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A Critical Oscillation Constant as a Variable of Time Scales for Half-Linear Dynamic Equations

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Abstract

We present criteria of Hille-Nehari type for the half-linear dynamic equation $(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0$ on time scales. As a particular important case we get that there is a a (sharp) critical constant which may be different from what is known from the continuous case, and its value depends on the graininess of a time scale and on the coefficient r. As applications we state criteria for strong (non)oscillation, examine generalized Euler type equations, and establish criteria of Kneser type. Examples from q-calculus and further possibilities for study are presented as well. Our results unify and extend many existing results from special cases, and are new even in the well-studied discrete case.

Keywords: dynamic equation, time scale, half-linear equation, (non)oscillation criteria, Hille-Nehari criteria, Kneser criteria, critical constant, oscillation constant

AMS Classification: 34C10, 34K11, 39A11, 39A12, 39A13.

1 Introduction

Consider the half-linear dynamic equation

$$(r(t)\Phi(y^{\Delta}))^{\Delta}+p(t)\Phi(y^{\sigma})=0, \tag{1}$$

where $\Phi(u)=|u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha>1$, $r(t)>0$ and $p(t)$ are rd-continuous functions defined on a time scale interval $[a,\infty)$, $a\in\mathbb{T}$, and a time scale $\mathbb{T}$ is assumed to be unbounded from above. This equation covers a large variety of well-studied and important equations, for example: If $\mathbb{T}=\mathbb{R}$, then (1) reduces to the half-linear differential equation

$$(r(t)\Phi(y'))'+p(t)\Phi(y)=0, \tag{2}$$

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see e.g. the initiating papers [9, 17] and the monograph [8]. Recall that the investigation of equations of the form (2) is extremely useful for example in studying spherically symmetric solutions of elliptic equations with p-laplacian and also finds applications when examining more general, quasilinear equations. The choice $\mathbb{T} = \mathbb{Z}$ yields the discrete counterpart of (2), namely the equation

$$\Delta(r_k \Phi(\Delta y_k)) + p_k \Phi(y_{k+1}) = 0, \tag{3}$$

where $\Delta y_k = y_{k+1} - y_k$, see e.g. the paper [19] and the survey on discrete oscillation theory [1]. If $\alpha = 2$, then from (1) we get

$$(r(t)y^{\Delta})^{\Delta} + p(t)y^{\sigma} = 0, \tag{4}$$

see e.g. the initiating paper [10] and the monograph [3]. Moreover, the latter equation covers the linear differential equation

$$(r(t)y')' + p(t)y = 0 \tag{5}$$

provided $\mathbb{T} = \mathbb{R}$, see e.g. the survey [24], the linear difference equation

$$\Delta(r_k \Delta y_k) + p_k y_{k+1} = 0 \tag{6}$$

provided $\mathbb{T} = \mathbb{Z}$ see e.g. the survey [1], and the linear q-difference equation

$$D_q(r(t)D_q y(t)) + p(t)y(qt) = 0, \tag{7}$$

provided $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, where $D_q y(t) = (y(qt) - y(t))/((q-1)t)$, see e.g. [6].

In particular, equation (5) has been deeply studied and there is an extensive literature devoted to it. The theory of (5) can serve to an extension to various more general cases. A very large part of the qualitative theory of (1) is formed by the oscillation theory, in which, very roughly speaking, one studies the sign and zeros of solutions. There is known a huge amount of various conditions guaranteeing oscillation or nonoscillation of (5). The so-called Hille-Nehari ones (see e.g. [18, 24, 25]) belong to the most famous ones: If

$$\liminf_{t\to\infty} \left(\int_a^t \frac{1}{r(s)}\, ds \right) \int_t^\infty p(s)\, ds > \frac{1}{4}, \tag{8}$$

then (5) is oscillatory (i.e., every its solution oscillates). If

$$\limsup_{t\to\infty} \left(\int_a^t \frac{1}{r(s)}\, ds \right) \int_t^\infty p(s)\, ds < \frac{1}{4}, \tag{9}$$

then (5) is nonoscillatory (i.e., every its solution is of eventually one sign). Clearly, the constant 1/4 is sharp since it is the same in (8) and in (9). Here we assume that $\int_a^\infty 1/r(s)\, ds = \infty$ and $\int_t^\infty p(s)\, ds \geq 0$ ($\not\equiv 0$) for large t. Note that there is a close connection of these results to the oscillatory behavior of the Euler differential equation $y'' + \lambda t^{-2} y = 0$. Indeed, the Hille-Nehari criteria provide the information about the oscillation constant $\lambda = \lambda_0$ (i.e., Euler's equation is oscillatory for $\lambda > \lambda_0$ and it is nonoscillatory for $\lambda < \lambda_0$; here it holds $\lambda_0 = 1/4$) without any need of solving the equation explicitly. Conversely, if we have an information about the oscillation constant (which can be simply obtained by solving the equation), then using the transformation of independent variable and the so-called Hille-Wintner (integral) comparison theorem, we get (8) and (9).

In recent years, there have been established quite many results that are related to an extension of Hille-Nehari theorems (or to an examination of Euler type equations) to other or more general settings, in particular, discrete, half-linear or time scales ones, see [5, 6, 7, 8, 11, 13, 14, 15, 16, 21, 22, 26]. However, the results presented there usually contain certain restrictions that disable examination of many important cases. Those restrictions are of the following two types: First, the constants on the right-hand sides are not the best possible. Second, there is a strict requirement on the choice of time scales.

Recall that for (2) Hille-Nehari criteria read as: If

$$\liminf_{t\to\infty} \left(\int_a^t r^{1-\beta}(s)\, ds \right)^{\alpha-1} \int_t^\infty p(s)\, ds > \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1},$$

then (2) is oscillatory. If

$$\limsup_{t\to\infty} \left(\int_a^t r^{1-\beta}(s)\, ds \right)^{\alpha-1} \int_t^\infty p(s)\, ds < \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1},$$

then (2) is nonoscillatory. Denote the constant on the right-hand sides as

$$\gamma_\alpha = \frac{1}{\alpha} \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1}$$

and notice that γ_α is again sharp and $\gamma_\alpha = 1/4$ provided $\alpha = 2$.

What we want in our paper is to show how the above mentioned restrictions can substantially be removed. We derive criteria which are generalizations of (8) and (9) to equation (1). Our results unify and extend (to new time scales) many of known criteria and observations from e.g. [1, 6, 8, 7, 14, 15, 16, 22, 23, 26]. In addition, we show how the constants on the right-hand sides depend on time scales. As a special case, when the limit

$$M = \lim_{t\to\infty} \frac{\mu(t) r^{1/(1-\alpha)}(t)}{\int_a^t r^{1/(1-\alpha)}(s)\, \Delta s}$$

exists, we get that the above mentioned sharp constant γ_α is replaced by other constant, which is again sharp but its value belongs to the interval $[0, \gamma_\alpha]$ and depends on the value of M. The word "sharp" means that such a constant forms a sharp borderline between oscillation and nonoscillation of (1). We will call it the *critical constant* or the *critical oscillation constant*.

Our new results lead to interesting conclusions. For example, the critical constant is equal to γ_α in all situations where $M = 0$; if $M > 0$, then the critical constant is strictly less than γ_α (this may happen even in the discrete case ($\mathbb{T} = \mathbb{Z}$) when $r(t) \not\equiv 1$); the critical constant can actually be equal to zero (this happens when $M = \infty$). It should be emphasized that the results are new even in the well-studied dicrete case.

The proof of the main results is based on the so-called function sequence technique which exploits a generalized Riccati technique. A crucial point which leads to a "variable critical constant", is that we do not force the "problematic term" occuring in a generalized Riccati dynamic equation (see (13)) to behave like almost in the continuous case which was a usual procedure in previous approaches.

The paper is organized as follows. In the next section we recall some important concepts and state preliminaries that are key to prove the main results. Generalized Hille-Nehari theorems are presented in Section 3. In Section 4 we discuss the concepts of critical and oscillation constant. As applications we derive criteria for strong (non)oscilation of (1) (Section 5), further we examine generalized Euler equations and establish Kneser type criteria (Section 6) and present examples from q-calculus (Section 7). In the last section we indicate some directions for a future research.

Throughout the paper we assume

$$\int_a^\infty r^{1-\beta}(s)\,\Delta s = \infty, \tag{10}$$

where β is the conjugate number of α, i.e., $1/\alpha + 1/\beta = 1$, and

$$\int_t^\infty p(s)\,\Delta s \text{ exists, is nonnegative and eventually nontrivial} \tag{11}$$

for large t, say $t \in [a, \infty)$, without loss of generality.

2 Preliminaries

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}$, σ, f^σ, μ, f^Δ and $\int_a^b f^\Delta(s)\,\Delta s$ stand for time scale, forward jump operator, $f \circ \sigma$, graininess, delta derivative of f, and delta integral of f from a to b, respectively. Further, $D_q f$ denotes f^Δ when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$ (this is the so-called *Jackson derivative* of f). See [12], which is the initiating paper of the time scale theory written by Hilger, and the monographs [3, 4] by Bohner and Peterson containing a lot of information on time scale calculus.

We will proceed with some essentials of oscillation theory of (1). First note that we are interested only in nontrivial solutions of (1). We say that a solution y of (1) has a *generalized zero* at t in case $y(t) = 0$. If $\mu(t) > 0$, then we say that y has a *generalized zero* in $(t, \sigma(t))$ in case $y(t)y^\sigma(t) < 0$. A nontrivial *solution* y of (1) is called *oscillatory* if it has infinitely many generalized zeros; note that the uniqueness of IVP excludes the existence of a cluster point which is less than ∞. Otherwise it is said to be *nonoscillatory.* In view of the fact that the Sturm type separation theorem extends to (1) (see e.g. [20]), we have the following equivalence: One solution of (1) is oscillatory if and only if every solution of (1) is oscillatory.

Hence we may speak about *oscillation* or *nonoscillation of equation* (1). Recall that the principal statements, like the Sturmian theory (Reid type roundabout theorem, Sturm type separation and comparison theorems) for (1), can be established under the mere assumption $r(t) \neq 0$ and the basic concepts, especially generalized zero, have to be adjusted, see the initiating paper [20] or [2]. However, our method requires the positivity of $r(t)$; equation (1) is viewed as a perturbation of the *nonoscillatory* equation $(r(t)\Phi(y^\Delta))^\Delta = 0$. Note that we do not require the positivity of $p(t)$ though many approaches even in special cases need this assumption.

The proofs of our main results is based on the below described *function sequence technique*. This technique utilizes very useful *Riccati type transformation* involving the *generalized Riccati dynamic equation*

$$w^\Delta + p(t) + S(w, r) = 0, \tag{12}$$

where

$$S(w, r) = \lim_{\lambda \to \mu} \frac{w}{\lambda} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda \Phi^{-1}(w))} \right) \tag{13}$$

Note that (1) and (12) are connected by the substitution $w = r\Phi(y^\Delta/y)$. Observe that for $\mathbb{T} = \mathbb{R}$, using L'Hospital's rule, S takes the form $(\alpha - 1) r^{1-\beta}(t)|w|^\beta$, which is the expression known from the continuous case, see [8].

Lemma 1 ([22]). *Define the function sequence* $\{\varphi_k(t)\}$ *by*

$$\varphi_0(t) = \int_t^\infty p(s)\,\Delta s, \quad \varphi_k(t) = \varphi_0(t) + \int_t^\infty S(\varphi_{k-1}, r)(s)\,\Delta s, \quad k = 1, 2, \dots.$$

The equation (1) *is nonoscillatory if and only if there exists* $t_0 \in [a, \infty)$ *such that* $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$, $t \geq t_0$, *i.e., the sequence* $\{\varphi_k(t)\}$ *is well defined and pointwise convergent.*

3 Main results — Hille–Nehari type criteria

In this section we prove an extension of Hille-Nehari criteria to equation (1). Recall that we assume (10) and (11). Note that (10) and $\int_a^\infty p(s)\,\Delta s = \infty$ implies (1) to be oscillatory (see [20, Theorem 5]), and thus it is reasonable to assume that $\int_a^\infty p(s)\Delta s$ is convergent.

Theorem 1. *Denote*

$$M_* = \liminf_{t\to\infty} \frac{\mu(t) r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\,\Delta s}, \quad M^* = \limsup_{t\to\infty} \frac{\mu(t) r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\,\Delta s}, \tag{14}$$

$$\gamma(x) = \lim_{t\to x} \left(\frac{(t+1)^{\frac{\alpha-1}{\alpha}} - 1}{t} \right)^{\alpha-1} \left(1 - \frac{1 - (t+1)^{-\frac{(\alpha-1)^2}{\alpha}}}{1 - (t+1)^{1-\alpha}} \right), \tag{15}$$

and

$$\mathcal{A}(t) = \left(\int_a^t r^{1-\beta}(s)\,\Delta s \right)^{\alpha-1} \int_t^\infty p(s)\,\Delta s.$$

If

$$\liminf_{t\to\infty} \mathcal{A}(t) > \gamma(M_*), \tag{16}$$

then (1) *is oscillatory. If*

$$\limsup_{t\to\infty} \mathcal{A}(t) < \gamma(M^*), \tag{17}$$

then (1) *is nonoscillatory.*

Proof. Oscillatory part. We will use the notation of Lemma 1. In addition, denote $R(t) = \int_a^t r^{1-\beta}(s)\,\Delta s$. Condition (16) can be rewritten as $\varphi_0(t) \ge \gamma_0 R^{1-\alpha}(t)$ for large t, say $t \ge t_0 > a$, where $\gamma_0 > \gamma(M_*)$. Then, using the monotone nature of S, see [20], and the equalities $\sigma(t) = \mu(t) + t$, $R^\sigma(t) = R(t) + \mu(t)R^\Delta(t) = R(t) + \mu(t)r^{1-\beta}(t)$, we have

$$\begin{aligned}
\varphi_1(t) &= \varphi_0(t) + \int_t^\infty S(\varphi_0, r)(s)\,\Delta s \\
&\ge \gamma_0 R^{1-\alpha}(t) + \int_t^\infty S(\gamma_0 R^{1-\alpha}, r)(s)\,\Delta s \\
&\ge \gamma_0 R^{1-\alpha}(t) + +\Gamma_*(t_0,\gamma_0)\int_t^\infty (-R^{1-\alpha})^\Delta(s)\,\Delta s \\
&= \gamma_1 R^{1-\alpha}(t),
\end{aligned}$$

where $\gamma_1 = \gamma_0 + \Gamma_*(t_0,\gamma_0)$ with

$$\begin{aligned}
\Gamma_*(t_0,u) &= \inf_{t\ge t_0}\lim_{\lambda\to\mu(t)}\left\{\frac{uR^{1-\alpha}(t)}{\lambda}\left(1-\frac{r(t)}{(r^{\beta-1}(t)+\lambda u^{\beta-1}/R(t))^{\alpha-1}}\right)\right. \\
&\qquad\qquad\left.\times\frac{\lambda}{R^{1-\alpha}(t)-R^{1-\alpha}(t+\lambda)}\right\} \\
&= \inf_{t\ge t_0}\lim_{\lambda\to\mu(t)}\left\{\frac{u}{1-(1+\lambda r^{1-\beta}(t)/R(t))^{1-\alpha}}\right. \\
&\qquad\qquad\left.\times\left(1-\frac{1}{(1+\lambda u^{\beta-1}r^{1-\beta}(t)/R(t))^{\alpha-1}}\right)\right\}.
\end{aligned}$$

Note that using the L'Hospital rule, we have

$$\lim_{\lambda\to 0}\left\{\frac{u}{1-(1+\lambda r^{1-\beta}(t)/R(t))^{1-\alpha}}\left(1-\frac{1}{(1+\lambda u^{\beta-1}r^{1-\beta}(t)/R(t))^{\alpha-1}}\right)\right\} = u^\beta,$$

which is what is known from the continuous case. Similarly arguments as above, by induction, yield$\varphi_k(t) \ge \gamma_k R^{1-\alpha}(t)$, where

$$\gamma_k = \gamma_0 + \Gamma_*(t_0,\gamma_{k-1}), \quad k = 1,2,\dots. \tag{18}$$

Since $\Gamma_*(t_0,\gamma_k) > 0$ for any t_0 and $k\in\mathbb{N}$, we have $\gamma_k < \gamma_{k+1}$, $k = 0,1,2,\dots$. We claim that $\lim_{k\to\infty}\gamma_k = \infty$. If not, let $\lim_{k\to\infty}\gamma_k = L < \infty$. Then from (18) we have

$L = \gamma_0 + \Gamma_*(t_0, L)$. First assume that $M := M_* = M^*$. Letting t_0 to ∞ in $\Gamma_*(t_0, L)$ we obtain

$$\Gamma_*(\infty, L) = \begin{cases} L^\beta & \text{for } M = 0, \\ \frac{L}{1-(1+M)^{1-\alpha}} \left(1 - \frac{1}{(1+L^{\beta-1}M)^{\alpha-1}}\right) & \text{for } 0 < M < \infty, \\ L & \text{for } M = \infty. \end{cases}$$

Next we show that the equation (18) after this limit process, i.e., the equation

$$L = \gamma_0 + \Gamma_*(\infty, L) \tag{19}$$

has no real solution. Indeed, if $M = \infty$, then (19) yields $L = \gamma_0 + L$, but we assume $\gamma_0 > \gamma(\infty) = 0$, contradiction. If $M = 0$, then $L = \gamma_0 + L^\beta$, but in this case there is again contradiction. Indeed, an application of the L'Hospital rule in (15) gives $\gamma(0) = \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. Moreover, we assume $\gamma_0 > \gamma(0)$, and then it is not difficult to see that the graph of $L \mapsto \gamma_0 + L^\beta$ does not cross the graph of $L \mapsto L$. If $0 < M < \infty$, then we proceed similarly as for $M = 0$. We will show that the parabola-like curve $\gamma(M) + \Gamma_*(\infty, L)$ touches the line L at $L = (((M+1)^{(\alpha-1)/\alpha} - 1)/M)^{\alpha-1}$, and since we have (19) with $\gamma_0 > \gamma(M)$, we obtain a contradiction. Recall that we are interested just in the first quadrant. After some computations we have

$$\frac{\partial}{\partial L}\Gamma_*(\infty, L) = \frac{1 - (1 + ML^{\beta-1})^{-\alpha}}{1 - (M+1)^{1-\alpha}}.$$

Now it is easy to see that

$$\frac{\partial}{\partial L}\Gamma_*(\infty, L) = 1 \quad \text{at} \quad L = \left(\frac{(M+1)^{\frac{\alpha-1}{\alpha}} - 1}{M}\right)^{\alpha-1}.$$

Moreover, we have $\frac{\partial}{\partial L}\Gamma_*(\infty, L) > 0$ and also $\frac{\partial^2}{\partial L^2}\Gamma_*(\infty, L) > 0$ for $L > 0$ and $M > 0$, from which the statement follows. It is interesting to observe how the computations for the case $M > 0$ resemble the case $M = 0$ when taking the limits as $M \to 0+$. In view of the above facts we must have $\gamma_k \to \infty$ as $k \to \infty$, which implies $\varphi_k(t) \to \infty$ as $k \to \infty$ for $t \geq t_0$, where t_0 is sufficiently large. Consequently, (1) is oscillatory by Lemma 1. Now we examine the case when $M_* < M^*$. We will show that (19) with $\gamma_0 > \gamma(M_*)$ has no real positive solution. Observe that

$$\Gamma_*(\infty, L) = \lim_{x \to \bar{M}} \frac{L}{1 - (1+x)^{1-\alpha}} \left(1 - \frac{1}{(1 + L^{\beta-1}x)^{\alpha-1}}\right),$$

where $\bar{M} \in [M_*, M^*]$. This follows from the fact that

$$x \mapsto \lim_{t \to x} \frac{L}{1 - (1+t)^{1-\alpha}} \left(1 - \frac{1}{(1 + L^{\beta-1}t)^{\alpha-1}}\right) \tag{20}$$

is decreasing or increasing (depending on the value of L). In any case, extrema occur at boundary points, i.e., at M_*, M^*. Using the arguments as in the previous part yields that the equation $L = \bar{\gamma}_0 + \Gamma_*(\infty, L)$ has no real positive solution provided $\bar{\gamma}_0 > \gamma(\bar{M})$. It is not difficult to see that $x \mapsto \gamma(x)$ is decreasing on $[0, \infty)$. Hence, $\gamma(\bar{M}) \le \gamma(M_*) < \gamma_0$, and so neither the equation (19) has a real positive solution. The rest of the proof is the same as in the case $M_* = M^*$. Note that M_* in condition (16) is the best value which can be attained when using the procedure of this proof. This follows from the fact that the function (20) is nondecreasing when $L \in [0, 1]$, and a closer examination of the proof shows that we are interested just in such L's.

Nonoscillatory part. First note that the examination of the case $M^* = \infty$ (i.e., $\gamma(M^*) = 0$) may obviously be excluded, in view of the assumptions of the theorem. Condition (17) can be rewritten as $\varphi_0(t) \le \delta_0 R^{1-\alpha}(t)$ for large t, say $t \ge t_0 > a$, where $0 < \delta_0 < \gamma(M^*)$. Similarly as in the previous part, we get

$$\varphi_k(t) \le \delta_k R^{1-\alpha}(t), \tag{21}$$

where

$$\delta_k = \delta_0 + \Gamma^*(t_0, \delta_{k-1}), \tag{22}$$

$k = 1, 2, \ldots$, with

$$\Gamma^*(t_0, u) = \sup_{t \ge t_0} \lim_{\lambda \to \mu(t)} \left\{ \frac{u}{1 - (1 + \lambda r^{1-\beta}(t)/R(t))^{1-\alpha}} \right. \\ \left. \times \left(1 - \frac{1}{(1 + \lambda u^{\beta-1} r^{1-\beta}(t)/R(t))^{\alpha-1}} \right) \right\}.$$

Clearly, $\{\delta_k\}$ is increasing. We claim that it converges. First assume $M = M_* = M^*$. To show the convergence, consider the fixed point problem $x = g(x)$, where

$$g(x) = \omega + \lim_{s \to M} \frac{x}{1 - (1 + s)^{1-\alpha}} \left(1 - \frac{1}{(1 + x^{\beta-1} s)^{\alpha-1}} \right)$$

with a positive parameter ω, and the "perturbed" problem $x = \tilde{g}(x)$, where $\tilde{g}(x) = \omega + \Gamma^*(t_0, x)$. Note that we are particularly interested in the first quadrant. First we examine the problem $x = g(x)$. Recall that $g(x) = \omega + x^\beta$ if $M = 0$. The fixed point will be found by means of th iteration scheme $x_k = g(x_{k-1})$, $k = 1, 2, \ldots$. The graph of g with $\omega > 0$ is a parabola-like curve where g is positive and increasing on $[0, \infty)$. If $\omega = \gamma(M)$, then the graph of g touches the line x at $x = \lim_{s \to M}(((M+1)^{(\alpha-1)/\alpha} - 1)/M)^{\alpha-1}$ (note that this value is equal to $((\alpha-1)/\alpha)^{\alpha-1}$ when $M = 0$). Therefore if we choose $x_0 = \omega = \gamma(M)$ (noting that $\gamma(M) < \lim_{s \to M}(((M+1)^{(\alpha-1)/\alpha} - 1)/M)^{\alpha-1}$), then we see that the approximating sequence $\{x_k\}$ for the problem $x = g(x)$ is strictly increasing and converges to $\lim_{s \to M}(((M+1)^{(\alpha-1)/\alpha} - 1)/M)^{\alpha-1}$. Clearly, if $0 < y_0 = \omega < \gamma(M)$, then the approximating sequence $\{y_k\}$ for the same problem, i.e., satisfying $y_k = g(y_{k-1})$, $k = 1, 2, \ldots$, is increasing as well and permits $y_k < x_k < \lim_{s \to M}(((M+1)^{(\alpha-1)/\alpha} - 1)/M)^{\alpha-1}$, $k \in \mathbb{N}$. Therefore $\{y_k\}$ converges. Now we take into account that $\lim_{t_0 \to \infty} \Gamma^*(t_0, x) = g(x) - \omega$. Hence the function $\tilde{g}$ in the perturbed problem can be made as close to g as we need (locally, on the interval under consideration) provided t_0 is sufficiently large. This closeness of g to $\tilde{g}$ along with the inequality $\delta_0 < \gamma(M)$ lead to the fact that the sequence $\{\delta_k\}$ for the original problem (22) converges for t_0 large. Thus $\{\varphi_k(t)\}$ converges by (21), and so (1) is nonoscillatory by Lemma 1. The case when $M_* < M^*$ can be treated similarly, combining the ideas of the last part of "oscillatory proof" with the previous part of this "nonoscillatory proof". $\square$

If there exists the limit of the expression in (14), then we may establish Hille-Nehari type criteria with the critical constant which is sharp. The L'Hospital rule is utilized in (15) when evaluating $\gamma(0)$ and $\gamma(\infty)$.

Corollary 1. *Let $M = M_* = M^*$ in Theorem 1. Then $\gamma(M)$ is the critical constant, i.e., the constants on the right-hand sides of* (16) *and* (17) *are the same. In particular,*

$$\gamma(M) = \begin{cases} \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} & \textit{if}\ \ M = 0, \\ \left(\frac{(M+1)^{\frac{\alpha-1}{\alpha}}-1}{M}\right)^{\alpha-1}\left(1 - \frac{1-(M+1)^{-\frac{(\alpha-1)^2}{\alpha}}}{1-(M+1)^{1-\alpha}}\right) & \textit{if}\ \ 0 < M < \infty, \\ 0 & \textit{if}\ \ M = \infty. \end{cases}$$

Remark 1. Note that $x \mapsto \gamma(x)$ is decreasing on $[0, \infty)$, and hence

$$\left(\frac{(M+1)^{\frac{\alpha-1}{\alpha}}-1}{M}\right)^{\alpha-1}\left(1 - \frac{1-(M+1)^{-\frac{(\alpha-1)^2}{\alpha}}}{1-(M+1)^{1-\alpha}}\right) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{23}$$

if $M > 0$. Therefore we see that the critical constant is not invariant with respect to time scales, and it may be strictly less than the constant known from the continuous theory.

As a special case of the previous corollary we get the criteria that match what is known from the linear theory, see [23].

Corollary 2. *Let $M = M_* = M^*$ and $\alpha = 2$ in Theorem 1. Then $\gamma(M)$ is the critical constant satisfying*

$$\gamma(M) = \begin{cases} \frac{1}{4} & \textit{if}\ \ M = 0, \\ \frac{1}{\left(\sqrt{M+1}+1\right)^2} & \textit{if}\ \ 0 < M < \infty, \\ 0 & \textit{if}\ \ M = \infty. \end{cases}$$

4 Critical and oscillation constant

As we already said, if the constants on the right-hand sides of (16) and (17) are the same, then it is called a *critical constant* (or a *critical oscillation constant*). This happens when $M = M_* = M^*$ in (14). Sometimes, in similar situations in the literature, this constant is said to be an *oscillation constant*. However, we prefer to use the former terminology since the latter one has sometimes another meaning, which concerns conditionally oscillatory equations discussed in the next section. As we will see, there is a connection between critical and oscillation constants: Hille–Nehari criteria involving the critical constant can be used to derive the oscillation constant. The term "critical" constant reflects the fact that this constant cannot be improved and forms a sharp border between oscillation and nonoscillation. Note that the strict inequalitities in Hille–Nehari criteria cannot be replaced by nonstrict ones since no conclusion can be drawn if either $\liminf_{t\to\infty} \mathcal{A}(t)$ or $\limsup_{t\to\infty} \mathcal{A}(t)$ equals the critical constant; both oscillation and nonoscillation may happen, as it has already been shown in the linear continuous case, see e.g. [24]. Our results show that if $\liminf_{t\to\infty} \mathcal{A}(t) > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$, then (1) is oscillatory (no matter what time scale is, since (23) holds). However, in addition, our Theorem 1 says that $\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$ is not the best possible constant which is universal for all time scales (in particular, it may not be critical at all). In fact, the constant depends on a time scale and also on the coefficient r; the cases happen where it is strictly less than $\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. If $M = M_* = M^*$ is satisfied, then for the critical constant it holds $\gamma(M) \in \left[0, \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\right]$. Later we will present examples where $\gamma(M) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$. We conclude this item with noting that oscillation of (1) is still possible even when $\liminf_{t\to\infty} \mathcal{A}(t) < \gamma(M)$. This follows from the following theorem, and we emphasize that there is needed no additional condition on a time scale.

Theorem 2 ([22, Theorem 4]). *If* $\limsup_{t\to\infty} \mathcal{A}(t) > 1$*, then* (1) *is oscillatory.*

5 Strong and conditional oscillation

Consider the equation

$$(r(t)\Phi(y^\Delta))^\Delta + \lambda p(t)\Phi(y^\sigma) = 0, \tag{24}$$

where $r(t) > 0$, $p(t) > 0$, and λ is a real parameter. In the linear continuous case, the concept of strong and conditional oscillation was introduced by Nehari [18]. We say that (24) is *conditionally oscillatory* if there exists a constant $0 < \lambda_0 < \infty$ such that (24) is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. The value λ_0 is called the *oscillation constant* of (24). If equation (24) is oscillatory (resp. nonoscillatory) for every $\lambda > 0$, then this equation is said to be *strongly oscillatory* (resp. *strongly nonoscillatory*). Next we apply the results from the previous section to derive necessary and sufficient conditions for strong (non)oscillation.

Theorem 3. *Assume that* $M^* < \infty$*. Then* (24) *is strongly oscillatory if and only if* $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$*, and it is strongly nonoscillatory if and only if* $\lim_{t\to\infty} \mathcal{A}(t) = 0$.

Proof. Denote $R(t) := \int_a^t r^{1-\beta}(s)\,\Delta s$. If $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$ holds, then we have $\limsup_{t\to\infty} R^{\alpha-1}(t) \int_t^\infty \lambda p(s)\,\Delta s > 1$ for every $\lambda > 0$, and so (24) is oscillatory for every $\lambda > 0$ by Theorem 2. Conversely, if (24) is strongly oscillatory, then

$$\limsup_{t\to\infty} R^{\alpha-1}(t) \int_t^\infty \lambda p(s)\,\Delta s \geq \gamma(M^*) > 0 \tag{25}$$

for every $\lambda > 0$ by Theorem 1. This implies $\limsup_{t\to\infty} \mathcal{A}(t) = \infty$, otherwise (25) would be violated for sufficiently small λ. The proof of the part concerning strong nonoscillation is based on similar arguments. The details are left to the reader. $\square$

One could ask whether the condition $M^* < \infty$ in the last theorem may be dropped. In general, the answer is no. Realize that strong oscillation [strong nonoscillation] of (24) is nothing but $\lambda_0 = 0$ [$\lambda_0 = \infty$], where λ_0 is the oscillation constant. Now assume that $M^* = \infty = M_*$ and $\lim_{t\to\infty} \mathcal{A}(t) = L \in (0, \infty)$ exists. Then $\lim_{t\to\infty} R^{\alpha-1}(t) \int_t^\infty \lambda p(s)\,\Delta s = \lambda L > 0$ for every $\lambda > 0$. This implies strong oscillation of (24), however the condition $\limsup_{t\to\infty} R^{\alpha-1}(t) \int_t^\infty \lambda p(s)\,\Delta s = \infty$ does not hold. A particular example of such strongly oscillatory equation will be given later.

6 Euler type equations and Kneser type criteria

Consider the equation

$$(\Phi(y^\Delta))^\Delta + \lambda(-t^{1-\alpha})^\Delta \Phi(y^\sigma) = 0, \tag{26}$$

where λ is a positive parameter. Note that we are interested only in positive λ's since for $\lambda = 0$, equation (26) is readily explicitly solvable, it is nonoscillatory, and thus for $\lambda < 0$ is nonoscillatory as well by the Sturm type comparison theorem. Equation (26) will be called a *generalized Euler dynamic equation* since for $\mathbb{T} = \mathbb{R}$ and $\alpha = 2$ it reduces to the well known Euler differential equation $y'' + \lambda t^{-2} y = 0$. Also note that (26) yields the generalized (half-linear) Euler differential equation $(\Phi(y'))' + \lambda(\alpha - 1)t^{-\alpha}\Phi(y) = 0$ when $\mathbb{T} = \mathbb{R}$ and it yields the linear dynamic equation $y^{\Delta\Delta} + (\lambda/(t\sigma(t)))y^\sigma = 0$ when $\alpha = 2$. Applying Theorem 1, we get that (26) is oscillatory provided $\lambda > \gamma(M_*)$ and nonoscillatory provided $\lambda > \gamma(M^*)$. Assume that $M = M_* = M^*$. Then $M = \lim_{t\to\infty} \mu(t)/t$, $\gamma(M)$ is the critical constant and $\lambda_0 = \gamma(M)$ is the oscillation constant. Now if, for example $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, then $M = 0$ and $\gamma(M) = ((\alpha - 1)/\alpha)^{\alpha-1}/\alpha$. Note that $\gamma(M) = ((\alpha - 1)/\alpha)^{\alpha-1}/\alpha$ for all time scales whose graininess $\mu(t)$ is asymptotically less than t; for example, $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$ (then $\mu(t) = 1 + 2\sqrt{t}$). If we assume that $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, then (26) reduces to the generalized Euler q-difference equation, $\mu(t) = (q-1)t$, and $M = q - 1 > 0$. Hence the critical constant is

$$\gamma(M) = \left(\frac{q^{\frac{\alpha-1}{\alpha}} - 1}{q + 1}\right)^{\alpha-1} \left(1 - \frac{1 - q^{-\frac{(\alpha-1)^2}{\alpha}}}{1 - q^{1-\alpha}}\right) < \frac{1}{\alpha}\left(\frac{\alpha - 1}{\alpha}\right)^{\alpha-1}.$$

Finally assume that $\mathbb{T} = 2^{\omega^{\mathbb{N}_0}} := \{2^{\omega^k} : k \in \mathbb{N}_0\}$ with $\omega > 1$. Then $\mu(t) = t^\omega - t$ and so $M = \infty$. Hence, the critical constant is $\gamma(M) = 0$. This implies that (26) on $\mathbb{T} = 2^{\omega^{\mathbb{N}_0}}$ is oscillatory for all $\lambda > 0$. Therefore, equation (26) is strongly oscillatory when $\mathbb{T} = 2^{\omega^{\mathbb{N}_0}}$ while it is conditionally oscillatory in all other previous cases. One may wish to consider a different type of generalized Euler equation, namely

$$(\Phi(y^\Delta))^\Delta + \frac{\bar{\lambda}}{\sigma^\alpha(t)} y^\sigma = 0 \tag{27}$$

or $(\Phi(y^\Delta))^\Delta + \tilde{\lambda} t^{-\alpha} y^\sigma = 0$. Note that (27) may arise as a quite natural form of the equation where the process of discretization of

$$(\Phi(y'))' + \frac{\bar{\lambda}}{t^\alpha}\Phi(y) = 0 \tag{28}$$

is reflected. Indeed, a discrete counterpart of the last equation may be of the form

$$\Delta(\Phi(\Delta y_k)) + \frac{\bar{\lambda}}{(k+1)^\alpha}\Phi(y_{k+1}) = 0. \tag{29}$$

Next we compare oscillatory behavior of equations (26) and (27) in some special cases. Assume that λ_0 is the oscillation constant of (26). If $\mathbb{T} = \mathbb{R}$, then

$$\bar{\lambda}_0 = (\alpha - 1)\lambda_0 = \left(\frac{\alpha - 1}{\alpha}\right)^\alpha$$

is the oscillation constant of (28). Since $\Delta(-t^{1-\alpha}) = (\alpha - 1)\xi^{-\alpha}(t) \sim (\alpha - 1)(t+1)^{-\alpha}$ with $t \le \xi(t) \le t+1$, the oscillation constant of (29) is the same as in the continuous case. If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then (27) reduces to the q-difference equation

$$D_q(\Phi(D_q y(t))) + \frac{\bar{\lambda}}{(qt)^\alpha}\Phi(y(qt)) = 0.$$

Since $D_q(-t^{1-\alpha}) = (1 - q^{1-\alpha})/(t^\alpha(q-1))$, for the oscillation constant of this equation we have

$$\bar{\lambda}_0 = \frac{q^\alpha - q}{q - 1}\lambda_0 = \frac{q^\alpha - q}{q - 1}\left(\frac{q^{\frac{\alpha-1}{\alpha}} - 1}{q + 1}\right)^{\alpha - 1}\left(1 - \frac{1 - q^{-\frac{(\alpha-1)^2}{\alpha}}}{1 - q^{1-\alpha}}\right).$$

Note how this result resembles the continuous counterpart when taking the limit as $q \to 1$. In particular $(q^\alpha - q)/(q - 1) \to \alpha - 1$ as $q \to 1$. One can see how a seemingly insignificant change from (26) to (27) may have an influence to the oscillatory behavior in a general case.

Next we will consider a generalized form of (26), namely

$$(r(t)\Phi(y^\Delta))^\Delta + \lambda(-R^{1-\alpha}(t))^\Delta \Phi(y^\sigma) = 0, \tag{30}$$

where λ is a positive parameter and $R(t) = \int_a^t r^{1-\beta}(s)\,\Delta s$ with $R(\infty) = \infty$. If $r(t) \equiv 1$, then (30) reduces to (26). In the continuous oscillation theory, i.e., $\mathbb{T} = \mathbb{R}$, there is no substantial difference between (26) and (30) owing to the transformation of independent variable $t \mapsto R(t)$, and so without loss of generality it is sufficient to examine just (26). However, in a general time scale case (even when e.g. $\mathbb{T} = \mathbb{Z}$) such a transformation is not available, and so considering the case (30) with $r(t) \not\equiv 1$ brings new observations. According to Corollary 1, the critical constant is equal to $\gamma(M)$, where $M = \lim_{t\to\infty} \mu(t) r^{1-\beta}(t)/R(t)$, and a simple calculation shows that for the associated oscillation constant of (30) one has $\lambda_0 = \gamma(M)$. Similarly as for (26), a discussion can be made for (30) distinguishing the cases where $\mu(t)$ is asymptotically less than or equal to or greater than $r^{\beta-1}(t)R(t)$. The details are left to the reader. It is not difficult to find an example of r showing that $M > 0$ may occur even in the well-studied discrete case that is the result which has not been known before. Equations of the form (30) may be very useful for comparison purposes. In particular, applying the generalized Sturm type theorem ([20, Theorem 3]), where (1) and (30) are compared, leads to the following theorem (where we do not need to assume (11)).

Theorem 4. *Let the (finite or infinite) limit*

$$M = \lim_{t\to\infty} \frac{\mu(t)r^{1-\beta}(t)}{R(t)}$$

exist. If

$$\liminf_{t\to\infty} \frac{p(t)}{(-R^{1-\alpha})^{\Delta}(t)} > \gamma(M),$$

then (1) *is oscillatory. If*

$$\limsup_{t\to\infty} \frac{p(t)}{(-R^{1-\alpha})^{\Delta}(t)} < \gamma(M),$$

then (1) *is nonoscillatory.*

Naturally, $\gamma(M)$ is again a sharp constant. For historical reasons, this theorem can be called a *generalized Kneser theorem*. Indeed, if $\mathbb{T} = \mathbb{R}$, $\alpha = 2$, and $r(t) \equiv 1$, then we obtain that $\liminf_{t\to\infty} t^2 p(t) > 1/4$ implies oscillation of $y'' + p(t)y = 0$, while $\limsup_{t\to\infty} t^2 p(t) < 1/4$ implies nonoscillation of $y'' + p(t)y = 0$ (cf. [24]). A slight modification of the above approach yields Kneser type criteria in the case when $M_* < M^*$. As before, the results are new even in the discrete case.

Similarly as the Sturm type theorem was applied to obtain Kneser type criteria, the Hille–Wintner type comparison theorem (see [22, Theorem 11]) can be used to obtain (back) Hille–Nehari type criteria. Recall that in contrast to the Sturm type theorem, where the comparison of the coefficients is pointwise, the comparison of the coefficients in the second terms of equations in a Hille-Wintner sense is “integral”, i.e., “on average”. Thus we have an alternative approach how to prove Corollary 1 (or Theorem 1). However, there is a disadvantage of this approach since in general we do not know how to describe solutions of the generalized Euler type equation (even when $r(t) \equiv 1$ and $\alpha = 2$) in such a way which would provide an exact information about the oscillation constant (and consequently critical constant). The approach from Section 3 does not require a knowledge of such constant, what's more that statement factually provides it. Finally note that there are situations where the Kneser type criteria do not apply while the Hille-Nehari ones do, see e.g. [23].

7 Examples from q-calculus

We have already presented some examples in the previous section. Here we add some more ones. We prefer to show a “non-classical” case (i.e., $\mathbb{T} \neq \mathbb{R}$ and $\mathbb{T} \neq \mathbb{Z}$), thus let us for example assume that $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$. We will compute the value of the critical constant for two different coefficients $r(t)$ to obtain nice examples of conditionally oscillatory Euler type q-difference equations of the form

$$D_q(r(t)\Phi(D_q y(t))) + \lambda D_q(-R^{1-\alpha}(t))\Phi(y(qt)) = 0 \tag{31}$$

with the known oscillation constant that can be further used for comparison purposes.

First assume that $r(t) = t^\delta$ with $\delta \in \mathbb{R}$, $\delta \le \alpha - 1$. Recall that for $f : \mathbb{T} \to \mathbb{R}$, it holds $\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t)f(t)$. Then, with $t = q^n$, $n \in \mathbb{N}_0$, and $\delta < \alpha - 1$ we have

$$\begin{aligned} M &= \lim_{t\to\infty} \frac{\mu(t)r^{1-\beta}(t)}{\int_1^t r^{1-\beta}(s)\,\Delta s} = \lim_{n\to\infty} \frac{(q-1)q^n((q^n)^\delta)^{1-\beta}}{\sum_{j=0}^{n-1} \mu(q^j)((q^j)^\delta)^{1-\beta}} \\ &= \lim_{n\to\infty} \frac{(q^{1+\delta-\delta\beta})^n}{\sum_{j=0}^{n-1}(q^{1+\delta-\delta\beta})^j} = \lim_{n\to\infty} \frac{q^{(1+\delta-\delta\beta)n}(q^{1+\delta-\delta\beta}-1)}{q^{(1+\delta-\delta\beta)n}} \\ &= q^{1+\delta-\delta\beta} - 1, \end{aligned}$$

where L'Hospital's rule is used. The same result is obtained when $\delta = \alpha - 1$ since $D_q(\ln t) = \ln q/((q-1)t)$. Thus the associated critical constant to equation (31) is equal to $\gamma(q^{1+\delta-\delta\beta} - 1)$.

Now assume that $r(t) = \zeta^{\log_q(1/t)}$ with $\zeta \ge q^{1-\alpha}$. Then, with $t = q^n$, $n \in \mathbb{N}_0$, we have $r(t) = \zeta^{-n}$. Applying similar arguments as above, we get $M = q\zeta^{\beta-1} - 1$, and so the associated critical constant to equation (31) is equal to $\gamma(q\zeta^{\beta-1} - 1)$.

The details how to specify the second coefficient in (31) are left to the reader. Note just that for general case we have

$$D_q(R^{1-\alpha}(t)) = (q-1)^{\alpha-2}\left[\left(\sum_{j=0}^{n} q^{j-\frac{n}{\alpha-1}} r^{1-\beta}(q^j)\right)^{\alpha-1} - \left(\sum_{j=0}^{n-1} q^{j-\frac{n}{\alpha-1}} r^{1-\beta}(q^j)\right)^{\alpha-1}\right]$$

with $t = q^n$, $n \in \mathbb{N}_0$, and in some particular cases this expression can substantially be simplified.

From the previous section we know that for the oscillation constant of (31) one has $\lambda_0 = \gamma(M)$, where $M = q^{1+\delta-\delta\beta} - 1$ for the former r while $M = q\zeta^{\beta-1} - 1$ for the latter r.

8 Concluding remarks

In this last section we indicate some directions for a future research related to the topic of this paper. We have established Hille-Nehari type criteria for half-linear dynamic equations. The theorems are new even in some special cases, and in view of the presence of the critical constant, the results are non-improvable in a certain sense. A much wider class of equations than before can now be examined by our criteria. In spite of these facts, there are still some problems to solve in this field. Moreover, our results could serve for possible establishing new results in related fields.

In the oscillation and asymptotic theory of second order equations, it is usual to distinguish between the divergence and the convergence of the integral of the coefficient in the differential term, i.e., in our case, $R = \int_a^\infty r^{1-\beta}(s)\,\Delta s$. The properties of this integral substantially affects the behavior of solutions. In our paper we assume the divergence. Thus it is natural to ask how about the case when $R < \infty$. In a linear case, there is no substantial difference between these two cases in view of the transformation of dependent variable (we transfer an equation with $R < \infty$ into an equation with $R = \infty$ and then we may apply known criteria). However, in a half-linear case such a transformation is not available (even in the continuous case), and hence equations with $R < \infty$ needs to be investigated by means of other methods.

Another question is what happens with oscillation or nonoscillation of (1) when $M_* < M^*$, i.e., $\gamma(M^*) < \gamma(M_*)$, and the limit values of the expression $\mathcal{A}(t)$ as $t \to \infty$ remains between $\gamma(M^*)$ and $\gamma(M_*)$. Such a situation is not very common in applications but generally it may occur.

The final remark concerns a utilization of our results in the theory of inequalities. The so-called *variational principle* (see [20]) says that there is an equivalence between the positive definiteness of the α-degree functional

$$\int_a^b \left\{ r(t)|\xi^\Delta(t)|^\alpha - p(t)|\xi^\sigma(t)|^\alpha \right\}\,\Delta s$$

on a certain class of the so-called admissible functions and the disconjugacy of (1) on $[a,b]$. Thanks to this relation we hope that the application of Hille-Nehari criteria could serve in establishing a Hardy type inequality (involving delta integrals) in quite general setting. In particular, we believe that the knowledge of the oscillation constant in the generalized Euler equation could yield the Hardy type inequality involving the best possible constant (depending on time scales). Till now there is known just that

$$\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t) - a} \right)^\alpha \Delta t < \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \int_a^\infty (f(t))^\alpha \Delta t,$$

where $F(t) = \int_a^t f(s)\,\Delta s$. The constant is the best possible provided $\lim_{t\to\infty} \mu(t)/t = 0$ (see [21]). A closer examination (in particular, applying the variational principle and the above inequality to an Euler type inequality) shows that this result coincides with our observations when $M = \lim_{t\to\infty} \mu(t)/t = 0$. Indeed, the fact that the constant in Hille-Nehari type criteria is critical corresponds to the fact that the constant in a Hardy type inequality is the best possible. What remains to solve and will hopefully be handled in our subsequent paper is the case $M \neq 0$. The cases with more general weight functions which then correspond to appropriate choices of r are also of interest.

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