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OPERATOR MACHINES ON DIRECTED GRAPHS

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ABSTRACT. We show that if an infinite-dimensional Banach space X has a symmetric basis then there exists a bounded, linear operator $R : X \longrightarrow X$ such that the set

$$A = \{x \in X \;:\; ||R^n x|| \to \infty\}$$

is non-empty and nowhere dense in X. Moreover, if $x \in X \setminus A$ then some subsequence of $(R^n x)_{n=1}^\infty$ converges weakly to x. This answers in the negative a recent conjecture of Prăjitură. The result can be extended to any Banach space containing an infinite-dimensional complemented subspace with a symmetric basis; in particular, all 'classical' Banach spaces admit such an operator.

1. Introduction

Given an infinite-dimensional Banach space X, a bounded linear operator $T : X \longrightarrow X$ and $x \in X$, we say that the *orbit of x with respect to T* is the set

$$\operatorname{orb}(x, T) \;=\; \{T^n x \;:\; n \in \mathbb{N}\}.$$

The study of orbits of points in infinite-dimensional linear spaces was initiated in [4]. In this paper, Rolewicz proved that in the infinite-dimensional case, it is possible to find examples of X, T and x as above, with the property that $\operatorname{orb}(x, T)$ is norm-dense in X. Such *hypercyclic* operators are a strictly infinite-dimensional phenomenon and have received considerable coverage in the recent literature, not least because their study is connected with the still open problem of whether every operator on ℓ_2 has a non-trivial closed, invariant subset. Indeed, an operator T on a Banach space X has such a subset if and only if $\operatorname{orb}(x, T)$ is not norm-dense for some non-zero $x \in X$. Orbits of points under operators have been the subject of study in other contexts. For example, in [3], it is shown that given an operator $T : X \longrightarrow X$, if the sequence $(||T^n||^{-1})_{n=1}^\infty$ is summable then there exists a vector $x \in X$ with the property that $||T^n x|| \to \infty$, and thus, T admits a non-trivial, closed invariant set. The various ways in which the sequences $(||T^n x||)_{n=1}^\infty$ can behave, as x ranges over X, is examined in [2]. Prăjitură makes the following conjecture.

Conjecture 1.1 ([2, Conjecture 2.9])**.** *Given an operator T on a Banach space, if x is a vector such that $||T^n x|| \to \infty$, then the set of all vectors with this property is norm-dense in X.*

Of course, by the Uniform Boundedness Principle, if $||T^n x|| \to \infty$ for some x then the set of y with the property that $\sup_n ||T^n y|| = \infty$ is a norm-dense G_δ in X,

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but this fact only implies that $||T^n y|| \to \infty$ if the limit exists. The object of this note is to provide a negative answer to Conjecture 1.1. Here follows a more precise statement of that given in the abstract.

Theorem 1.2. *Let X have a symmetric basis $(e_i)_{i=1}^{\infty}$. Then there exists a bounded, linear operator $R : X \longrightarrow X$ such that given $x = \sum_{i=1}^{\infty} x_i e_i \in X$ satisfying $x_1 \neq 0$ and $x_2 = 0$, we have*

$$||R^n x|| \to \infty.$$

On the other hand, if $x_1 = 0$ or $x_2 \neq 0$ then there exists a subsequence of $(R^n x)_{n=1}^{\infty}$ which converges weakly to x.

We also obtain the following corollary, which shows that all 'classical' Banach spaces admit such an operator.

Corollary 1.3. *If Y has an infinite-dimensional complemented subspace X with a symmetric basis then there exists a bounded linear operator $W : Y \longrightarrow Y$ such that the set*

$$B = \{y \in Y \ : \ ||W^n y|| \to \infty\}$$

is non-empty and nowhere dense. Moreover, if $y \in Y \setminus B$ then there exists a subsequence of $(W^n y)$ which converges weakly to y.

2. Local estimates

Our map R in Theorem 1.2 is going to be a block diagonal operator on X. In this section, we build the template for the operators acting on the blocks and gather together some basic estimates. Let $m, T \in \mathbb{N}$, $\varepsilon > 0$ and $Y = \ell_p^T$, where $4m \leq T$ and $1 \leq p \leq \infty$. Define the operators $S : Y \longrightarrow Y$ and $F : \mathbb{R} \longrightarrow Y$ by

$$S(y) = (y_T, y_1, \ldots, y_{T-1})$$

where $y = (y_1, \ldots, y_T)$, and

$$F(a) = (\underbrace{\varepsilon a, \ldots, \varepsilon a}_{m \text{ times}}, \underbrace{-\varepsilon a, \ldots, -\varepsilon a}_{m \text{ times}}, 0, \ldots, 0).$$

In this way, S can be described as a shift operator and F a 'feed' operator. Let $R : \mathbb{R} \oplus Y \longrightarrow \mathbb{R} \oplus Y$ be defined by $R(a, y) = (a, S(y) + F(a))$. We are interested in the behaviour of $R^t(a, 0)$ at time $t \in \mathbb{N}$. We can imagine that S drives a circular conveyor belt in a factory and F deposits the factory's product (albeit some of it negative) onto the belt at a fixed set of positions. The amount of product deposited depends on the value of the first coordinate. Using this analogy, we can see that the result of repeated applications of R to the vector $(a, 0)$ can be viewed as the sum of two bumps: one stationary bump of height εam and base width $2m$, and a moving bump of height $-\varepsilon am$ and base width again $2m$. The moving bump's motion is periodic, with period T. Let us denote by P the map $(a, y) \mapsto y$.

Lemma 2.1. *Suppose that $1 \leq p \leq \infty$. There exists a constant L, depending only on p, such that*

(1) *if* $m \le t \le T - m$ *then*

$$\big|\big|PR^t(a,0)\big|\big| \ge \begin{cases} \left(\frac{2}{p+1}\right)^{p^{-1}} \varepsilon|a|m^{(p+1)p^{-1}} & \textit{if } p < \infty \\ \varepsilon|a|m & \textit{if } p = \infty; \end{cases} \tag{1}$$

(2) *at all times* t *we have*

$$\big|\big|PR^t(a,0)\big|\big| \le \begin{cases} L\varepsilon|a|m^{(p+1)p^{-1}} & \textit{if } p < \infty \\ L\varepsilon|a|m & \textit{if } p = \infty; \end{cases} \tag{2}$$

(3) *if* $t \le m$ *then*

$$\big|\big|PR^t(a,0)\big|\big| \le \begin{cases} L\varepsilon|a|m^{p^{-1}}t & \textit{if } p < \infty \\ L\varepsilon|a|t & \textit{if } p = \infty. \end{cases} \tag{3}$$

Proof. We estimate the norm of the sum of the standing and moving bumps. If $p = \infty$ we simply measure the absolute height of the sum of the bumps to obtain the values listed above, with $L = 1$. From now on, we shall assume that $p < \infty$. Set

$$L \;=\; \left(\frac{2^{p+3}}{p+1}\right)^{\frac{1}{p}} \;>\; \left(2 + \frac{2^{p+2}+1}{p+1}\right)^{\frac{1}{p}}.$$

In case (1), we have

$$\big|\big|PR^t(a,0)\big|\big|^p \;\ge\; 2\varepsilon^p|a|^p \int_0^m s^p \,\mathrm{d}s \;=\; \frac{2\varepsilon^p|a|^p}{p+1} m^{p+1}.$$

In case (2), we note that the maximum value of the norm is attained when the supports of the standing and moving bumps are disjoint, which occurs if and only if $2m \le t \le T - 2m$. Thus we estimate

$$\big|\big|PR^t(a,0)\big|\big|^p \;\le\; 4\varepsilon^p|a|^p \int_0^{m+1} s^p \,\mathrm{d}s \;=\; \frac{4\varepsilon^p|a|^p}{p+1}(m+1)^{p+1} \;\le\; \frac{2^{p+3}\varepsilon^p|a|^p}{p+1} m^{p+1}.$$

For (3), when $t \le m$, we have

$$\begin{aligned} \big|\big|PR^t(a,0)\big|\big|^p &\le 2\varepsilon^p|a|^p \left\{ (m-t)t^p + \int_0^{t+1} s^p \,\mathrm{d}s + \int_0^{\frac{t}{2}} (2s)^p \,\mathrm{d}s \right\} \\ &= 2\varepsilon^p|a|^p \left\{ (m-t)t^p + \frac{(t+1)^{p+1}}{p+1} + \frac{t^{p+1}}{2(p+1)} \right\} \\ &\le \left(2 + \frac{2^{p+2}+1}{p+1}\right) \varepsilon^p|a|^p m t^p. \end{aligned}$$

□

In order to build our operator R on a Banach space X with a symmetric basis, we will need some reasonably precise estimates the norms of certain vectors in X. In order to do this, we combine the estimates of Lemma 2.1 with a result closely based on a theorem of Tzafriri [5]. We have altered the statement of the next result to suit our purposes.

Proposition 2.2 ([5, Proposition 5]). *Let V be a 2^n-dimensional vector space with basis $(v_\sigma)_{\sigma\in G}$, where G is the set of all functions from $\{1,\ldots,n\}$ to $\{-1,1\}$. Suppose that there are constants $K>0$ and $r>2$ such that given scalars a_σ, $\sigma\in G$, we have*

$$\frac{K^{-1}}{(2^n)^{\frac{1}{s}}}\left(\sum_{\sigma\in G}|a_\sigma|^s\right)^{\frac{1}{s}} \le \left|\left|\sum_{\sigma\in G} a_\sigma v_\sigma\right|\right| \Big/ \left|\left|\sum_{\sigma\in G} v_\sigma\right|\right| \le \frac{K}{(2^n)^{\frac{1}{r}}}\left(\sum_{\sigma\in G}|a_\sigma|^r\right)^{\frac{1}{r}}$$

where $r^{-1}+s^{-1}=1$. Then there exists M, dependent on K and r, but independent of V and n, with the property that if we define

$$w_l = \sum_{\sigma\in G}\sigma(l)v_\sigma$$

for $1\le l\le n$, then $d([w_l]_{l=1}^n,\ell_2^n) < M$.

The proof of the next result closely follows that of [5, Theorem 1], although we note that the assumed symmetry of the norm allows us to bypass the Ramsey arguments that feature in [5]. Tzafriri's notation has also been modified slightly to suit our requirements.

Lemma 2.3. *Let X have a normalised symmetric basis $(e_i)_{i=1}^\infty$ with conjugate system $(e_i^*)_{i=1}^\infty$ and symmetric norm $||\cdot||$. Then there exists $M>0$ and $p\in\{1,2,\infty\}$, a pairwise disjoint family of finite subsets $F_n\subseteq\mathbb{N}$, $n\in\mathbb{N}$, vectors $w_{l,n}$, $1\le l\le n$, supported on F_n and permutations π_n of F_n with three properties:*

(1) *given n, if a linear operator S on X satisfies $Se_i = e_{\pi_n(i)}$ for all $i\in F_n$, then $Sw_{l,n} = w_{\tau(l),n}$, where τ is the cycle $(1,\ldots,n)$;*
(2) *$d([w_{l,n}]_{l=1}^n,\ell_p^n) < M$ for all n;*
(3) *π_n has order n.*

Proof. Define

$$\lambda(n) = ||e_1+\ldots+e_n|| \quad \text{and} \quad \mu(n) = ||e_1^*+\ldots+e_n^*||\,.$$

We follow the proof of [5, Theorem 1] in distinguishing three cases.

Case I: for every $n\in\mathbb{N}$ there exists $m_n\in\mathbb{N}$ such that $\lambda(nm_n)/\lambda(m_n) < 2$. Put $p=\infty$. Set $k_1=0$ and, given k_n, define $k_{n+1}=k_n+nm_n$. Let

$$F_n = \{k_n+1,\ldots,k_n+nm_n\}$$

and define

$$w_{l,n} = (e_{k_n+(l-1)m_n+1}+\ldots e_{k_n+lm_n})/\lambda(m_n)$$

for $1\le l\le n$, $n\in\mathbb{N}$. Finally, define

$$\pi_n(k_n+(l-1)m_n+r) = \begin{cases} k_n+lm_n+r & \text{if } 1\le l<n \text{ and } 1\le r\le m_n \\ k_n+r & \text{if } l=n \text{ and } 1\le r\le m_n. \end{cases}$$

It is clear that the F_n are pairwise disjoint and properties (1) and (3) hold. Now we prove (2). By the symmetry of the norm, we have $||w_{l,n}||=1$. Since

$$\max_{l=1}^n |a_l| \;\le\; \left|\left|\sum_{l=1}^n a_l w_{l,n}\right|\right| \;\le\; \max_{l=1}^n |a_l| \left|\left|\sum_{l=1}^n w_{l,n}\right|\right|$$

$$\leq \max_{l=1}^n |a_l| \frac{\lambda(nm_n)}{\lambda(m_n)} \leq 2\max_{l=1}^n |a_l|$$

for any scalars $a_1, \dots, a_n$, we can see that (2) holds for any $M > 2$.

Case II: for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\mu(nm_n)/\mu(m_n) < 2$. Now put $p = 1$ and set k_n, F_n and π_n exactly as in case I. If we set

$$w^*_{l,n} = (e^*_{k_n+(l-1)m_n+1} + \dots e^*_{k_n+lm_n})/\mu(m_n).$$

then we have

$$\max_{l=1}^n |a_l| \leq \left|\left|\sum_{l=1}^n a_l w^*_{l,n}\right|\right| \leq 2\max_{l=1}^n |a_l|$$

just as above. Let $w_{1,n}$ satisfy $||w_{1,n}|| = 1$ and $w^*_{1,n}(w_{1,n}) > \frac{1}{2}$, and have support contained in $\{k_n+1, k_n+m_n\}$, i.e., the support of $w^*_{1,n}$. If we let S be a linear operator satisfying $Se_i = e_{\pi_n(i)}$ for $i \in F_n$, and define $w_{l,n} = S^{l-1}w_{1,n}$ for $1 < l \leq n$, then it follows by the symmetry of the norm that $||w_{l,n}|| = 1$ and $w^*_{l,n}(w_{l,n}) = w^*_{1,n}(w_{1,n})$ whenever $1 \leq l \leq n$. By design, we have ensured that (1) holds. To check (2), we observe that

$$\left|\left|\sum_{l=1}^n a_l w_{l,n}\right|\right| \leq \sum_{l=1}^n |a_l| \leq 2\left(\sum_{l=1}^n (\operatorname{sgn} a_l) w^*_{l,n}\right)\left(\sum_{k=1}^n a_k w_{k,n}\right) \leq 4\left|\left|\sum_{l=1}^n a_l w_{l,n}\right|\right|.$$

Therefore (2) holds whenever $M > 4$.

Case III: if neither case I nor case II hold then, following the proof of [5, Theorem 1] in case III, we obtain constants $K > 0$ and $r > 2$ such that for all $n \in \mathbb{N}$ and scalars $a_1, \dots, a_n$, we have

$$\frac{K^{-1}}{n^{\frac{1}{s}}}\left(\sum_{l=1}^n |a_l|^s\right)^{\frac{1}{s}} \leq \frac{1}{\lambda(n)}\left|\left|\sum_{l=1}^n a_l e_{n+l}\right|\right| \leq \frac{K}{n^{\frac{1}{r}}}\left(\sum_{l=1}^n |a_l|^r\right)^{\frac{1}{r}}$$

where $r^{-1} + s^{-1} = 1$. We set $p = 2$ and

$$F_n = \{2^n + 1, \dots, 2^{n+1}\}.$$

Fix n and let f be a bijection from $F = F_n$ to G, where G is as in Proposition 2.2. Put $v_\sigma = e_{f^{-1}(\sigma)}$ for $\sigma \in G$, and let w_l, $1 \leq l \leq n$, be as in Proposition 2.2. Let τ be the cycle $(1, \dots, n)$, define a permutation $\hat{\pi}$ on G by $\hat{\pi}(\sigma) = \sigma \circ \tau^{-1}$, and then set $\pi = f^{-1} \circ \hat{\pi} \circ f$. We have (3). If S is an operator on X satisfying $Se_i = e_{\pi(i)}$ then we calculate

$$\begin{aligned} Sw_l &= S\left(\sum_{\sigma\in G}\sigma(l)v_\sigma\right) = S\left(\sum_{\sigma\in G}\sigma(l)e_{f^{-1}(\sigma)}\right) \\ &= \sum_{\sigma\in G}\sigma(l)e_{f^{-1}(\hat{\pi}(\sigma))} \\ &= \sum_{\sigma\in G}\sigma(l)v_{\hat{\pi}(\sigma)} = \sum_{\sigma\in G}(\sigma\circ\tau)(l)v_\sigma = w_{\tau(l)}. \end{aligned}$$

Moreover, by construction, we have ensured that $d([w_l]_{l=1}^n, \ell_p^n) < M$. $\square$

We remark that we can follow the proof of [5, Theorem 1] a little more to show that the subspaces $[w_{l,n}]_{l=1}^n$, $n \in \mathbb{N}$, are uniformly complemented in X, that is, they are the images of a sequence of projections which are uniformly bounded in norm. However, we do not require this particular property of the $[w_{l,n}]_{l=1}^n$.

3. Proofs of the main results

We shall prove Theorem 1.2 using a sequence of lemmas. We take constants $m_k, T_k \in \mathbb{N}$, $\varepsilon_k > 0$ and $\lambda_k \in \mathbb{R}$. The values of these constants will be chosen in due course. Let X have a normalised symmetric basis $(e_i)_{i=1}^\infty$ with symmetric norm $||\cdot||$, and let M, p, F_n, $w_{l,n}$ and π_n be as in Lemma 2.3, with the additional constraint that $F_n \subseteq \mathbb{N} \setminus \{1, 2\}$ for all n. Define

$$Se_i = \begin{cases} e_{\pi_{T_k}(i)} & \text{if } i \in F_{T_k} \text{ for some } T_k \\ e_i & \text{otherwise} \end{cases}$$

and extend S linearly to X. As $||\cdot||$ is symmetric, S is an isometry. Define operators $S_k : [w_{l,T_k}]_{l=1}^{T_k} \longrightarrow [w_{l,T_k}]_{l=1}^{T_k}$ and $F_k : \mathbb{R} \longrightarrow [w_{l,T_k}]_{l=1}^{T_k}$ by

$$S_k \left(\sum_{l=1}^{T_k} y_l w_{l,T_k} \right) = \sum_{l=1}^{T_k} y_l w_{\tau(l),T_k}$$

where τ is the cycle $(1, \ldots, T_k)$, and

$$F_k(a) = a\varepsilon_k \sum_{l=1}^{m_k} w_{l,T_k} - a\varepsilon_k \sum_{l=m_k+1}^{2m_k} w_{l,T_k}.$$

Then define R_k on $\mathbb{R} \oplus [w_{l,T_k}]_{l=1}^{T_k}$ by

$$R_k(a, y) = (a, S_k(y) + F_k(a))$$

and let $P_k(a, y) = y$ for $y \in [w_{l,T_k}]_{l=1}^{T_k}$. Let P be the projection

$$P \left(\sum_{i=1}^\infty x_i e_i \right) = x_1 e_1 + x_2 e_2$$

and define an operator R on X by

$$Rx = Sx + \sum_{k=1}^\infty F_k(x_1 - \lambda_k x_2).$$

where $x = \sum_{i=1}^\infty x_i e_i$. Of course, it is necessary to choose the various constants so that R is bounded and maps into X.

First of all, we define the constants λ_k. Let $\lambda_1 = 0$. Define $f(t) = t/(1-t)$ for $0 \leq t < 1$. Given $n \geq 1$, we set

$$\lambda_k = \begin{cases} f(\frac{k-2^n}{2^{n-1}}) & \text{if } 2^n \leq k < 2^n + 2^{n-1} \\ -f(\frac{k-2^n-2^{n-1}}{2^{n-1}}) & \text{if } 2^n + 2^{n-1} \leq k < 2^{n+1}. \end{cases}$$

Before defining m_k, T_k and ε_k, we observe two important inequalities concerning the λ_k. The first identifies an overall bound for the quantity $|x_1 - \lambda_k x_2|$, for k in the range $2^n \leq k < 2^{n+1}$. The second shows that $|x_1 - \lambda_k x_2|$ is small for some k in this range, provided $x_1 = 0$ or $x_2 \neq 0$, and n is large enough. The idea behind the second inequality is that if $x_2 \neq 0$, we have an infinite supply of the λ_k which can approximate the solution to the equation $x_1 - \lambda x_2 = 0$ with arbitrary precision.

Lemma 3.1. *Let $x \in X$. First*

$$|x_1 - \lambda_k x_2| \leq 2^{n-1} ||x|| \quad \textit{whenever } 2^n \leq k < 2^{n+1}. \tag{4}$$

Second, if $x_2 \neq 0$ then, for every n large enough, there exists k so that $2^n \leq k < 2^{n+1}$ and

$$|x_1 - \lambda_k x_2| \leq \frac{2^{4-n} ||x||^2}{|x_2|}. \tag{5}$$

Proof. For (4), we simply observe that

$$\begin{aligned} |x_1 - \lambda_k x_2| &\leq (1 + \lambda_{2^n+2^{n-1}-1}) ||x|| \\ &= (1 + (2^{n-1} - 1)) ||x|| \\ &= 2^{n-1} ||x|| \end{aligned}$$

whenever $2^n \leq k < 2^{n+1}$.

To show (5), we first let λ satisfy $x_1 - \lambda x_2 = 0$. We shall assume first that $\lambda \geq 0$. We take n large enough so that $\lambda < f(1 - 2^{n-1}) = \lambda_{2^n+2^{n-1}-1}$. This allows us to find k in the range $2^n \leq k < 2^n + 2^{n-1} - 1$ such that $\lambda_k \leq \lambda < \lambda_{k+1}$. Hence

$$\begin{aligned} 0 \leq \lambda - \lambda_k &< \lambda_{k+1} - \lambda_k \\ &< 2^{1-n} f'(f^{-1}(\lambda_{k+1})) \quad \text{by the Mean Value Theorem} \\ &\leq 2^{1-n} f'(f^{-1}(\lambda) + 2^{1-n}) \quad \text{since } f^{-1}(\lambda_{k+1}) = f^{-1}(\lambda_k) + 2^{1-n} \\ &= 2^{1-n} \left(\frac{1}{1 - 2^{1-n} - f^{-1}(\lambda)} \right)^2. \end{aligned}$$

Therefore

$$2^{n-1}(\lambda - \lambda_k) < \left(\frac{1}{1 - 2^{1-n} - f^{-1}(\lambda)} \right)^2 \to \left(\frac{1}{1 - f^{-1}(\lambda)} \right)^2 = (\lambda + 1)^2$$

as $n \to \infty$, bearing in mind that $f^{-1}(t) = t/(t+1)$ for $t \geq 0$. We conclude that for large enough n, we can select k in the range $2^n \leq k < 2^n + 2^{n-1} - 1$ so that

$$0 \leq \lambda - \lambda_k < 2^{2-n}(\lambda + 1)^2.$$

Finally

$$\begin{aligned} |x_1 - \lambda_k x_2| &= (\lambda - \lambda_k)|x_2| \\ &\leq 2^{2-n} \left(\frac{x_1 + x_2}{x_2} \right)^2 |x_2| \\ &\leq \frac{2^{4-n} ||x||^2}{|x_2|} \end{aligned}$$

for such k, as required. If $\lambda < 0$ then we can appeal to symmetry and repeat the process, using k in the range $2^n + 2^{n-1} \leq k < 2^{n+1} - 1$. $\square$

Now we define the constants m_k, T_k and ε_k. First let $m_1 = 1$, $T_1 = 4$ and $\varepsilon_1 = 0$. Then set $T_k = (5^n + 1)T_{k-1}$, $m_k = T_{k-1} - m_{k-1}$ and

$$\varepsilon_k = \begin{cases} \frac{n}{m_k^{(p+1)p^{-1}}} & \text{if } p = 1 \text{ or } p = 2 \\ \frac{n}{m_k} & \text{if } p = \infty \end{cases}$$

whenever $n \geq 1$ and $2^n \leq k < 2^{n+1}$. Our first task is to show that, with respect to these constants, R is a bounded operator mapping into X.

Lemma 3.2. *The operator R is bounded and maps into X.*

Proof. We show that $\sum_{k=1}^{\infty} F_k(x_1 - \lambda_k x_2)$ is absolutely summable. By Lemma 2.3, part 2, we have

$$\sqrt{M^{-1}}\,||y|| \leq \left|\left|\sum_{l=1}^{T_k} y_l w_{l,T_k}\right|\right| \leq \sqrt{M}\,||y|| \tag{6}$$

where $y = (y_1, \ldots, y_{T_k}) \in \ell_p^{T_k}$ and $p \in \{1, 2, \infty\}$ is as in Lemma 2.3. Let L be as in Lemma 2.1. Note that $F_k(a) = S_k(0) + F_k(a) = P_k R_k(a, 0)$. Therefore, from (3) with $t = 1$, (4) and the definition of ε_k, we have

$$\begin{aligned} ||F_k(x_1 - \lambda_k x_2)|| &\leq \begin{cases} \sqrt{M} L \varepsilon_k |x_1 - \lambda_k x_2| m_k^{p^{-1}} & \text{if } p = 1 \text{ or } p = 2 \\ \sqrt{M} L \varepsilon_k |x_1 - \lambda_k x_2| & \text{if } p = \infty. \end{cases} \\ &\leq \sqrt{M} L n 2^{n-1}\,||x||\,m_k^{-1} \end{aligned}$$

whenever $2^n \leq k < 2^{n+1}$.

From the definitions of m_k and T_k, we obtain

$$m_{k+1} = T_k - m_k = T_k - T_{k-1} + m_{k-1} \geq 5^n T_{k-1} \geq 5^n m_k \tag{7}$$

whenever $n \geq 1$ and $2^n \leq k < 2^{n+1}$. In particular, $m_{k+1} \geq 5 m_k$. Therefore

$$\begin{aligned} \sum_{k=2}^{\infty} ||F_k(x_1 - \lambda_k x_2)|| &\leq \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \sqrt{M} L n 2^{n-1}\,||x||\,m_k^{-1} \\ &\leq \sqrt{M} L m_2^{-1}\,||x|| \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} \frac{n 2^{n-1}}{5^{k-2}} \\ &\leq \sqrt{M} L m_2^{-1}\,||x|| \sum_{n=1}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} n(\tfrac{2}{5})^{n-1} \\ &= \tfrac{5}{2} \sqrt{M} L m_2^{-1}\,||x|| \sum_{n=1}^{\infty} n(\tfrac{4}{5})^n \end{aligned}$$

bearing in mind that $n - 1 \leq k - 2$. Hence $Rx \in X$ and R is bounded. ∎

In order to analyse the behaviour of $R^m x$, it will help to consider separately $R^m P x$ and $R^m (I - P) x$.

Lemma 3.3. *We have*

$$R^m(I-P)x = S^m(I-P)x \tag{8}$$

and

$$R^m Px = Px + \sum_{k=1}^{\infty} P_k R_k^m (x_1 - \lambda_k x_2, 0) \tag{9}$$

for all m.

Proof. Clearly $R(I-P)x = S(I-P)x$. Since $(I-P)S = S(I-P)$, if (8) holds for $m \geq 1$ then

$R^{m+1}(I-P)x = RS^m(I-P)x = R(I-P)S^m x = S(I-P)S^m x = S^{m+1}(I-P)x.$

Now

$$P_k R_k(a,0) = S_k(0) + F_k(a) = F_k(a)$$

and $SPx = Px$, so (9) holds for $m = 1$. Assume that (9) holds for some $m \geq 1$. Suppose that

$$P_k R_k^m(a,0) = y = \sum_{l=1}^{T_k} y_l w_{l,T_k}.$$

By Lemma 2.3, we have ensured that $S(y) = S_k(y)$. Furthermore, we observe

$$\begin{aligned} P_k R_k^{m+1}(a,0) \;=\; P_k R_k(R_k^m(a,0)) &= P_k R_k(a,y) \\ &= S_k(y) + F_k(a) \\ &= S(y) + F_k(a) \;=\; SP_k R_k^m(a,0) + F_k(a). \end{aligned}$$

Therefore

$$\begin{aligned} R^{m+1}Px &= R\left(Px + \sum_{k=1}^{\infty} P_k R_k^m(x_1 - \lambda_k x_2, 0)\right) \\ &= S\left(Px + \sum_{k=1}^{\infty} P_k R_k^m(x_1 - \lambda_k x_2, 0)\right) + \sum_{k=1}^{\infty} F_k(x_1 - \lambda_k x_2) \\ &= Px + \sum_{k=1}^{\infty} SP_k R_k^m(x_1 - \lambda_k x_2, 0) + F_k(x_1 - \lambda_k x_2) \\ &= Px + \sum_{k=1}^{\infty} P_k R_k^{m+1}(x_1 - \lambda_k x_2, 0) \end{aligned}$$

as required. $\square$

The consequence of Lemma 3.3 is that we can split the analysis of $R^m x$ into two parts: the 'shift' and the 'perturbation'. First, we examine the behaviour of the shift.

Lemma 3.4. *Given $x \in X$, we have $||R^m(I-P)x|| = ||(I-P)x||$ for all m. Moreover, $R^{T_k}(I-P)x \xrightarrow{w} (I-P)x$.*

Proof. Given (8) and the fact that S is an isometry, the first assertion is trivial. Now consider the weak convergence. Let $f \in X^*$ with $||f|| = 1$ and $\varepsilon > 0$. We take $k \in \mathbb{N}$ such that

$$\left|\left|\sum_{l=k+1}^{\infty}\sum_{i\in F_{T_l}} x_i e_i\right|\right| < \varepsilon.$$

Since T_l divides T_j whenever $l \le j$, we can see that $\pi_{T_l}^{T_j}$ is the identity for such l. Therefore, if $j \ge k$, we estimate

$$\begin{aligned}
|f(S^{T_j}(I-P)x - (I-P)x)| &= \left| f\left(\sum_{l=j+1}^{\infty}\sum_{i\in F_{T_l}} x_i e_{\pi_{T_l}^{T_j}(i)} - \sum_{l=j+1}^{\infty}\sum_{i\in F_{T_l}} x_i e_i \right)\right| \\
&\le 2\left|\left|\sum_{l=j+1}^{\infty}\sum_{i\in F_{T_l}} x_i e_i\right|\right| \\
&\le 2\left|\left|\sum_{l=k+1}^{\infty}\sum_{i\in F_{T_l}} x_i e_i\right|\right| < 2\varepsilon
\end{aligned}$$

by symmetry of the norm. □

Now we analyse the behaviour of the perturbation. Ultimately, it is the perturbation that drives the behaviour of the system as a whole.

Lemma 3.5. *If $x_1 \neq 0$ and $x_2 = 0$ then*

$$||R^m Px|| \to \infty.$$

On the other hand, if $x_1 = 0$ or $x_2 \neq 0$ then there exists k_n in the range $2^n \le k_n < 2^{n+1}$ with the property that

$$\left|\left|R^{T_{k_n-1}}Px - Px\right|\right| \to 0.$$

Proof. If $x_1 \neq 0$ and $x_2 = 0$ then by (9), (1), (6) and the definition of ε_k, we have

$$\begin{aligned}
&||R^m Px|| \\
\ge\ & ||P_k R_k^m(x_1, 0)|| \\
\ge\ & \begin{cases} \sqrt{M^{-1}}\left(\frac{2}{p+1}\right)^{p^{-1}} \varepsilon_k |x_1| m_k^{(p+1)p^{-1}} = \sqrt{M^{-1}}\left(\frac{2}{p+1}\right)^{p^{-1}} |x_1| n & \text{if } p = 1, 2 \\ \sqrt{M^{-1}}\varepsilon_k |x_1| m_k = \sqrt{M^{-1}}|x_1| n & \text{if } p = \infty \end{cases}
\end{aligned}$$

whenever $m_k \le m < T_k - m_k = m_{k+1}$ and $2^n \le k < 2^{n+1}$.

Instead, if $x_2 \neq 0$ then by (5), for large enough n there exists k_n in the range $2^n \le k_n < 2^{n+1}$, such that

$$|x_1 - \lambda_{k_n} x_2| \le \frac{2^{4-n}\,||x||^2}{|x_2|}.$$

By (2), (6) and the definition of ε_k, we have

$$\left|\left|P_{k_n} R_{k_n}^{T_{k_n-1}}(x_1 - \lambda_{k_n} x_2, 0)\right|\right| \leq \begin{cases} \sqrt{M} L \varepsilon_{k_n} |x_1 - \lambda_{k_n} x_2| m_{k_n}^{(p+1)p^{-1}} & \text{if } p = 1, 2 \\ \sqrt{M} L \varepsilon_{k_n} |x_1 - \lambda_{k_n} x_2| m_{k_n} & \text{if } p = \infty \end{cases}$$

$$(10) \qquad \leq \frac{2^{4-n}\sqrt{M} L n \left|\left|x\right|\right|^2}{|x_2|}.$$

Then we notice that if $j \leq k_n - 1$, we have

$$(11) \qquad \left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right| = 0$$

because $R_j^{T_j}$ is the identity and T_j divides T_{k_n-1} whenever $j \leq k_n - 1$. Now we have to estimate $\left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right|$ for $j \geq k_n + 1$. If $j \geq k_n + 1$ then from (7), we have

$$m_j \geq 5^{j-(k_n+1)} m_{k_n+1} \geq 5^{j-(k_n+1)} 5^n T_{k_n-1}.$$

Take $l \geq n$ such that $2^l \leq j < 2^{l+1}$. We apply (3), (4) and (6) to obtain

$$(12) \qquad \begin{aligned} & \left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right| \\ \leq & \begin{cases} \sqrt{M} L \varepsilon_j |x_1 - \lambda_j x_2| m_j^{p^{-1}} T_{k_n-1} & \text{if } p = 1 \text{ or } p = 2 \\ \sqrt{M} L \varepsilon_j |x_1 - \lambda_j x_2| T_{k_n-1} & \text{if } p = \infty \end{cases} \\ \leq & \sqrt{M} L \left|\left|x\right|\right| T_{k_n-1} \frac{2^{l-1} l}{m_j} \\ \leq & \sqrt{M} L \left|\left|x\right|\right| \frac{2^{l-1} l}{5^n 5^{j-(k_n+1)}} \\ \leq & \sqrt{M} L \left|\left|x\right|\right| \frac{2^{l-1} l}{5^n 5^{l-(n+1)}} \quad \text{since } l-(n+1) \leq j-(k_n+1) \\ = & \sqrt{M} L \left|\left|x\right|\right| l (\tfrac{2}{5})^{l-1} \end{aligned}$$

where L is defined as in the proof of Proposition 3.2. Combining (9) with (10), (11) and (12) gives

$$\begin{aligned} \left|\left|R^{T_{k_n-1}} P x - P x\right|\right| & \leq \sum_{j=1}^{\infty} \left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right| \\ & = \sum_{j=k_n}^{\infty} \left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right| \\ & = \frac{2^{4-n}\sqrt{M} L n \left|\left|x\right|\right|^2}{|x_2|} + \sum_{j=k_n+1}^{\infty} \left|\left|P_j R_j^{T_{k_n-1}}(x_1 - \lambda_j x_2, 0)\right|\right| \\ & \leq \frac{2^{4-n}\sqrt{M} L n \left|\left|x\right|\right|^2}{|x_2|} + \sqrt{M} L \left|\left|x\right|\right| \sum_{l=n}^{\infty} 2^l l (\tfrac{2}{5})^{l-1} \end{aligned}$$

$$
\begin{aligned}
&= \quad \frac{2^{4-n}\sqrt{M}Ln\,||x||^2}{|x_2|} + \tfrac{5}{2}\sqrt{M}L\,||x||\sum_{l=n}^{\infty} l(\tfrac{4}{5})^l \\
&\rightarrow \quad 0
\end{aligned}
$$

as $n \rightarrow \infty$. This concludes the proof in the case $x_2 \neq 0$. Finally, if $x_1 = 0$ then we repeat the above with $k_n = 2^n$ to reach the same conclusion. □

Proof of Theorem 1.2. Let $x_1 \neq 0$ and $x_2 = 0$. Then by Lemmas 3.4 and 3.5 we have

$$||R^m x|| \geq ||R^m Px|| - ||R^m(I-P)x|| = ||R^m x|| - ||(I-P)x|| \rightarrow \infty$$

as $m \rightarrow \infty$.

Now let $x_1 = 0$ or $x_2 \neq 0$. Again by Lemmas 3.4 and 3.5, we can pick suitable k_n such that

$$R^{T_{k_n-1}}x = R^{T_{k_n-1}}Px + R^{T_{k_n-1}}(I-P)x \xrightarrow{w} Px + (I-P)x = x.$$

□

Proof of Corollary 1.3. Let Q be a projection onto X and let X have symmetric basis $(e_i)_{i=1}^{\infty}$. Using Theorem 1.2, we can find an operator $R : X \longrightarrow X$ such that if

$$A = \{x = \sum_{i=1}^{\infty} x_i e_i \in X \ : \ x_1 \neq 0 \text{ and } x_2 = 0\}$$

then $||R^n x|| \rightarrow \infty$ whenever $x \in A$, and $(R^n x)$ has a subsequence converging weakly to x if $x \in X \setminus A$. Define $W = RQ + (I - Q)$ and let $B = Q^{-1}A$. It is easy to check that B satisfies the required properties. □

If $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$, then we can simplify the proof of Theorem 1.2 by replacing the $w_{l,n}$ with unit vectors and replacing the corresponding π_n with cycles. Since there is a Banach space with a symmetric basis, but containing no isomorphic copy of c_0 or ℓ_p, $p \geq 1$, [1], it is not possible to obtain Theorem 1.2 by proving it in the cases $X = c_0$ and $X = \ell_p$, and then applying Corollary 1.3.

4. Problems

Since the operators constructed in this note rely fundamentally on permutations of basis vectors, it makes sense to pose the following question.

Problem 4.1. *If X is a Banach space with an unconditional basis, does there exist an operator $R : X \longrightarrow X$ with the property that $||R^n x|| \rightarrow \infty$ for some $x \in X$, and $||R^n y|| \not\rightarrow \infty$ for all y in some open subset of X?*

Also, given the fact that the operators which feature above are not compact, the next question seems natural to us.

Problem 4.2. *If T is compact, can T or $I + T$, where I is the identity, satisfy the properties given in the abstract? In particular, does the Argyros-Haydon space admit such an operator?*

If no sum $I+T$, where T is compact, satisfies the properties given in the abstract, then this suggests to us that some kind of unconditional structure is necessary in order to construct such operators.

Finally, we make a remark about the title of this note. The operator R constructed above can be viewed as a machine which acts on a countable family of disjoint cycles. This family of disjoint cycles can be viewed as a countable directed graph. We speculate that it may be possible to construct other operators with interesting properties by basing them on more complicated directed graphs.

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