 Perturbated problem

We consider that the velocity field $V(\tau, x)$ is divergence free, which implies that also our $\mathbf{u}$ and $\mathbf{H}$ also conserve the divergenceless. This simplifies the situation and we do not need to apply Bogovskii operator, since for pressure we use the standard Rham theorem.

**Definition 5.1.** We say a perturbated problem to the model (1)-(6) in a perturbated domain to the following system of equations in $\Omega_\tau$

$$-\nabla \cdot (\nu D\mathbf{u}^\tau) + (\mathbf{u}^\tau \cdot \nabla)\mathbf{u}^\tau + \nabla p^\tau = \mu \mathrm{rot} \mathbf{H}^\tau \times \mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau; \quad (32)$$

$$\nabla \times (\nabla \times \mathbf{H}^\tau) = \nabla \times (\mathbf{J}_0^\tau + \sigma\mu \mathbf{u}^\tau \times \mathbf{H}^\tau); \quad (33)$$

$$\mathrm{div} \mathbf{u}^\tau = \mathrm{div} \mathbf{H}^\tau = 0; \quad (34)$$

$$-\nabla \cdot (k \nabla T^\tau) + \mathbf{u}^\tau \cdot \nabla T^\tau = f^\tau; \quad (35)$$

with the boundary conditions:

$$\mathbf{u}^\tau = \mathbf{g}^\tau, \qquad \mathbf{H}^\tau \cdot \mathbf{n}^\tau = 0 \quad \text{on } \partial\Omega_\tau; \tag{36}$$

$$T^\tau = 0 \quad \text{on } \Gamma_D^\tau; \quad k\frac{\partial T^\tau}{\partial \mathbf{n}^\tau} + \alpha T^\tau = h^\tau \quad \text{on } \Gamma_N^\tau. \tag{37}$$

We introduce the Hilbert spaces

$$\begin{aligned}
\mathbf{V}^\tau &= \{\mathbf{v} \in \mathbf{H}_0^1(\Omega_\tau) : \quad \text{div } \mathbf{v} = 0 \text{ in } \Omega_\tau\} \\
\mathbf{V}^\tau(rot) &= \{\mathbf{v} \in \mathbf{L}^2(\Omega_\tau) : \text{ rot } \mathbf{v} \in \mathbf{L}^2(\Omega_\tau), \\
&\qquad \text{div } \mathbf{v} = 0 \text{ in } \Omega_\tau, \ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega_\tau\} \\
Z^\tau &= \{\xi \in H^1(\Omega_\tau) : \quad \xi = 0 \text{ on } \Gamma_D^\tau\}
\end{aligned}$$

equipped with their standard inner products.

**Theorem 5.2.** *Let* $\mathbf{f}^\tau \in \mathbf{L}^2(\Omega_\tau)$, $\mathbf{J}_0^\tau \in \mathbf{L}^2(\Omega_\tau)$, $f^\tau \in L^2(\Omega_\tau)$ *and* $h^\tau \in L^2(\Gamma_N^\tau)$. *Assuming moreover*

$$\begin{aligned}
b > 0 \quad \textit{and} \quad & \mu a^2 < b^3 \\
& a = \frac{\nu}{\mu\sigma}\|\mathbf{J}_0^\tau\| \\
& b = \frac{\nu}{\mu\sigma} - \left(\|\mathbf{f}^\tau\| + \frac{G^\#}{k}(\|f^\tau\| + \|h^\tau\|_{\Gamma_N})\right),
\end{aligned}$$

*then the problem (32)-(37) has a weak solution in the following sense:*
*The triple* $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau) \in \mathbf{V}^\tau \times \mathbf{V}^\tau(rot) \times Z^\tau$ *and it satisfies*

$$\nu\int_{\Omega_\tau} D\mathbf{u}^\tau : D\mathbf{v}^\tau dx_\tau + \int_{\Omega_\tau} D\mathbf{u}^\tau : (\mathbf{u}^\tau \otimes \mathbf{v}^\tau) dx_\tau =$$

$$= \int_{\Omega_\tau} \Big(\mu(\nabla\times\mathbf{H}^\tau)\times\mathbf{H}^\tau + \mathbf{f}^\tau - \mathbf{G}(T^\tau)T^\tau\Big)\cdot\mathbf{v}^\tau dx_\tau, \quad \forall \mathbf{v}^\tau \in \mathbf{V}^\tau;$$

$$k\int_{\Omega_\tau} \nabla T^\tau\cdot\nabla\eta^\tau dx_\tau + \int_{\Omega_\tau} \mathbf{u}^\tau\cdot\nabla T^\tau \eta^\tau dx_\tau + \alpha\int_{\Gamma_N^\tau} T^\tau\eta^\tau ds_\tau =$$

$$= \int_{\Omega_\tau} f^\tau\eta^\tau dx_\tau + \int_{\Gamma_N^\tau} h^\tau\eta^\tau ds_\tau, \quad \forall \eta^\tau \in Z^\tau;$$

$$\int_{\Omega_\tau} (\nabla\times\mathbf{H}^\tau)\cdot(\nabla\times\mathbf{w}^\tau)dx_\tau = \sigma\mu\int_{\Omega_\tau} (\mathbf{u}^\tau\times\mathbf{H}^\tau)\cdot(\nabla\times\mathbf{w}^\tau)dx_\tau +$$

$$+ \int_{\Omega_\tau} \mathbf{J}_0^\tau\cdot(\nabla\times\mathbf{w}^\tau)dx_\tau, \quad \forall \mathbf{w}^\tau \in \mathbf{V}^\tau(rot).$$

**Proof.** See the proof of Theorem 2.1.

**Theorem 5.3.** *If the assumptions of Theorem 5.2 are fulfilled, the solution* $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ *in accordance to Theorem 5.2 is such that* $(\mathbf{H}^\tau, T^\tau)$ *belongs to* $\mathbf{W}^{1,2+\epsilon}(\Omega_\tau) \times W^{1,2+\varepsilon}(\Omega_\tau)$ *for some* $\epsilon$, $\varepsilon > 0$. *Moreover, if we assume* $\mathbf{f}^\tau \in \mathbf{L}^{2+\delta_1}(\Omega_\tau)$ *for some* $\delta_1 > 0$ *and* $\mathbf{J}_0^\tau \in \mathbf{L}^q(\Omega_\tau)$ *with* $q$ *given as in (17), then* $\mathbf{u}^\tau \in \mathbf{W}^{1,2+\delta}(\Omega_\tau)$ *for some* $\delta > 0$, *and* $(\mathbf{u}^\tau, \mathbf{H}^\tau, T^\tau)$ *is unique under small data.*

**Proof.** See the proof of Theorems 2.1 and 2.3.

## 5.2 Transported problem

The transported solution to the fixed domain is denoted by $\mathbf{u}_\tau = \mathbf{u}^\tau \circ \mathcal{T}_\tau$, $\mathbf{H}_\tau = \mathbf{H}^\tau \circ \mathcal{T}_\tau$, $T_\tau = T^\tau \circ \mathcal{T}_\tau$ with data $\mathbf{f}_\tau = \mathbf{f}^\tau \circ \mathcal{T}_\tau$, $\mathbf{G}_\tau = \mathbf{G}^\tau \circ \mathcal{T}_\tau$, $\mathbf{J}_{0\tau} = \mathbf{J}_0^\tau \circ \mathcal{T}_\tau$, $f_\tau = f^\tau \circ \mathcal{T}_\tau$ and $h_\tau = h^\tau \circ \mathcal{T}_\tau$.

We begin by recalling the result.

**Proposition 5.4.** *The unit normal vector field on* $\Gamma_\tau$ *is given by*

$$\mathbf{n}_\tau(\mathcal{T}_\tau(X)) = (\|{}^*J\mathcal{T}_\tau^{-1} \cdot \mathbf{n}\|_{\mathbb{R}^3}^{-1}{}^*(D\mathcal{T}_\tau)^{-1} \cdot \mathbf{n})(X)$$

*for* $X \in \Gamma$. *Here we denote by* $J\mathcal{T}_\tau$ *the Jacobian of* $\mathcal{T}_\tau$ *and for any matrix* $B$ *the transposed matrix is denoted by* ${}^*B$. *For any* $f \in L^1(\Gamma_\tau)$,

$$\int_{\Gamma_\tau} f ds_\tau = \int_\Gamma f \circ \mathcal{T}_\tau \|M(\mathcal{T}_\tau) \cdot \mathbf{n}\|_{\mathbb{R}^3} ds,$$

*where* $M(\mathcal{T}_\tau) = det\ (J\mathcal{T}_\tau)^* J\mathcal{T}_\tau^{-1}$ *is the cofactor matrix of the Jacobian matrix* $J\mathcal{T}_\tau$.

We introduce the following notations

$$\begin{aligned}
\zeta(\tau) &= \det(J\mathcal{T}_\tau),\\
\varrho(\tau) &= {}^*J\mathcal{T}_\tau^{-1},\\
A(\tau) &= \zeta(\tau)\ {}^*\varrho(\tau)\varrho(\tau),\\
B(\tau) &= \zeta(\tau)\varrho(\tau),\\
\omega(\tau) &= \|M(J\mathcal{T}_\tau) \cdot \mathbf{n}\|_{\mathbb{R}^3}.
\end{aligned}$$

**Definition 5.5.** We call the transported problem to the following system of equations

$$\nu\int_\Omega A(\tau):(D\mathbf{u}_\tau D\mathbf{v}_\tau)dx+\int_\Omega B(\tau)\nabla\mathbf{u}_\tau:(\mathbf{v}_\tau\otimes\mathbf{u}_\tau)dx=$$

$$=\int_\Omega\zeta(\tau)\Big(\mu((\varrho(\tau)\nabla)\times\mathbf{H}_\tau)\times\mathbf{H}_\tau+\mathbf{f}_\tau-\mathbf{G}(T_\tau)T_\tau\Big)\cdot\mathbf{v}_\tau dx,\quad\forall\mathbf{v}_\tau\in\mathbf{V};$$

$$\int_\Omega((\varrho(\tau)\nabla)\times\mathbf{H}_\tau)\cdot((\varrho(\tau)\nabla)\times\mathbf{w}_\tau)=$$

$$=\sigma\mu\int_\Omega\zeta(\tau)(\mathbf{u}_\tau\times\mathbf{H}_\tau+\mathbf{J}_{0\tau})\cdot((\varrho(\tau)\nabla)\times\mathbf{w}_\tau)dx,\quad\forall\mathbf{w}_\tau\in\mathbf{V}(rot);$$

$$k\int_\Omega A(\tau):(\nabla T_\tau\otimes\nabla\eta_\tau)dx+\int_\Omega B(\tau):(\mathbf{u}_\tau\otimes\nabla T_\tau)\eta_\tau dx+$$

$$+\alpha\int_{\Gamma_N}T_\tau\eta_\tau\omega(\tau)ds=\int_\Omega f_\tau\eta_\tau\zeta(\tau)dx+\int_{\Gamma_N}h_\tau\eta_\tau\omega(\tau)ds,\quad\forall\eta_\tau\in Z.$$

**Theorem 5.6.** *Suppose that the assumptions on* $\mathbf{G}$ *and (12) are fulfilled and additionally assuming that*

$$b>0\quad\textit{and}\quad\mu a^2<b^3$$
$$a=\frac{\nu}{\mu\sigma}\|\mathbf{J}_{0\tau}\|$$
$$b=\frac{\nu}{\mu\sigma}-\left(\|\mathbf{f}_\tau\|+\frac{G^{\#}}{k}(\|f_\tau\|+\|h_\tau\|_{\Gamma_N})\right),$$

*then the triple* $(\mathbf{u}_\tau,\mathbf{H}_\tau,T_\tau)\in\mathbf{V}\times\mathbf{V}(rot)\times Z$ *is a weak solution in the sense of Definition 5.5. Moreover, the solution* $(\mathbf{u}_\tau,\mathbf{H}_\tau,T_\tau)$ *is such that* $(\mathbf{H}_\tau,T_\tau)$ *belongs to* $\mathbf{W}^{1,2+\epsilon}(\Omega)\times W^{1,2+\varepsilon}(\Omega)$ *for some* $\epsilon$, $\varepsilon>0$, *and if* $\mathbf{f}_\tau\in\mathbf{L}^{2+\delta_1}(\Omega)$ *for some* $\delta_1>0$ *and* $\mathbf{J}_{0\tau}\in\mathbf{L}^q(\Omega)$ *with* $q$ *given as in (17) then* $\mathbf{u}_\tau\in\mathbf{W}^{1,2+\delta}(\Omega)$ *for some* $\delta>0$. *Furthermore* $(\mathbf{u}_\tau,\mathbf{H}_\tau,T_\tau)$ *is unique under small data.*

**Proof.** See the proof of Theorems 2.1 and 2.3.
Introducing the forms as

(F1) $\alpha_0(\tau,\mathbf{u},\mathbf{v})=\nu\int_\Omega\zeta(\tau)(\varrho(\tau)D\mathbf{u}):(\varrho(\tau)D\mathbf{v})dx=\nu\int_\Omega A(\tau):(D\mathbf{u}D\mathbf{v})dx$

(F2) $\alpha_1(\tau,\mathbf{u},\mathbf{v})=\int_\Omega\zeta(\tau)(\varrho(\tau)\nabla\mathbf{u}):(\mathbf{v}\otimes\mathbf{u})dx=\int_\Omega B(\tau)\nabla\mathbf{u}:(\mathbf{v}\otimes\mathbf{u})dx$

(F3) $\alpha_2(\tau,\mathbf{H},\mathbf{v})=\mu\int_\Omega\zeta(\tau)\Big(((\varrho(\tau)\nabla)\times\mathbf{H})\times\mathbf{H}\Big)\cdot\mathbf{v}dx$

(F4) $\alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) = \int_\Omega \zeta(\tau)\Big(\mathbf{f} - \mathbf{G}(T)T\Big) \cdot \mathbf{v} dx$

(F5) $\beta_1(\tau, \mathbf{H}, \mathbf{w}) = \int_\Omega ((\varrho(\tau)\nabla) \times \mathbf{H}) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w})\ dx$

(F6) $\beta_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \sigma\mu \int_\Omega \zeta(\tau)(\mathbf{u} \times \mathbf{H}) \cdot ((\varrho(\tau)\nabla) \times \mathbf{w})\ dx$

(F7) $\beta_3(\tau, \mathbf{J}_0, \mathbf{w}) = \int_\Omega \zeta(\tau)\mathbf{J}_0 \cdot ((\varrho(\tau)\nabla) \times \mathbf{w})\ dx$

(F8) $\gamma_1(\tau, T, \eta) = k \int_\Omega A(\tau) : (\nabla T \otimes \nabla \eta) dx$

(F9) $\gamma_2(\tau, \mathbf{u}, T, \eta) = \int_\Omega \zeta(\tau)\mathbf{u} \cdot (\varrho(\tau)\nabla) T\eta dx = \int_\Omega B(\tau) : (\mathbf{u} \otimes \nabla T)\eta dx$

(F10) $\gamma_3(\tau, T, \eta) = \alpha \int_{\Gamma_N} T\eta\omega(\tau) ds$

(F11) $\gamma_4(\tau, f, \eta) = \int_\Omega f\eta\zeta(\tau) dx$

(F12) $\gamma_5(\tau, h, \eta) = \int_{\Gamma_N} h\eta\omega(\tau) ds$

the following corollary can be stated.

**Corollary 5.7.** *Let $|\tau| \le \tau_1$ and $\tau_1$ be small enough, then there exists realvalued functions $g_i$ satisfying $g_i(\tau) = o(\tau)$, $i = 0, ..., 11$ and forms $\tilde{\alpha}_i(\tau, ...), i = 0, 1, 2, 3$, $\tilde{\beta}(\tau, ...)$, $i = 1, 2, 3$, and $\tilde{\gamma}(\tau, ...)$, $i = 1, \cdots, 5$, such that the following statements are valid.*

(B1) *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\begin{aligned}\alpha_0(\tau, \mathbf{u}, \mathbf{v}) \quad &= \quad \alpha_0(0, \mathbf{u}, \mathbf{v}) + \tau\alpha_{0,\tau}(0, \mathbf{u}, \mathbf{v}) + \widetilde{\alpha_0}(\tau, \mathbf{u}, \mathbf{v}) \\ &\quad\quad \alpha_{0,\tau}(0, \mathbf{u}, \mathbf{v}) = \nu \int_\Omega A'(0) : (D\mathbf{u} D\mathbf{v}) dx \\ &\quad\quad \widetilde{\alpha_0}(\tau, \mathbf{u}, \mathbf{v}) \le g_0(\tau)\|\mathbf{u}\|_1 \|\mathbf{v}\|_1.\end{aligned}$$

(B2) *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\begin{aligned}\alpha_1(\tau, \mathbf{u}, \mathbf{v}) \quad &= \quad \alpha_1(0, \mathbf{u}, \mathbf{v}) + \tau\alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) + \widetilde{\alpha_1}(\tau, \mathbf{u}, \mathbf{v}) \\ &\quad\quad \alpha_{1,\tau}(0, \mathbf{u}, \mathbf{v}) = \int_\Omega B'(0)\nabla\mathbf{u} : (\mathbf{v} \otimes \mathbf{u}) dx \\ &\quad\quad \widetilde{\alpha_1}(\tau, \mathbf{u}, \mathbf{v}) \le g_1(\tau)\|\mathbf{u}\|_1^2 \|\mathbf{v}\|_1.\end{aligned}$$

(B3) *For all* $\mathbf{H} \in \mathbf{H}^1(\Omega)$ *and* $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \alpha_2(\tau, \mathbf{H}, \mathbf{v}) &= \alpha_2(0, \mathbf{H}, \mathbf{v}) + \tau\alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) + \widetilde{\alpha_2}(\tau, \mathbf{H}, \mathbf{v}) \\ &\quad \alpha_{2,\tau}(0, \mathbf{H}, \mathbf{v}) = \mu \int_\Omega \Big(\zeta'(0)(\nabla \times \mathbf{H}) \times \mathbf{H} \\ &\quad +((\varrho'(0)\nabla) \times \mathbf{H}) \times \mathbf{H}\Big) \cdot \mathbf{v} dx \\ &\quad \widetilde{\alpha_2}(\tau, \mathbf{H}, \mathbf{v}) \leq g_2(\tau)\|\nabla \times \mathbf{H}\|\|\mathbf{H}\|_1\|\mathbf{v}\|_1. \end{aligned}$$

(B4) *For all* $\mathbf{f} \in \mathbf{L}^2(\Omega)$*,* $T \in Z$ *and* $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \alpha_3(\tau, \mathbf{f}, T, \mathbf{v}) &= \alpha_3(0, \mathbf{f}, T, \mathbf{v}) + \tau\alpha_{3,\tau}(0, \mathbf{f}, T, \mathbf{v}) + \widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}) \\ &\quad \alpha_{3,\tau}(0, \mathbf{f}, T, \mathbf{v}) = \int_\Omega \zeta'(0)\Big(\mathbf{f} - \mathbf{G}(T)T\Big) \cdot \mathbf{v} dx \\ &\quad \widetilde{\alpha_3}(\tau, \mathbf{f}, T, \mathbf{v}) \leq g_3(\tau)\big(\|\mathbf{f}\| + G^{\#}\|T\|\big)\|\mathbf{v}\|. \end{aligned}$$

(B5) *For all* $\mathbf{H}, \mathbf{w} \in \mathbf{V}(rot)$

$$\begin{aligned} \beta_1(\tau, \mathbf{H}, \mathbf{w}) &= \beta_1(0, \mathbf{H}, \mathbf{w}) + \tau\beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) + \widetilde{\beta_1}(\tau, \mathbf{H}, \mathbf{w}) \\ &\quad \beta_{1,\tau}(0, \mathbf{H}, \mathbf{w}) = \int_\Omega A'(0) : (\nabla \times \mathbf{H}) \otimes (\nabla \times \mathbf{w}) dx \\ &\quad \widetilde{\beta_1}(\tau, \mathbf{H}, \mathbf{w}) \leq g_4(\tau)\|\mathbf{H}\|_1\|\mathbf{w}\|_1. \end{aligned}$$

(B6) *For all* $\mathbf{u} \in \mathbf{V},\ \mathbf{H} \in \mathbf{V}(rot),\ \mathbf{w} \in \mathbf{V}(rot)$

$$\begin{aligned} \beta_2(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) &= \beta_2(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \tau\beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) + \widetilde{\beta_2}(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) \\ &\quad \beta_{2,\tau}(0, \mathbf{u}, \mathbf{H}, \mathbf{w}) = \sigma\mu \int_\Omega \Big(\zeta'(0)(\mathbf{u} \times \mathbf{H}) \cdot (\nabla \times \mathbf{w}) + \\ &\quad +(\mathbf{u} \times \mathbf{H}) \cdot ((\varrho'(0)\nabla) \times \mathbf{w})\Big) dx \\ &\quad \widetilde{\beta_2}(\tau, \mathbf{u}, \mathbf{H}, \mathbf{w}) \leq g_5(\tau)\|\mathbf{u} \times \mathbf{H}\|_2\|\mathbf{w}\|_{\mathbf{V}(rot)}. \end{aligned}$$

(B7) *For all* $\mathbf{J}_0 \in \mathbf{L}^2(\Omega),\ \mathbf{w} \in \mathbf{V}(rot)$

$$\beta_3(\tau, \mathbf{J}_0, \mathbf{w}) = \beta_2(0, \mathbf{J}_0, \mathbf{w}) + \tau\beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) + \widetilde{\beta_3}(\tau, \mathbf{J}_0, \mathbf{w})$$

$$\beta_{3,\tau}(0, \mathbf{J}_0, \mathbf{w}) = \int_\Omega \Big(\zeta'(0)\mathbf{J}_0 \cdot (\nabla \times \mathbf{w}) + \mathbf{J}_0 \cdot ((\varrho'(0)\nabla) \times \mathbf{w})\Big) dx$$

$$\widetilde{\beta_3}(\tau, \mathbf{J}_0, \mathbf{w}) \leq g_6(\tau)\|\mathbf{J}_0\|\|\nabla \times \mathbf{w}\|.$$

(B8) *For all* $T, \eta \in Z$

$$\begin{aligned}\gamma_1(\tau, T, \eta) &= \gamma_1(0, T, \eta) + \tau\gamma_{1,\tau}(0, T, \eta) + \widetilde{\gamma_1}(\tau, T, \eta)\\ &\quad \gamma_{1,\tau}(0, T, \eta) = k\int_\Omega A'(0) : (\nabla T \otimes \nabla\eta)dx\\ &\quad \widetilde{\gamma_1}(\tau, T, \eta) \le g_7(\tau)\|T\|_1\|\eta\|_1.\end{aligned}$$

(B9) *For all* $\mathbf{u} \in \mathbf{V}$ *and* $T, \eta \in Z$

$$\begin{aligned}\gamma_2(\tau, \mathbf{u}, T, \eta) &= \gamma_2(0, \mathbf{u}, T, \eta) + \tau\gamma_{2,\tau}(0, \mathbf{u}, T, \eta) + \widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta)\\ &\quad \gamma_{2,\tau}(0, \mathbf{u}, T, \eta) = \int_\Omega B'(0) : (\mathbf{u} \otimes \nabla T)\eta dx\\ &\quad \widetilde{\gamma_2}(\tau, \mathbf{u}, T, \eta) \le g_8(\tau)\|\mathbf{u}\|_1\|T\|_1\|\eta\|_1.\end{aligned}$$

(B10) *For all* $T \in Z, \eta \in Z$

$$\begin{aligned}\gamma_3(\tau, T, \eta) &= \gamma_3(0, T, \eta) + \tau\gamma_{3,\tau}(0, T, \eta) + \widetilde{\gamma_3}(\tau, T, \eta)\\ &\quad \gamma_{3,\tau}(0, T, \eta) = \alpha\int_{\Gamma_N} T\eta\omega'(0)ds\\ &\quad \widetilde{\gamma_3}(\tau, T, \eta) \le g_9(\tau)\|T\|_1\|\eta\|_1.\end{aligned}$$

(B11) *For all* $f \in L^2(\Omega)$ *and* $\eta \in Z$

$$\begin{aligned}\gamma_4(\tau, f, \eta) &= \gamma_4(0, f, \eta) + \tau\gamma_{4,\tau}(0, f, \eta) + \widetilde{\gamma_4}(\tau, f, \eta)\\ &\quad \gamma_{4,\tau}(0, f, \eta) = \int_\Omega f\eta\zeta'(0)dx\\ &\quad \widetilde{\gamma_4}(\tau, f, \eta) \le g_{10}(\tau)\|f\|\|\eta\|.\end{aligned}$$

(B12) *For all* $h \in L^2(\Gamma_N)$ *and* $\eta \in Z$

$$\begin{aligned}\gamma_5(\tau, h, \eta) &= \gamma_5(0, h, \eta) + \tau\gamma_{5,\tau}(0, h, \eta) + \widetilde{\gamma_5}(\tau, h, \eta)\\ &\quad \gamma_{5,\tau}(0, h, \eta) = \int_{\Gamma_N} h\eta\omega'(0)ds\\ &\quad \widetilde{\gamma_5}(\tau, h, \eta) \le g_{11}(\tau)\|h\|_{\Gamma_N}\|\eta\|_1.\end{aligned}$$

Applying Taylor polynomials of degree one we can prove the stability results.

**Proposition 5.8.** *Under the assumptions of Theorem 5.6, if* $(\mathbf{u}_\tau, \mathbf{H}_\tau, T_\tau)$ *is the transported solution correspondent to* $(\mathbf{u}, \mathbf{H}, T)$ *and the following assumptions are fulfilled*

(M1) $\|\mathbf{f}_\tau - \mathbf{f}\| \leq c|\tau|$

(M2) $\|\mathbf{G}(T_\tau)T_\tau - \mathbf{G}(T)T\|_{6/5} \leq c|\tau|$

(M3) $\|\mathbf{J}_{0\tau} - \mathbf{J}_0\| \leq c|\tau|$

(M4) $\|h_\tau - h\|_{\Gamma_N} \leq c|\tau|$

(M5) $\|f_\tau - f\| \leq c|\tau|$

*then we have*

$$\begin{aligned}\|\mathbf{u}_\tau - \mathbf{u}\|_1 &\leq C|\tau|;\\ \|\mathbf{H}_\tau - \mathbf{H}\|_1 &\leq C|\tau|;\\ \|T_\tau - T\|_1 &\leq C|\tau|,\end{aligned}$$

*with* $C$ *denoting different constants.*

**Proof.** The proof is similar to the proof in [5, Proposition ] and we will skip the proof.

Finally, we are in the position to formulate the existence theorem for the material derivative of our problem.

**Definition 5.9.** The following limit in the function space norm $\mathcal{H}$

$$\dot{f} = \lim_{\tau \to 0} \frac{f(\tau) - f(0)}{\tau}$$

is called the strong material derivative $\dot{f}$ of $f$ in $\mathcal{H}$.

**Definition 5.10.** The shape derivative $u'$ of $u(\tau)$ in the direction of the vector field $V$ is defined by the formula

$$u' = \dot{u} - \nabla u \cdot V$$

provided that there exists the material derivative $\dot{u}$.

We recall that $A(0) = B(0) = \varrho(0) = I$ , $\zeta(0) = \omega(0) = 1$, $\dot{\zeta} = \zeta'(0)$, $\dot{\varrho} = \varrho'(0)$, $\dot{A} = A'(0)$, $\dot{B} = B'(0)$, $\dot{\zeta} = \zeta'(0)$ and $\dot{\omega} = \omega'(0)$, and we state the following result on the existence of material derivatives.

**Theorem 5.11.** *Under the assumptions* $\dot{\mathbf{f}} \in \mathbf{L}^2(\Omega), \dot{\mathbf{J}} \in \mathbf{L}^2(\Omega), \dot{f} \in L^2(\Omega), \dot{h} \in L^2(\Gamma_N)$. *Assuming moreover*

$$\tilde{b} > 0 \quad \textit{and} \quad \mu\tilde{a}^2 < \tilde{b}^3$$
$$\tilde{a} = \frac{\nu}{\mu\sigma}\|\dot{\mathbf{J}}_0\|$$
$$\tilde{b} = \frac{\nu}{\mu\sigma} - \left(\|\dot{\mathbf{f}}\| + \frac{\dot{G}^{\#}}{k}(\|\dot{f}\| + \|\dot{h}\|_{\Gamma_N})\right),$$

*the triple* $(\dot{\mathbf{u}}, \dot{\mathbf{H}}, \dot{T}) \in \mathbf{V} \times \mathbf{V}(rot) \times Z$ *and it satisfies*
**the momentum equations**

$$\nu \int_\Omega (\dot{A}D\mathbf{u} + D\dot{\mathbf{u}}) : D\mathbf{v}dx + \int_\Omega (\dot{B}\nabla\mathbf{u} + \nabla\dot{\mathbf{u}}) : (\mathbf{v} \otimes \mathbf{u})dx + \int_\Omega \nabla\mathbf{u} : (\mathbf{v} \otimes \dot{\mathbf{u}})dx$$

$$= \mu \int_\Omega \dot{\zeta}\Big((\nabla \times \mathbf{H}) \times \mathbf{H}\Big) \cdot \mathbf{v}dx + \mu \int_\Omega \Big(((\dot{\varrho}\nabla) \times \mathbf{H}) \times \mathbf{H}\Big) \cdot \mathbf{v}dx +$$
$$+\mu \int_\Omega \Big((\nabla \times \dot{\mathbf{H}}) \times \mathbf{H} + (\nabla \times \mathbf{H}) \times \dot{\mathbf{H}}\Big) \cdot \mathbf{v}dx +$$

$$+ \int_\Omega \Big(\dot{\mathbf{f}} - \dot{\mathbf{G}}(T)T - \mathbf{G}(T)\dot{T}\Big) \cdot \mathbf{v}dx + \int_\Omega \Big((\mathbf{f} - \mathbf{G}(T)T)\dot{\zeta}\Big) \cdot \mathbf{v}dx, \quad \forall \mathbf{v} \in \mathbf{V};$$

**the equation for the electric field**

$$\int_\Omega ((\dot{\varrho}\nabla) \times \mathbf{H} + \nabla \times \dot{\mathbf{H}}) \cdot (\nabla \times \mathbf{w})\ dx + \int_\Omega (\nabla \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w})\ dx =$$

$$= \sigma\mu \int_\Omega \Big(\dot{\zeta}(\mathbf{u} \times \mathbf{H}) + \dot{\mathbf{u}} \times \mathbf{H} + \mathbf{u} \times \dot{\mathbf{H}}\Big) \cdot (\nabla \times \mathbf{w})dx +$$
$$+\sigma\mu \int_\Omega (\mathbf{u} \times \mathbf{H}) \cdot ((\dot{\varrho}\nabla) \times \mathbf{w})dx +$$

$$+ \int_\Omega (\dot{\mathbf{J}}_0 + \mathbf{J}_0\dot{\zeta}) \cdot (\nabla \times \mathbf{w})dx + \int_\Omega \mathbf{J}_0 \cdot (\dot{\varrho}\nabla) \times \mathbf{w}dx, \quad \forall \mathbf{w} \in \mathbf{V}(rot);$$

**the energy equation**

$$k \int_\Omega (\dot{A}\nabla T + \nabla\dot{T}) \cdot \nabla\eta dx + \int_\Omega (\dot{B} : \mathbf{u} \otimes \nabla T + \dot{\mathbf{u}} \cdot \nabla T + \mathbf{u} \cdot \nabla\dot{T})\eta dx$$
$$+\alpha \int_{\Gamma_N} (\dot{T} + T\dot{\omega})\eta ds = \int_\Omega (\dot{f} + f\dot{\zeta})\eta dx + \int_{\Gamma_N} (\dot{h} + h\dot{\omega})\eta ds, \quad \forall \eta \in Z;$$

*and the following estimates*

$$\|\dot{T}\|_1 \leq C\Big((1+\|\mathbf{u}\|_1+\|\dot{\mathbf{u}}\|_1)\|T\|_1+\|\dot{f}\|+\|f\|+\|\dot{h}\|_{\Gamma_N}+\|h\|_{\Gamma_N}\Big);$$

$$\|\dot{\mathbf{u}}\|_1 \leq C\Big((\|\dot{\mathbf{H}}\|_1+\|\mathbf{H}\|_1)\|\mathbf{H}\|_1+\|\dot{\mathbf{f}}\|+\|\mathbf{f}\|+ \\ +\dot{G}^{\#}\|T\|_1+G^{\#}(\|\dot{T}\|_1+\|T\|_1)+\|\mathbf{u}\|_1\Big);$$

$$\|\dot{\mathbf{H}}\|_1 \leq C\Big(\|\mathbf{H}\|_1+\mu\sigma(\|\mathbf{u}\times\mathbf{H}\|+\|\dot{\mathbf{u}}\times\mathbf{H}\|+\|\mathbf{u}\times\dot{\mathbf{H}}\|)+\|\dot{\mathbf{J}}_0\|+\|\mathbf{J}_0\|\Big).$$

**Proof.** We subtract the perturbated solution and the transported solution and we pass to the limit with $\tau$ tending to 0 (for details see [5] for analogous proof).

# 6 Concluding remarks

As we mention in introduction to overcome problem of loosing divergence free behavior we can apply Piola transform which is given by the following mapping:

$$P_I : \mathbf{V} \to \mathbf{V}^\tau;$$
$$\mathbf{v} \mapsto (J\mathcal{T}_\tau \cdot \mathbf{v}) \circ \mathcal{T}_\tau^{-1}.$$

Denoting

$$\hat{\mathbf{u}}_\tau := (J\mathcal{T}_\tau)^{-1} \cdot (\mathbf{u}_\tau \circ \mathcal{T}_\tau) \text{ defined on } \Omega$$

and

$$\mathbf{u}_\tau = P_I(\hat{\mathbf{u}}_\tau) \text{ is defined on } \Omega_\tau,$$

the mapping $P_I$ can be applied on velocity field and also on magnetic field to conserve the divergenceless and that $\mathbf{u}\cdot\mathbf{n}=0$ and $\mathbf{H}\cdot\mathbf{n}=0$. By the same method as in Section 5 we get the stability and material derivative for $\hat{\mathbf{u}}$ and then we just apply the inverse mapping to conclude the results in [5].

**Remark 6.1.** In [5] we get the stability depending not only on the data but also on assumption of behavior of $\mathbf{H}$, but it is not the case in our present problem.

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