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Extrapolation of the Sobolev theorem and dimension-free imbeddings

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Abstract

We prove dimension-invariant imbedding theorems for Sobolev spaces using extrapolation means.

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1 Introduction, notations

In this paper we study inequalities

$$\left(\int_{\Omega} |f(x)|^p V(x)\, dx\right)^{1/p} \le c\left(\int_{\Omega} |\nabla f(x)|^p\, dx\right)^{1/p}, \qquad f \in W_0^{1,p}(\Omega), \quad (1.1)$$

and

$$\|f|L^p \log^{\alpha}(1+L)\| \le c\|\nabla f|L^p\|, \qquad f \in W_0^{1,p}(\Omega), \quad (1.2)$$

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where either Ω is a bounded domain in $\mathbb{R}^N$ (specifically, Ω will be the unit cube in $\mathbb{R}^N$) or $\Omega = \mathbb{R}^N$ and V is a weight on Ω, that is, a.e. non-negative and locally integrable function on Ω, and the constant c on the right hand sides is independent of f and N (and might depend on V).

Inequalities of the type (1.1) with a weighted L^p norm on the left hand side are of importance in various areas of analysis and have been studied in numerous papers. Sometimes they come under names as the *trace inequality* or the *uncertainty principle.* Necessary and sufficient conditions (in $\mathbb{R}^N$ or sometimes with some restrictions on N) are known in terms of behaviour of the Riesz potential of V or of capacities (see e.g. Kerman and Sawyer [15], Maz'ya [25], [26]) and amenable conditions in terms of function spaces are of no smaller importance (see e.g. Fefferman [8], Chiarenza and Frasca [6], Adams [1]). We make no attempt here to make a representative list of references.

Dimension-invariant (or shortly dimension-free) estimates of type (1.2) answer the natural question about existence of some residual improvement of the integrability properties independent of the dimension. As is well known the Sobolev space $W^{1,p}$ $(1 \le p < N)$ in a sufficiently smooth domain in $\mathbb{R}^N$ is imbedded into the Lebesgue space L^{p^*}, where $p^* = Np/(N-p)$ is the Sobolev exponent. If $N \to \infty$, then $p^* \to p$. Should there exists a target space into which $W^{1,p}$ is imbedded for all N, then clearly we must step beyond the scale of the Lebesgue spaces. Note in passing that dimension-free estimates are also linked with other interesting concepts concerning the Sobolev spaces, for instance, properties of contraction semigroups and find applications even in quantum physics (see e.g. [22] for some of the references).

In [17]–[19] we used the logarithmic Gross inequality [11] and its generalization due to Gunson [12] to establish imbeddings (1.2) for $\alpha = 1$ and Sobolev spaces $W_0^{1,p}$ (spaces with zero traces) on the unit ball, the unit cube, and $\mathbb{R}^N$, with some restrictions on p. As to (1.1) one can ask whether and under which conditions imposed on V there exists an imbedding $L^p \log(1+L)$ into the weighted space $L^p(V)$, whose norm depends on the norm of V but otherwise it is independent of f and N. Conditions for V, derived from a variant of Ishii's imbedding theorem for generalized Orlicz-Musielak spaces [14] and [28], yield an exponential integrability of (a multiple of) V.

In [17] we have employed the Gross theorem to show that

$$\int_B |f(x)|^2 \log(1 + |f(x)|/\|\nabla f\|_2)\, dx \le c\|f|W_0^{1,2}(B)\|^2$$

($W_0^{1,2}(B) = \overline{C_0^\infty(B)}^{W^{1,2}(B)}$, B being the unit ball in $\mathbb{R}^N$) with a constant c independent of f and N. This was generalized later in [19] to

$$\int_Q |f(x)|^p \left[\log\left(1 + \frac{|f(x)|}{\|\nabla f\|_p}\right)\right]^\alpha dx \le c\|\nabla f|L^p(Q)\|^p \tag{1.3}$$

for $2 \le p < \infty$ (or $1 < p < \infty$), $0 < \alpha \le p/2$, and $f \in W_0^{1,p}(\Omega)$, and the weighted dimension-free imbedding of the form

$$\int_Q |f(x)|^p V(x)\, dx \le c\|V|L_{\exp t^{2/p}}(Q)\|\, \|\nabla f|L^p(Q)\|^p,$$

for $f \in W_0^{1,p}(Q)$ (similarly on $\mathbb{R}^N$), where Q is the unit ball in $\mathbb{R}^N$ and $L_{\exp t^{2/p}}(Q)$ is the Orlicz space generated by the Young function $t \mapsto \exp|t|^{2/p} - 1$.

Let us point out that in [17] we studied the case of functions living in the unit ball of $\mathbb{R}^N$, whose measure tends to zero as $N \to \infty$; this is an advantage leading to the whole range $1 < p < \infty$.

Here we will follow an alternative approach to (1.1) and (1.2) based on extrapolation of standard Sobolev imbedding theorem and knowledge of the best constant (the norm of the imbedding) to overcome limitations given by the form of the Gross inequality and, in particular, the power at the logarithmic function. In Section 2 we establish conditions for validity of (1.2) and in Section 3 we will tackle weighted consequences of (1.2) and discuss alternative approaches to (1.1).

We shall tacitly assume that all functions here are real-valued (complex-valued functions can be considered, too). Various constants independent of f will be denoted by the same generic symbol c, C etc. if no misunderstanding can arise.

We shall use the standard notation $\|\,.\,\|_{k,p}$ for the norm in $W^{k,p}$; if $k = 0$, then $W^{k,p} = L^p$ with the norm denoted by $\|\,.\,\|_p$; sometimes we shall use symbols like $\|f|L^p\|$ etc. for the sake of better legibility. If Ω is a domain in $\mathbb{R}^N$, then the norm in $W_0^{1,p}(\Omega)$, in the Sobolev space of functions with zero traces, will be taken as $\|\nabla f|L^p(\Omega)\|$. We define the space $W_0^{1,p}(\Omega)$ itself as a completion of $C_0^\infty(\Omega)$. Note that one should be cautious here: Generally this space does not coincide with the space of functions in $W^{1,p}(\mathbb{R}^N)$ whose support is contained in $\overline{\Omega}$. If, nevertheless, Ω has a Lipschitz boundary, then both concepts coincide. The latter space is sometimes denoted by $\widetilde{W}_0^{1,p}(\Omega)$

to emphasise the difference (see Triebel [32] for a detailed discussion). If V is a weight in a domain $\Omega \subset \mathbb{R}^N$ then the *weighted Lebesgue space* $L^p(V) = L^p(V)(\Omega)$ is defined as the space of all measurable f on Ω with the finite norm $\|f|L^p(V)\| = \left(\int_\Omega |f(x)|^p V(x)\,dx\right)^{1/p}$. If f is a measurable function in $\mathbb{R}^N$, then f^* will denote its *non-increasing rearrangement*. The symbol $L^{p,q}$ will stand for the usual *Lorentz space* ($1 \le p, q < \infty$, or $1 \le p < \infty$ and $q = \infty$).

If Φ is a Young function, that is, Φ is even, convex, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t)/t = \infty$, and $\Omega \subset \mathbb{R}^N$ is measurable, then $m(\Phi, f) = \int_\Omega \Phi(f(x))\,dx$ is the *modular* and the (quasi)norm in the corresponding Orlicz space $L_\Phi = L_\Phi(\Omega)$ is the Minkowski functional of the modular unit ball, namely, $\|f|L_\Phi\| = \inf\{\lambda > 0 : m(\Phi, f/\lambda) \le 1\}$ (the Luxemburg norm). One can release the assumptions on Φ, for instance Φ can be just increasing rather than convex. In particular, if such a function Φ is convex on some interval (a, ∞) $(a > 0)$ and is equivalent to some convex function on $(0, \infty)$, then we get the same space (with an equivalent norm). If, moreover, Ω has finite measure and Φ is convex on some interval (a, ∞), then it is always possible to find an equivalent function which is convex on $(0, \infty)$. We refer to [16] and [28] for the theory of classical Orlicz spaces and of general modular spaces, respectively. We shall restrict ourselves to a characterization of weighted Orlicz spaces $L_\Phi(V) = L_\Phi(\Omega, V)$, generated by the modular $\int_\Omega \Phi(f(x))V(x)\,dx$ as special Musielak-Orlicz spaces. Let us recall the latter concept in a form adapted to our needs (see [28] for the general case). Let us assume that $\Phi = \Phi(x,t) : \Omega \times \mathbb{R} \to [0, \infty)$ is a Young function of the variable t for each fixed $x \in \Omega$ and a measurable function of the variable x for each fixed $t \in \mathbb{R}$. The function Φ with these properties is called the *generalized Young function* or the *Musielak-Orlicz function*. Then

$$m(\Phi, f) = \int_\Omega \Phi(x, f(x))\,dx$$

is a *modular* on the set of all measurable functions on Ω so that we can consider the corresponding Orlicz space.

The weighted Orlicz spaces can be described in this language. Let V be a weight on Ω and let Φ be a Young function. Define

$$\Phi_1(x,t) = \Phi(t)V(x), \qquad x \in \Omega,\ t \in \mathbb{R}.$$

Then Φ_1 is a generalized Young function and the resulting Musielak-Orlicz space $L_{\Phi_1}(\Omega)$ is nothing but the *weighted Orlicz space* $L_\Phi(V) = L_\Phi(V)(\Omega)$

with the modular

$$m(f,V) = \int_{\Omega} \Phi(f(x))V(x)\,dx$$

with the corresponding Luxemburg norm.

A suitable reformulation of the imbedding theorem for the Musielak-Orlicz spaces due to Ishii [14] (see also [28] for still a more general claim) will be used later in Section 2—we state it explicitly for reader's convenience: The Orlicz space L_{Φ} is imbedded into the Musielak-Orlicz space $L_{\Phi}(V)(\Omega)$ if there exists $K > 1$ such that

$$\sup_{t\geq 0}\,[\Phi(t)V(x) - K\Psi(x)] \in L_1(\Omega).$$

In the sequel we will work with special Orlicz spaces, usually denoted by $L^p \log^{\alpha}(1+L)$ $(1 \leq p < \infty,\ \alpha > 0)$. The generating function here is $t \mapsto |t|^p \log^{\alpha}(1+|t|)$, $t \in \mathbb{R}$. Further, $L_{\exp t^{\alpha}}$ for $\alpha > 0$ will stand for the space with the generating function $t \mapsto \exp(|t|^{\alpha}) - 1$, $t \in \mathbb{R}$. For $\alpha = 1$ we shall simply write $L^p \log(1+L)$ and $L_{\exp}$. Note that the function $t \mapsto |t|^p \log^{\alpha}(1+|t|)$ is not generally convex near the origin. It is, however, a purely technical problem to consider an equivalent Young function (convex on the whole of $\mathbb{R}^1$), yielding the same space with an equivalent norm. No confusions can arise so that we will stick to the traditional notations $L^p \log^{\alpha}(1+L)$.

2 Extrapolation of Sobolev imbeddings

In this section we will use extrapolation of Sobolev imbeddings to get the residual dimension-free imbeddings for functions with no constraints on their support.

The symbol $W_0^{1,p}$ will denote either $\widetilde{W}^{1,p}(\Omega)$ with some domain $\Omega \subset \mathbb{R}^N$ or the space $W^{1,p}(\mathbb{R}^N)$ (which coincides with $W_0^{1,p}(\mathbb{R}^N)$).

Theorem 2.1. *Let $1 < p < N$. There exists a constant c independent of N, α, and $f \in W_0^{1,p}$ such that*

$$\|f|L^p \log^{\alpha}(1+L)\| \leq c\|\nabla f|L^p\| \tag{2.1}$$

for every $f \in W_0^{1,p}$ and $\alpha \in (p^2/(N-p), p/2]$.

Corollary 2.2. *Let V be a weight function on Ω, $1 < p < N$, and assume that V belongs to the Orlicz space L_Ψ, where Ψ is a Young function complementary to $t \mapsto K|t|\log^\alpha(1+|t|^{1/p})$ for some $K > 1$, that is,*

$$\int_\Omega \Psi(\mu V(x))\,dx < \infty,$$

for some $\mu > 0$. Then

$$\|f|L^p(V)\| \le c\|\nabla f|L^p\|$$

for every $f \in \widetilde{W}_0^{1,p}$, with a constant c depending on V but independent of f and N. If $|\Omega| < \infty$, it is enough to assume that $V \in L_{\exp t^{1/\alpha}}$.

Proof of Theorem 2.1. Let $\alpha > 0$. Hölder's inequality combined with the Sobolev imbedding gives

$$\begin{aligned}
&\left(\int_\Omega |f(x)|^p \log^\alpha(1+|f(x)|)\,dx\right)^{1/p} \\
&\le \left(\int_\Omega |f(x)|^{Np/(N-p)}\,dx\right)^{(N-p)/Np} \left(\int_\Omega \log^{N\alpha/p}(1+|f(x)|)\,dx\right)^{1/N} \qquad (2.2)\\
&\le \frac{c}{N^{1/2}}\|\nabla f|L^p\| \left(\int_\Omega \log^{N\alpha/p}(1+|f(x)|)\,dx\right)^{1/N}.
\end{aligned}$$

It is not difficult to see that for $\varepsilon \in (0,1]$,

$$\log(1+\xi) \le \frac{1}{\varepsilon}\xi^\varepsilon, \qquad \xi > 0.$$

Indeed, consider $h(\xi) = \log(1+\xi) - c_\varepsilon \xi^\varepsilon$ with c_ε to be specified later. Then $h(0) = 0$ and

$$h'(\xi) = \frac{1}{1+\xi} - \varepsilon c_\varepsilon \xi^{\varepsilon-1}.$$

We wish to find c_ε such that $h'(\xi) \le 0$, i.e.,

$$\varepsilon c_\varepsilon \xi^{\varepsilon-1} + \varepsilon c_\varepsilon \xi^\varepsilon \ge 1.$$

Plainly it is sufficient that $c_\varepsilon = 1/\varepsilon$ so that

$$[\log(1+|f(x)|)]^{N\alpha/p} \le \left(\frac{1}{\varepsilon}\right)^{N\alpha/p} |f(x)|^{N\alpha\varepsilon/p},$$

and

$$\left(\int_\Omega \log^{N\alpha/p}(1+|f(x)|)\,dx\right)^{1/N} \leq \left(\frac{1}{\varepsilon}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{N\alpha\varepsilon/p}\,dx\right)^{1/N}. \tag{2.3}$$

This means that the appropriate choice is

$$\frac{N\alpha\varepsilon}{p} = \frac{Np}{N-p},$$

in another terms,

$$\varepsilon = \frac{p^2}{\alpha(N-p)}.$$

Inserting this into (2.3) and applying Sobolev's inequality again we get

$$\begin{aligned}
&\left(\int_\Omega \log^{N\alpha/p}(1+|f(x)|)\,dx\right)^{1/N} \\
&\leq \left(\frac{1}{\varepsilon}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{N\alpha\varepsilon/p}\,dx\right)^{1/N} \\
&\leq \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{Np/(N-p)}\,dx\right)^{1/N} \\
&\leq \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\int_\Omega |f(x)|^{Np/(N-p)}\,dx\right)^{((N-p)/Np)(p/(N-p))} \\
&\leq \left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\frac{c}{N^{1/2}}\right)^{p/(N-p)}\|\nabla f|L^p\|^{p/(N-p)}.
\end{aligned}$$

Together with (2.2) this yields

$$\left(\int_\Omega |f(x)|^p\log^\alpha(1+|f(x)|)\,dx\right)^{1/p} \leq \frac{c}{N^{1/2}}\left(\frac{\alpha(N-p)}{p^2}\right)^{\alpha/p}\left(\frac{1}{N^{1/2}}\right)^{p/(N-p)}\|\nabla f|L^p\|^{1+p/(N-p)}.$$

Hence for $N > p$ we have

$$\left(\int_\Omega |f(x)|^p\log^\alpha(1+|f(x)|)\,dx\right)^{1/p} \leq c\frac{N^{\alpha/p}}{N^{1/2}}\|\nabla f|L^p\|^{1+p/(N-p)} \tag{2.4}$$

with some c independent of f and N. To achieve independence of the right hand side of N we have to choose

$$\alpha \le \frac{p}{2}.$$

Let $\alpha \le p/2$ and $\|\nabla f|L^p\| = 1/c^{(N-p)/N}$ with c from (2.4). We get

$$\int_\Omega |f(x)|^p \log^\alpha(1+|f(x)|)\,dx \le 1. \tag{2.5}$$

For general $f \in W_0^{1,p}$ insert $f/(c^{(N-p)/N}\|\nabla f|L^p\|)$ into (2.5). We get

$$\int_\Omega \frac{|f(x)|^p}{c^{p(N-p)/N}\|\nabla f|L^p\|^p} \log^\alpha\left(1+\frac{|f(x)|}{c^{(N-p)/N}\|\nabla f|L^p\|}\right)dx \le 1.$$

According to the definition of the Luxemburg norm this yields

$$\|f|L^p\log^\alpha(1+L)\| \le c^{(N-p)/N}\|\nabla f|L^p\| \le c\|\nabla f|L^p\|.$$

□

Remark 2.3. Note that the best choice for spaces on bounded domains is $\alpha = p/2$ and we have then

$$\left(\int_\Omega |f(x)|^p \log^{p/2}(1+|f(x)|)\,dx\right)^{1/p} \le c\|\nabla f|L^p\|^{1+p/(N-p)}$$

and consequently the corresponding norm estimate holds, too. This is worse estimate than that in [19] if $1 < p < 2$ but it is an improvement for $p > 2$.

Proof of Corollary 2.2. According to Ishii's theorem [14], [28] we have $L^p\log^\alpha(1+L) \hookrightarrow L^p(V)$ if the function

$$H(x) = \sup_{t>0}\left(t^pV(x) - Kt^p\log^\alpha(1+t)\right) \tag{2.6}$$

is integrable over Ω. But (2.6) can be rewritten as

$$H(x) = \sup_{t>0}\left(tV(x) - Kt\log^\alpha(1+t^{1/p})\right). \tag{2.7}$$

A careful analysis of Ishii's proof shows that the norm of the imbedding $L^p\log^\alpha(1+L) \hookrightarrow L^p(V)$ depends on V but it is otherwise independent of N. We refer to [19] for details.

Hence the sufficient condition for the imbedding is

$$\int_{\Omega} \Psi(\mu V(x))\, dx < \infty,$$

where Ψ is a Young function complementary to $Kt \log^{\alpha}(1 + t^{1/p})$ and μ is a suitable positive constant.

The function $t \mapsto \log^{\alpha}(1 + t^{1/p})$ is an inverse to the Δ_3-function

$$\widetilde{\Psi}(\xi) = \left(\exp \xi^{1/\alpha} - 1\right)^p;$$

so that (see [16, I/§6]) we have $\Psi(\xi) \sim \exp\left(\xi^{1/\alpha}\right) - 1$ for ξ bounded away from zero, say for $\xi \geq 1$ (in the sense of the equivalence of Young functions). As to values of V belonging to $(0, 1)$ we have to look directly at the integrability of the function in (2.7). Elementary calculations show that if $V(x) \leq 1$, then the expression on the right hand side of (2.7) is negative if $t > K^{-1}\left(\exp K^{-1/\alpha} - 1\right)$ and the sup becomes a (fixed) multiple of $V(x)$.

□

3 More on the weighted imbeddings

The condition on the weight function V in Corollary 2.2 are derived via another imbedding and a natural question is whether one can get a stronger weighted result by a suitable direct method. We will show that this is indeed the case. Of interest in this connection is also the special choice of weights $V = V(x)$ such that $V^*(t) = (\log(1/t))^{\alpha}$, that is, inequalities of type

$$\int_0^1 f^*(t)^p \log^{\alpha} \frac{1}{t}\, dt \leq c \int_Q |\nabla f(x)|^p\, dx. \tag{3.1}$$

Note that by the Hardy-Littlewood inequality for product of functions the left hand side of (3.1) majorizes $\int_Q |f(x)|^p V(x)\, dx$ for such weights V. Moreover, the weighted integral on the left can be interpreted as a modular and it is well known (cf. e.g. [4]) that it generates the space $L^p \log^{\alpha}(1+L)$ with a norm equivalent to the standard Luxemburg norm there—this, however, does not automatically implies an inequality generalizing (1.2).

It is not difficult to see that $V \in L^{N/p}$ is a sufficient condition for (1.1) in $\mathbb{R}^N$ or in $Q \subset \mathbb{R}^N$; it is enough to apply Hölder's inequality with conjugate exponents $N/(N-p)$ and N/p to the left hand side of (1.1). One can do a bit better: Since $W_0^{1,p}(Q)$ is imbedded into the Lorentz space $L^{Np/(N-p),p}(Q)$ we have, using the Hardy-Littlewood rearrangement inequality,

$$\begin{aligned}
\int_Q |f(x)|^p V(x)\,dx &\le \int_0^1 (f^*(t))^p V^*(t)\,dt \\
&\le \int_0^1 t^{(N-p)/N} f^*(t)^p t^{p/N} V^*(t)\,\frac{dt}{t} \\
&\le \sup_{0<s<\infty} s^{p/N} V^*(s) \int_0^1 \left(t^{(N-p)/Np} f^*(t)\right)^p \,\frac{dt}{t} \\
&= \|V|L^{N/p,\infty}\|\,\|f|L^{Np/(N-p),p}\|^p \\
&\le c(N)\|V|L^{N/p,\infty}\|\,\|\nabla f|L^p\|^p,
\end{aligned}$$

where $c(N)$ is the best constant for the imbedding of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{Np/(N-p),p}$. Hence (1.1) holds in $Q \subset \mathbb{R}^N$ if $V \in L^{N/p,\infty}(Q)$, similarly in $\mathbb{R}^N$. The behaviour of the constant $c(N)$ is known thanks to Alvino [3]: a bit surprisingly the best constant for the refined Sobolev imbedding into $L^{Np/(N-p),p}(\mathbb{R}^N)$ behaves in the same way with respect to $N \to \infty$ as the best constant in [31], that is, the norm of the imbedding is equivalent to $N^{-1/2}$ as $N \to \infty$ (see [18] or [19] for detailed calculations). Consequently

$$\begin{aligned}
\int_Q |f(x)|^p V(x)\,dx &\le c\frac{\|V|L^{N/p,\infty}\|}{N^{p/2}}\,\|\nabla f|L^p\|^p \\
&\le c\frac{\|V|L^{N/p}\|}{N^{p/2}}\,\|\nabla f|L^p\|^p \\
&\le cc(p)\frac{\|V|L^{N/p}\|}{(N/p)^{p/2}}\,\|\nabla f|L^p\|^p \\
&\le cc(p)\sup_{q\ge 1}\frac{\|V|L^q\|}{q^{p/2}}\,\|\nabla f|L^p\|^p.
\end{aligned} \tag{3.2}$$

Let us recall the standard extrapolation fact that

$$\|V|L_{\exp t^\beta}(Q)\| \sim \sup_{q\ge 1}\frac{\|V|L^q(Q)\|}{q^{1/\beta}} < \infty. \tag{3.3}$$

The equivalence of norms here is independent of N; indeed, one can pass to rearrangements of V in (3.3) and the (one-dimensional) norms of V^* are equal to the corresponding norms of V. For more on extrapolation conditions using L^q and $L^{q,\infty}$ norms see, e.g. [7] or [27].

Hence we have proved the following

Theorem 3.1. *Let $1 < p < N$ and let V be a weight function in Q. Then there exists a constant c independent of N and f such that*

$$\|f|L^p(V)(Q)\| \le c\|V|L_{\exp t^{2/p}}(Q)\|\|\nabla f|L^p\| \tag{3.4}$$

for every $f \in W_0^{1,p}(Q)$.

An analog Theorem 3.1 holds in $\mathbb{R}^N$ (and similarly on sets with an infinite measure). As to a characterization as in (3.3) one has to be careful. In (3.2) we require in fact that

$$\sup_{q\ge N/p} \frac{\|V|L^q(\mathbb{R}^N)\|}{q^{p/2}} < \infty,$$

which in contrast of the situation in Q is not equivalent to finiteness of $\sup_{q\ge 1}$.

A remedy is to use Orlicz spaces $E_{r,\beta}(\mathbb{R}^N)$ generated by the Young function $t \mapsto |t|^r \left(\exp t^\beta - 1\right)$ $(\beta > 0)$, investigated in Schmeisser and Sickel [30]. The Luxemburg norm in $E_{r,\beta}(\mathbb{R}^N)$ is equivalent to the (extrapolation) norm

$$\sup_{q\ge r} \frac{\|V|L^q(\mathbb{R}^N)\|}{q^{1/\beta}}$$

and the constants of the equivalence are independent of N.

We get

Theorem 3.2. *Let $1 < p < N$ and let V be a weight function in $\mathbb{R}^N$. Then there exists a constant c independent of N and f such that*

$$\|f|L^p(V)(\mathbb{R}^N)\| \le c\|V|E_{N/p,p/2}(\mathbb{R}^N)\|\|\nabla f|L^p(\mathbb{R}^N)\| \tag{3.5}$$

for every $f \in W^{1,p}(\mathbb{R}^N)$.

An analogous statement holds true for a domain $\Omega \subset \mathbb{R}^N$ and functions in $\widetilde{W}_0^{1,p}(\Omega)$.

For the special choice of the weight function V on Q, when $V^*(t) = (\log(1/t))^\alpha$, Theorem 2.1 yields the condition $\alpha \le p/2$.

Inequalities of type

$$\int_0^1 f^*(t)^p \log^\alpha \frac{1}{t}\, dt \le c\,(\|\nabla f|L^p(Q)\|^p + \|f|L^p(Q)\|^p) \tag{3.6}$$

have been recently studied by Martín and Milman [24]). Such relations are of interest because the expression on the left hand side leads to an equivalent norm in $L^p \log^\alpha(1+L))$, say, on the unit cube, nevertheless, dimension-free relations of the usual modular (corresponding to the Young function $|t|^p \log^\alpha(1+|t|)$ to $\|\nabla f|L^p\|$ are not immediately clear.

In particular, in some response to [18], which handled dimension-free imbeddings of Sobolev spaces on a unit ball, Triebel [33] raised a question about inequalities of the above type for spaces on a unit cube and the dependence of c on the dimension. The answer was given in [24, subsect. 7.1], giving $\alpha = p/2$ as the "best" exponent in (3.6). The proof is based on the isoperimetric inequality.

Note that the dimension-free estimate (3.6) with $\alpha = p/2$ is contained in Corollary 2.2 or Theorem 3.1. Methods of proofs are different. It is not, however, necessary to use an additional deeper technique as in [24] (isoperimetric theorem) or here in Section 3 (extrapolation) to get (3.6). A simple short proof can be given, using just claim on the best constant for the Sobolev imbedding and properties of the Gamma function ($\Gamma(\xi)^{1/\xi} \sim \xi$ as $\xi \to \infty$). Indeed, for $f \in W_0^{1,p}(Q)$ we have

$$\begin{aligned}
&\int_0^1 f^*(t)^p \left(\log\frac{1}{t}\right)^\alpha dt \\
&\qquad \le \left(\int_0^1 f^*(t)^{Np/(N-p)}\, dt\right)^{(N-p)/N} \left(\int_0^1 \left(\log\frac{1}{t}\right)^{N\alpha/p} dt\right)^{p/N} \\
&\qquad \le \frac{c}{N^{p/2}} \|\nabla f|L^p\|^p \left[\Gamma\left(1+\frac{N\alpha}{p}\right)\right]^{p/N} \\
&\qquad \le \frac{c}{N^{p/2}} \|\nabla f|L^p\|^p \left[\left(\frac{N\alpha}{p}\right)^{p/(N\alpha)}\right]^\alpha \left(\left[\Gamma\left(\frac{N\alpha}{p}\right)\right]^{p/(N\alpha)}\right)^\alpha \\
&\qquad \le \frac{c}{N^{p/2}} \|\nabla f|L^p\|^p \left(\frac{N\alpha}{p}\right)^\alpha .
\end{aligned}$$

Once more we recover the condition $\alpha \le p/2$ for the independence of N.

The problem remains whether the exponent $p/2$ is sharp. We return to this question in a forthcoming paper.

Once more we recover the condition $\alpha \leq p/2$ for the independence of N.

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